PARAMETRIC RESONANCE IN A PLASMA IN A MAGNETIC FIELD

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A theory is developed for the oscillations and stability of a plasma in a high-frequency electric and stationary magnetic spatially homogeneous field. Small deviations of the system from the ground state in which relative motion of the electrons and ions is produced by the presence of external fields are considered by employing a self-consistant interaction kinetic equation. A dispersion equation for the potential-oscillation spectrum of such a system is derived. The equation is solved for the case of a cold plasma. The frequency range of an external magnetic field in which the plasma is unstable is determined. Expressions for the increments of growing potential oscillations are obtained in the instability region. Since the proper oscillations of a plasma in a magnetic field occur only in a finite frequency range, the region of instability of plasma with relatively large increments turns out to be much broader than in the case of a plasma in the absence of a magnetic field. The maximum value of the instability increment is plotted as a function of the external field frequency (in this case the ratio of the square of the electron cyclotron frequency to the square of the Langmuir frequency is assumed to be 3.5).

1. In experiments on radiative acceleration [1] it is necessary to deal with a fully ionized plasma situated in a strong high-frequency electric field.[1)]

It was shown in [2] that a plasma situated in a high frequency electric field with frequency close to the Langmuir frequency of the electrons or smaller, turns out to be stable against buildup of potential oscillations. The reason for the occurrence of such an instability is relative motion of the electrons and the ions under the influence of the electric field. In this sense, the situation is similar to that in the ordinary current instability of a plasma (see, for example, [3]). The theory developed in [2] cannot satisfy completely the experimental demands, primarily because it considers a plasma without an external magnetic field. To the contrary, the present paper is devoted to the theory of instability of a plasma situated both in a high-frequency electric field and in a constant magnetic field. Just as in [2], we consider potential oscillations with wavelengths much shorter than the characteristic dimensions of variation of the external field. This is precisely why the high-frequency electric and constant magnetic fields will be assumed to be spatially homogeneous (see also [4]).

The theory developed below makes it possible to determine the alternating electric field frequencies at which the plasma is unstable. In the instability region, we obtain expressions for the increments of the growing potential oscillations of the plasma. By virtue of the fact that in a magnetic field the natural high-frequency oscillations of the plasma occur in a finite region of frequencies, of width determined by the relation between the gyroscopic and the Langmuir frequencies of the ions, and the region of instability of the plasma with relatively large increments, turn out to be much larger than in the case of a plasma without a magnetic field.

2. The fundamental state of a plasma in a constant magnetic field and a high-frequency electric field, both spatially homogeneous, is described by the particle distribution functions satisfying the equations

$$\frac{\partial f_a^{(0)}}{\partial t} + e_a\left\{\mathbf{E}(t) + \frac{1}{c}[\mathbf{v}_a\mathbf{B}]\right\}\frac{\partial f_a^{(0)}}{\partial \mathbf{p}_a} = 0. \qquad (2.1)^*$$

The corresponding solution can be written in the form

[1)] Indeed, in these experiments the electron temperature is ~ 1 eV and the frequency of the alternating field is $\omega_0 \approx 2 \times 10^{10}$ sec^{-1}. For such parameters the field $E = mv_T\,\omega_0/e$, at which the velocity of the oscillations v_E becomes comparable with the thermal velocity, turns out to be ~ 300 V/cm. In real devices, on the other hand, we are dealing with fields that are larger by one order of magnitude.

*$[\mathbf{v}_a\mathbf{B}] \equiv \mathbf{v}_a \times \mathbf{B}$.

$$f_a^{(0)}(\mathbf{p}_a, t) = f_{a0}\Big(\mathbf{p}_a - e_a \int_{-\infty}^{t} dt' \,\{\mathbf{b}(\mathbf{bE}(t')) + [\mathbf{E}(t')\mathbf{b}] \sin \Omega_a(t-t') + [\mathbf{b}[\mathbf{E}(t')\mathbf{b}]] \cos \Omega_a (t-t')\}\Big), \tag{2.2}$$

where $\mathbf{b} = \mathbf{B}/B$ and $f_{a0}(\mathbf{p}_a)$ is, for example, a Maxwellian distribution.

