SYMMETRY BREAKDOWN IN THE WIGHTMAN AXIOMATIC SCHEME

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Submitted to JETP editor September 20, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) 50, 954–957 (April, 1966)

The problem of symmetry breakdown within the Wightman axiomatic scheme is considered on the basis of the previously introduced[6] concept of a system of dynamic equations of a theory. Two possible types of symmetry breakdown are considered. The results of the previous paper[6] are used for the analysis of the possibility of both types of symmetry breakdown.

1. INTRODUCTION

LET a set of dynamical equations be given (i.e., a set of equations for the field operators) and consider the problem of determining a given field theory in terms of the system of dynamical equations. First to arise, of course, are the questions of existence and uniqueness of the solution.

Usually the equations obeyed by the field operators are derived from a Lagrangian, and their solutions are sought by means of the perturbation method. The existence and uniqueness of the solution are tacitly assumed. A consequence of the uniqueness of the solution is, among others, the fact that the symmetry of the solution is the same as the symmetry of the equations. Along with this usual scheme there have also appeared models in which the solutions of the dynamical equations are not found by means of perturbation theory and are not unique.[1,2] In particular, the theory of superconductivity is constructed in just this manner.[3]

A direct consequence of the non-uniqueness of the solution of the system of dynamical equations is the possibility of symmetry breakdown, i.e., of a situation in which the symmetry group of the solution does not coincide with the symmetry group of the dynamical equations. An arbitrary transformation from the symmetry group of the dynamical equations must, of course, transform a solution into a solution but need not necessarily transform each such solution into itself, as long as the solution is not unique. Therefore the symmetry group of the solution may be narrower than the symmetry group of the dynamical equations.

Indeed, parity is violated in this manner in the Goldstone model,[1] γ_5-invariance is violated in Nambu's model,[2] and gauge invariance is violated in the theory of superconductivity.[3]

In the present paper the problem of symmetry breakdown will be considered in the framework of the Wightman axiomatic scheme.

2. TWO TYPES OF SYMMETRY BREAKDOWN

Symmetry breakdown is one of the consequences of the non-uniqueness of the relation between the dynamics of a theory and the theory itself. Therefore it is necessary first to give a precise definition of the meaning of these concepts.

In the Wightman scheme a theory is considered given if the Wightman functional W (a positive linear functional on the fundamental algebra A, satisfying additional conditions which are physically motivated) is given.[4] It has been shown[4, 5] that all physically relevant quantities can be constructed in terms of this functional.

Since a given positive linear functional over an algebra A defines a representation R(A) of A up to unitary equivalence, one may consider that a physical theory is determined by giving a definite representation of the algebra A.

The concept of a family of dynamical equations in a theory of the Wightman type has been defined previously.[6] Such a family is determined by the kernel M of a given representation of the algebra A. The kernel of a representation is the set of all $g \in A$ such that $a(g) = 0$ (here g is an element of A, a(g) its representer in R(A)). Two representations with identical kernels are algebraically isomorphic, but the corresponding physical theories may differ (the theories coincide only if the representations are unitarily equivalent[1]).

[1]This statement is at variance with the concept of physical equivalence used by Haag and Kastler (Journ. Math. Phys. 5, 848 (1964)) where the requirement of physical equivalence coincides with the concept of "weak equivalence" introduced by Fell (translator's note).

The consideration of any symmetry within the Wightman framework starts with the introduction of a group of automorphisms $\mathcal{T}$ of the algebra A. Invariance of the dynamics under the group is the invariance of the kernel M with respect to the group of automorphisms $\mathcal{T}$ of A. For the invariance of M it is necessary and sufficient that for any $g \in M$, $\tau \in \mathcal{T}$ the action of the automorphism τ on the element g should leave that element inside the kernel M.

On the other hand, invariance of the theory with respect to the given symmetry means that the corresponding Wightman functional is invariant, i.e., $W(g_\tau) = W(g)$, where g_τ is the result of the action of $\tau \in \mathcal{T}$ on $g \in A$.

It is easy to see that the invariance of a theory implies the invariance of its kernel, but that the opposite is not generally true. The situation in which the dynamics is invariant under a symmetry whereas the functional is not, is called breakdown of symmetry.

Two types of symmetry breakdown are possible in principle.

The first type is trivial and corresponds to the case when the kernel is not sufficiently large, i.e., the case where the set of dynamical equations is incomplete. Mathematically this manifests itself in the reducibility of the representation with the given kernel. Let us consider an example.

On the basis of any theory with a dynamics which is not invariant under a given symmetry, one can construct a theory in which this symmetry manifests itself as a broken symmetry. Let us consider for simplicity, the case of a group $\mathcal{T}$ with a finite number of elements $\tau_1 \ldots \tau_n$; $\tau_1 = 1$. Let W be an irreducible functional, for which the dynamics is non-invariant under this symmetry. We now construct n functionals W_i in the following manner: $W_i(g_{\tau_i}) = W(g)$. Since $\mathcal{T}$ is a group of automorphisms of the *-algebra A we have

$$(g^+g)_\tau = (g^+)_\tau \cdot g_\tau = g_\tau^+ \cdot g_\tau$$

and consequently each W_i will be a positive functional over A. It is easy to show that the kernel M_i of the representation defined by the functional W_i is obtained from M by the action of the automorphism τ_i: $M_i = \{g_{\tau_i} : g \in M\}$. Neither of the kernels M_i is invariant with respect to the group $\mathcal{T}$.