To study the stability of such an equilibrium state let us consider the states of a plasma whose distribution functions over small δf_a differ from the equilibrium functions (2.2). Because of the spatial homogeneity of the fundamental state we can assume a coordinate dependence of the non-equilibrium increment $\delta f_a \sim \exp(i\mathbf{k}\cdot\mathbf{r})$. Then the linearized kinetic equation with self-consistent potential field can be written in the form

$$\frac{\partial \delta f_a}{\partial t} + i(\mathbf{kv}_a)\delta f_a + e_a\Big(\mathbf{E}(t) + \frac{1}{c}[\mathbf{v}_a\mathbf{B}]\Big)\frac{\partial \delta f_a}{\partial \mathbf{p}_a} - i\mathbf{k}\frac{\partial f_a^{(0)}(\mathbf{p}_a, t)}{\partial \mathbf{p}_a}\sum_b \frac{4\pi e_a e_b}{k^2}\int d\mathbf{p}_b\, \delta f_b = 0. \tag{2.3}$$

Having in mind the argument of the right side of (2.2), it is convenient to introduce the function (see [5])

$$\Psi_a(\mathbf{p}_a, t) = \delta f_a\Big(\mathbf{p}_a + e_a\int_{-\infty}^{t} dt'\,\{\mathbf{b}(\mathbf{bE}(t')) + [\mathbf{E}(t')\mathbf{b}]\sin\Omega_a(t-t') + [\mathbf{b}[\mathbf{E}(t')\mathbf{b}]]\cos\Omega_a(t-t')\}, t\Big)\, e^{i\mathbf{kr}_a(t)},$$

where

$$\mathbf{r}_a(t) = \frac{e_a}{m_a}\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt''\,\{\mathbf{b}(\mathbf{bE}(t'')) + [\mathbf{E}(t'')\mathbf{b}]\sin\Omega_a(t'-t'') + [\mathbf{b}[\mathbf{E}(t'')\mathbf{b}]]\cos\Omega_a(t'-t'')\}.$$

For such a function, in accord with (2.3), we have the equation

$$\frac{\partial \Psi_a}{\partial t} + i\mathbf{kv}_a\Psi_a + \frac{e_a}{c}[\mathbf{v}_a\mathbf{B}]\frac{\partial \Psi_a}{\partial \mathbf{p}_a} - i\mathbf{k}\frac{\partial f_{a0}(\mathbf{p}_a)}{\partial \mathbf{p}_a}\sum_b \frac{4\pi e_a e_b}{k^2}\exp\{i\mathbf{k}, \mathbf{r}_a(t) - \mathbf{r}_b(t)\} \times \int \Psi_b\, d\mathbf{p}_b = 0. \tag{2.4}$$

We confine ourselves further to the case of a plasma consisting of electrons and one species of ions. Let the time dependence of the electric field be $\mathbf{E}(t) = \mathbf{E}\sin\omega_0 t$.[2)] Then the dependence on the electric field in (2.4) appears in the form

$$\exp\{i\mathbf{k}, \mathbf{r}_e - \mathbf{r}_i\} = \exp\{ia_B\sin(\omega_0 t + \varphi)\} = \sum_{l=-\infty}^{\infty} J_l(a_B)\exp\{il(\omega_0 t + \varphi)\},$$

where J_l is a Bessel function and φ is the phase. In a rectangular coordinate system with z-axis directed along the magnetic field we have

$$a_B^2 = \Big[\frac{k_z E_z}{\omega_0^2}\Big(\frac{e}{m} - \frac{e_i}{m_i}\Big) + (E_x k_x + E_y k_y)\Big(\frac{e}{m(\omega_0^2 - \Omega_e^2)} - \frac{e_i}{m_i(\omega_0^2 - \Omega_i^2)}\Big)\Big]^2 + \frac{(E_x k_y - E_y k_x)^2}{\omega_0^2}\Big[\frac{e\Omega_e}{m(\Omega_e^2 - \omega_0^2)} - \frac{e_i\Omega_i}{m_i(\Omega_i^2 - \omega_0^2)}\Big]^2,$$

$$\operatorname{ctg}\varphi = \Big[\frac{k_z E_z}{\omega_0^2}\Big(\frac{e}{m} - \frac{e_i}{m_i}\Big) + (E_x k_x + E_y k_y)\Big(\frac{e}{m(\omega_0^2 - \Omega_e^2)} - \frac{e_i}{m_i(\omega_0^2 - \Omega_i^2)}\Big)\Big] \times (E_x k_y - E_y k_x)^{-1}\Big(\frac{e\Omega_e}{m(\Omega_e^2 - \omega_0^2)} - \frac{e_i\Omega_i}{m_i(\Omega_i^2 - \omega_0^2)}\Big)^{-1}, \tag{2.5*}$$

$\Omega_a = e_a B/m_a c$ is the gyroscopic frequency of the particle of species a.

Making in (2.4) a change of variable $\omega_0 t + \varphi \to \omega_0 t$, we seek a solution in the form

$$\Psi_a(\mathbf{p}_a, t) = \sum_{n=-\infty}^{\infty}\Psi_a^{(n)}(\mathbf{p}_a)\exp\{-i(n\omega_0 + \omega)t\}.$$

We then obtain for the functions $\Psi_a^{(n)}$ a system of equations

$$-i(n\omega_0 + \omega - \mathbf{kv}_e)\Psi_e^{(n)} + \frac{e}{c}[\mathbf{v}_e\mathbf{B}]\frac{\partial \Psi_e^{(n)}}{\partial \mathbf{p}_e} = i\mathbf{k}\frac{\partial f_{e0}}{\partial \mathbf{p}_e}\Big\{\frac{4\pi e^2}{k^2}\int d\mathbf{p}_e'\,\Psi_e^{(n)}(\mathbf{p}_e') + \frac{4\pi e e_i}{k^2}\sum_l J_{n-l}(a_B)\int d\mathbf{p}_i'\Psi_i^{(l)}(\mathbf{p}_i')\Big\},$$

$$-i(n\omega_0 + \omega - \mathbf{kv}_i)\Psi_i^{(n)} + \frac{e_i}{c}[\mathbf{v}_i\mathbf{B}]\frac{\partial \Psi_i^{(n)}}{\partial \mathbf{p}_i} = i\mathbf{k}\frac{\partial f_{i0}}{\partial \mathbf{p}_i}\Big\{\frac{4\pi e_i^2}{k^2}\int d\mathbf{p}_i'\,\Psi_i^{(n)}(\mathbf{p}_i') + \frac{4\pi e e_i}{k^2}\sum_l J_{l-n}(a_B)\int d\mathbf{p}_e'\Psi_e^{(l)}(\mathbf{p}_e')\Big\}.$$

From this system of integro-differential equations we can readily obtain a system of algebraic equations for the quantities

2) The case when the fields vary like

$$E(t) = \sum_r \mathbf{E}_r \sin(\omega_0 t + \delta_r)$$

is considered in Appendix I.

*ctg ≡ cot.

$$u_a^{(n)} = e_a \int d\mathbf{p}_a \, \Psi_a^{(n)}(\mathbf{p}_a).$$

Assuming that $f_{a0}(\mathbf{p}_a)$ depends only on the projection of the momentum on the direction of the magnetic field and on the absolute magnitude of the transverse projection of the momentum, and introducing the notation

$$\delta\varepsilon_a(\omega, \mathbf{k}) = \frac{4\pi e_a^2}{k^2} \int d\mathbf{p}_a \sum_{n=-\infty}^{\infty} \frac{1}{\omega + i0 - n\Omega_a - k_\parallel v_{a\parallel}} \times J_n^2\left(\frac{k_\perp v_{a\perp}}{\Omega_a}\right)\left(k_\parallel \frac{\partial f_{a0}}{\partial p_{a\parallel}} + \frac{n\Omega_a}{v_{a\perp}} \frac{\partial f_{a0}}{\partial p_{a\perp}}\right), \quad (2.6)$$

we obtain in this case

$$u_e^{(n)} + \frac{\delta\varepsilon_e(n\omega_0 + \omega, \mathbf{k})}{1 + \delta\varepsilon_e(n\omega_0 + \omega, \mathbf{k})} \sum_m J_{n-m}(a_B) u_i^{(m)} = 0,$$

$$u_i^{(n)} + \frac{\delta\varepsilon_i(n\omega_0 + \omega, \mathbf{k})}{1 + \delta\varepsilon_i(n\omega_0 + \omega, \mathbf{k})} \sum_m J_{m-n}(a_B) u_e^{(m)} = 0. \quad (2.7)$$