We now construct the reducible functional

$$\overline{W} = \sum_{i=1}^{n} \rho_i W_i, \quad \rho_i > 0, \quad \sum_{i=1}^{n} \rho_i = 1.$$

This functional represents a theory with symmetry breakdown of the first kind.

Indeed, the kernel $\overline{M}$ of the functional $\overline{W}$ is the intersection of all kernels M_i and is therefore invariant under the group $\mathcal{T}$, whereas the functional itself is not invariant under this symmetry, except in the case $\rho_1 = \rho_2 = \ldots = \rho_n$.

In this case the symmetry breakdown is caused by the fact that the kernel $\overline{M}$ is not sufficiently large, i.e., the family of dynamical equations $a(g) = 0$, $g \in \overline{M}$ is not sufficient to determine an irreducible theory. In order to specify an irreducible solution, for instance W_i, this family of dynamical equations must be completed, extending the kernel $\overline{M}$ to M_i. It is therefore clear that a symmetry breakdown of the first kind is always accompanied by vacuum degeneracy.

The second and more interesting type of symmetry breakdown corresponds to the case where there are two or more irreducible functionals having the same kernel M which is invariant with respect to the symmetry $\mathcal{T}$. If to a given set of dynamical equations, i.e., to a given kernel M there would correspond a single functional W, this functional would automatically be invariant under the symmetry $\mathcal{T}$. Indeed, since M is invariant under the symmetry $\mathcal{T}$, the kernel M_i of the functional W_i $(W_i(g_\tau^i) = W(g))$ coincides with M, and, since by assumption there is only one functional corresponding to M, we have $W_i = W$.

If however there are several different irreducible functionals corresponding to a given kernel, then there appears the possibility of symmetry breakdown. In this case the symmetry breakdown is a consequence of the dynamics itself, and not simply due to the incompleteness of the set of dynamical equations. This breakdown cannot be removed by simple adjunction of additional dynamical equations. Such a symmetry breakdown is not necessarily accompanied by vacuum degeneracy, since a theory with such a breakdown may be irreducible.

3. THE POSSIBILITY OF SYMMETRY BREAKDOWN

It follows from the considerations of the preceding section that symmetry breakdown of the first kind is a quite frequent phenomenon. More precisely: to any theory which does not possess a symmetry (i.e., for which the kernel is not invariant under this symmetry) one can associate a whole class of reducible theories for which this symmetry appears as a broken one (symmetry breakdown of the first kind). The only exception in this class is a reducible theory possessing the symmetry completely (the case $\rho_1 = \rho_2 = \ldots = \rho_n$).

Let us now consider symmetry breakdown of the second kind. The possibility of this breakdown is not so obvious as for the first kind. In order for such a breakdown to be possible it is necessary that there exist at least two different irreducible theories with the same kernel.

From this point of view, the selection of the initial algebra A, on which the Wightman functional is defined, plays an essential role. Let us assume that the Wightman functional is well defined also as a functional over the algebra A′ which contains the algebra A. Then two irreducible functionals which have the same kernel when considered as functionals over A may possess different kernels when considered as functionals over the algebra A′. It is clear from here that by enlarging the initial algebra the possibility of symmetry breakdown of the second kind becomes less likely.

The author[6] has constructed one of the possible extensions of the algebra A and has shown that the kernel of an irreducible functional (over the extended algebra) determines the extension uniquely. Consequently, if one selects as the initial algebra this extension, there is no room for symmetry breakdown of the second type.

Consideration of the extended algebra also simplifies the study of symmetry breakdown of the first type. One can show in this case, that the example of symmetry breakdown given in the preceding section represents a quite general symmetry breakdown scheme for a group with a finite number of elements. More precisely, we prove the following proposition.

Let W be a functional consisting of a finite number of irreducible functionals and representing a theory with symmetry breakdown of the first type. Then together with each irreducible W_α appearing in the decomposition of W, it will contain also the functionals $W_{\alpha, i}(g) = W_\alpha(g_i^{-1})$.

We denote by M, M_α, $M_{\alpha, i}$ the kernels of the functionals W, W_α, $W_{\alpha, i}$. As usual, we construct the functional

$$\overline{W}_\alpha = \sum_{i=1}^{n} \rho_i W_{\alpha, i}.$$

Since $M \subseteq M_\alpha$ and M is invariant with respect to the group $\mathscr{T}$, the kernel $\overline{M}_\alpha = \bigcap M_{\alpha, i}$ of the functional $\overline{W}_\alpha$ contains M. According to theorem 2 of [6] it follows that $\overline{W}_\alpha$ is composed of the same irreducible functionals as W. Consequently, all functionals $W_{\alpha, i}$ must enter into the decomposition of W.

It should be remembered that the preceding statement and the conclusion that symmetry breakdown of the second type is impossible are valid only if one starts from the extended algebra. For the usual algebra both these questions remain open.

[1] J. Goldstone, Nuovo Cimento **19**, 154 (1961).

[2] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

[3] R. Haag, Nuovo Cimento **25**, 287 (1962).

[4] H. J. Borchers, Nuovo Cimento **24**, 214 (1962).

[5] A. S. Wightman, Phys. Rev. **101**, 860 (1956).

[6] A. N. Vasil'ev, JETP **50**, 954 (1966), Soviet Phys. JETP **23**, 633 (1966).

Translated by M. E. Mayer
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