Finally, eliminating the ionic function, we can write the following system of equations for only the electronic functions $u_e^{(n)}$ ($n = 0, \pm 1; \pm 2, \ldots$):

$$u_e^{(n)} = R_e^{(n)} \sum_{m=-\infty}^{\infty} J_{n-m}(a_B) R_i^{(m)} \sum_{l=-\infty}^{\infty} J_{l-m}(a_B) u_e^{(l)}. \quad (2.8)$$

We have used here the notation

$$R_a^{(n)} = \frac{\delta\varepsilon_a(n\omega_0 + \omega, \mathbf{k})}{1 + \delta\varepsilon_a(n\omega_0 + \omega, \mathbf{k})}. \quad (2.9)$$

The infinite determinant of this system of equations, set equal to zero, constitutes the dispersion equation for the potential plasma oscillations.

It is obvious that the natural frequencies of the plasma oscillations are determined accurate to a term $n\omega_0$.

3. For the qualitative analysis of the spectrum of the natural oscillations of the plasma, which we are about to present, it is convenient to introduce the concept of resonant frequencies, of the electrons and ions, respectively, defined as frequencies satisfying the equations

$$1 + \delta\varepsilon_e(\omega_{re}, \mathbf{k}) = 0, \quad (3.1)$$

$$1 + \delta\varepsilon_i(\omega_{ri}, \mathbf{k}) = 0. \quad (3.2)$$

We shall distinguish below between the case where the overtone of the external frequency is close to the resonant frequency of the electrons ($|n^2\omega_0^2 - \omega_{re}^2| \ll \omega_{re}^2$), which we shall call the resonance at the overtone of the external frequency, and the nonresonant case when the overtones of the external frequency are not close to the resonant electron frequencies.

We confine ourselves further to an examination of a case in which the frequency of the external field greatly exceeds the gyroscopic and Langmuir frequencies of the ions, and therefore also the resonant frequencies of the ions. This enables us to state immediately that $R_i^{(m)} \ll 1$ when $m \neq 0$. We consider first the nonresonant case. For not too rarefied a plasma ($\omega_{Le}^2 \gg \Omega_i^2$) we can distinguish in this case between the low frequency branch of the oscillations with frequency ω of the order of ω_{ri}, and the high-frequency branch with frequency of the order of ω_{re}.

For the high-frequency oscillations $R_i^{(m)} \ll 1$ ($m = 0, \pm 1, \ldots$), the solutions of Eq. (2.8) exist therefore only when $R_e^{(n)} \gg 1$ for a certain n, whereas for the remaining $m \neq n$ we have $R_e^{(m)} \sim 1$. From Eq. (2.8) we see in this case that the largest is the function $u_e^{(n)}$, so that we can immediately write the following dispersion equation for the high frequency oscillations:

$$1 = R_e^{(n)} \sum_{m=-\infty}^{\infty} J_{n-m}^2(a_B) R_i^{(m)}. \quad (3.3)$$

For the low frequency oscillations $R_e^{(m)} \sim 1$ for all m. In the study of these oscillations we have confined ourselves to frequencies of the external electric field ω_0 greatly exceeding the resonant frequencies of the ions ω_{ri}. Therefore, bearing in mind the smallness of all the R_m when $m \neq 0$, we can replace (2.8) by the following approximate equation

$$u_e^{(n)} = R_e^{(n)} R_i^{(0)} J_n \sum_{l=-\infty}^{\infty} J_l u_e^{(l)}. \quad (3.4)$$

From this we get the dispersion equation

$$1 = R_i^{(0)} \sum_{n=-\infty}^{\infty} J_n^2(a_B) R_e^{(n)}. \quad (3.5)$$

In the limit when the magnetic field vanishes, Eq. (3.5) goes over into Eq. (4.8) of [2].

In the vicinity of the resonance at the overtone of the external frequency with $|\omega| \ll \omega_0$ it is obvious that not only $R_e^{(n)}$ but also $R_e^{(-n)}$ is large. It then follows from (2.8) that $u_e^{(\pm n)} \gg u_e^{(m)}$ ($m \neq \pm n$). We shall therefore use in place of (2.8) the following approximate equation

$$u_e^{(k)} = R_e^{(k)} \Big\{ J_k R_i^{(0)} \sum_{l=-\infty}^{\infty} J_l u_e^{(l)} + \sum_{m\neq 0} J_{k-m} R_i^{(m)} [J_{n-m} u_e^{(n)} + J_{-n-m} u_e^{(-n)}] \Big\}. \quad (3.6)$$

Expressing with the aid of this equation

$$\sum_{l=-\infty}^{\infty} J_l u_e^{(l)}$$

in terms of $u_e^{(\pm n)}$, we obtain a system of two equations, the vanishing of whose determinant leads to the following approximate dispersion equation (we take account here of the fact that in the vicinity of the resonance we have $R_e^{(\pm n)} \gg R_e^{(m)}$ ($m \neq \pm n$)):

$$1 = R_i^{(0)}\Big\{ \sum_{m=-\infty}^{\infty} J_m^2 R_e^{(m)} + 2R_e^{(n)} R_e^{(-n)} J_n^2 \times \sum_{m\neq 0} R_i^{(m)} (-J_{n-m}^2 + (-1)^n J_{n-m} J_{-n-m}) \Big\}. \quad (3.7)$$

The difference between this equation and Eq. (3.5) lies in the presence of an additional term in the right side of (3.7). For oscillation frequencies which greatly exceed ω_{ri}, this term is proportional to the small quantity $R_i^{(m)}$ ($m \neq 0$) and is therefore a small correction. As seen from formula (4.8), which is derived from (3.5), the oscillation frequencies greatly exceed ω_{ri} if

$$|n^2\omega_0^2 - \omega_{re}^2| \gg \omega_{ri}^2.$$

Thus, Eq. (3.5) is convenient also for the study of the natural oscillations of a plasma under conditions of parametric resonance, provided this resonance is not very exact. This is connected with the fact that when the overtone of the external frequency approaches resonance, we have together with the increase in $R_e^{(\pm n)}$ also an increase of the increment of the oscillations, and therefore $R_i^{(0)}$ decreases.

In the direct vicinity of the resonance, when $|n^2\omega_0^2 - \omega_{re}^2| \lesssim \omega_{ri}^2$, the first term in (3.7) decreases, because of the cancellation of large resonant terms, and becomes equal to the second. Then the frequencies of the oscillations also decrease appreciably, and become of the order of ω_{ri}.

4. In the case of a cold plasma, when the thermal motion of the particles is insignificant, we have

$$\delta\varepsilon_a(\omega, \mathbf{k}) = -\omega_{La}^2 \ (\omega^2 - \Omega_a^2 \cos^2\theta) \,/\, \omega^2(\omega^2 - \Omega_a^2), \quad (4.1)$$

where the θ is the angle between the magnetic field and the wave vector $\mathbf{k}$, and $\omega_{La} = (4\pi e_a^2 n_a/m_a)^{1/2}$ is the Langmuir frequency of the particles of species a. In this case, in accordance with Eq. (3.1), the electron resonance frequencies are of the form

$$(\omega_{re}^{\pm})^2 = {}^1\!/_2 \{\Omega_e^2 + \omega_{Le}^2 \pm [(\Omega_e^2 + \omega_e^2)^2 - 4\Omega_e^2\omega_{Le}^2 \cos^2\theta]^{1/2}\}. \quad (4.2)$$

Far from resonance of the overtones of the external frequency, i.e., in the case when $(n\omega_0)^2$ differs noticeably from ω_{re}^2, the high-frequency oscillations in the electric field differ little from the frequencies (4.2):

$$\omega^2 = (\omega_{re}^{\pm})^2 \Big\{ 1 \mp \frac{\omega_{Li}^2 [(\omega_{re}^{\pm})^2 - \Omega_e^2]}{[(\Omega_e^2 + \omega_{Le}^2)^2 - 4\Omega_e^2\omega_{Le}^2\cos^2\theta]^{1/2}} \times \sum_{n=-\infty}^{\infty} \frac{J_n^2(a_B)}{(n\omega_0 + \omega_{re}^{\pm})^2} \frac{[(n\omega_0 + \omega_{re}^{\pm})^2 - \Omega_i^2\cos^2\theta]}{[(n\omega_0 + \omega_{re}^{\pm})^2 - \Omega_i^2]} \Big\}. \quad (4.3)$$

To the contrary, the low-frequency oscillations depend essentially on the electric field. Thus, far from resonance, at the overtone of the external frequency we have for the spectrum of the low-frequency oscillations ($\theta \neq \pi/2$)

$$\omega^2 = {}^1\!/_2 \{\Omega_i^2 + \omega_{Li}^2 A(\omega_0, \mathbf{k}) \pm ([\Omega_i^2 + \omega_{Li}^2 A(\omega_0, \mathbf{k})]^2 - 4\Omega_i^2\omega_{Li}^2 A \cos^2\theta)^{1/2}\}, \quad (4.4)$$

where

$$A(\omega_0, \mathbf{k}) = \sum_{n=-\infty}^{\infty} \frac{n^2\omega_0^2(n^2\omega_0^2 - \Omega_e^2) J_n^2(a_B)}{[n^2\omega_0^2 - (\omega_{re}^+)^2][n^2\omega_0^2 - (\omega_{re}^-)^2]}$$

$$\equiv 1 - \frac{[(\omega_{re}^-)^2 - \omega_{Le}^2]}{[(\omega_{re}^-)^2 - (\omega_{re}^+)^2]} \Phi_{\omega_{re}^+/\omega_0}(a_B) + \frac{[(\omega_{re}^+)^2 - \omega_{Le}^2]}{[(\omega_{re}^-)^2 - (\omega_{re}^+)^2]} \Phi_{\omega_{re}^-/\omega_0}(a_B),$$

$$\Phi_x(z) = \frac{\pi x}{\sin \pi x} J_x(z) J_{-x}(z). \quad (4.5)$$

Formulas (4.4) corresponds to buildup of oscillations for a minus sign and for $A(\omega_0, \mathbf{k}) < 0$. Formula (4.5) assumes a specially simple form when $a_B \ll 1$. Then

$$A(\omega_0, \mathbf{k}) = \frac{a_B^2\omega_0^2(\omega_0^2 - \Omega_e^2)}{2[\omega_0^4 - \omega_0^2(\Omega_e^2 + \omega_{Le}^2) + \Omega_e^2\omega_{Le}^2\cos^2\theta]}. \quad (4.6)$$

It is easy to see that at an external-field frequency smaller than the gyroscopic electron frequency, the buildup of oscillations is possible when $\omega_0 < \omega_{Le}$. In the opposite case ($\omega_0 > \Omega_e$), the threshold of the buildup of the oscillations is determined by the inequality $\omega_0 < (\omega_{Le}^2 + \Omega_e^2)^{1/2}$. We note that at the threshold of the instability the growing waves are those whose wave vectors are almost perpendicular to the magnetic field.

In the vicinity of the resonance of the overtone of the external frequency, the increments increase appreciably. For the waves that grow most rapidly we have here $a_B \sim 1$. The greatest are the increments in the regions where the following equality holds

$$\Delta_n = (\omega_{re}^{\pm} / n\omega_0)^2 - 1 = |4g_n|^{1/3}\eta. \quad (4.7)$$

Here

$$g_n = \frac{\omega_{Li}^2(n^2\omega_0^2 - \Omega_e^2) J_n^2(a_B)}{n^2\omega_0^2(2n^2\omega_0^2 - \Omega_e^2 - \omega_{Le}^2)}, \quad (4.7')$$

η is a real number of the order of unity. In this region the solution of the dispersion equation takes the form

$$(\omega / n\omega_0)^2 = {}^1/_8 |4g_n|^{2/3} \{\eta^2 \pm \sqrt{\eta^4 + 8\eta}\}. \qquad (4.8)$$

Formula (4.8) was obtained under the assumption that $\omega^2 \gg \Omega_i^2$. If the opposite inequality holds, the right sides of formulas (4.7′) and (4.8) must be multiplied in addition by $(\cos\theta)^{2/3}$ and $(\cos\theta)^{4/3}$, respectively.

Let us consider certain consequences of formula (4.8). In the case of a minus sign, the maximum increment of the buildup takes place for $\eta = 1$. In the case of a plus sign, the buildup of the oscillations is possible if $-2 < \eta < 0$, with

$$(\omega / n\omega_0) = \pm (g_n / 16)^{1/3} \{[\sqrt{8|\eta|} + \eta^2]^{1/2} \pm i[\sqrt{8|\eta|} - \eta^2]^{1/2}\}. \qquad (4.9)$$

The maximum increment corresponding to formula (4.9) occurs when $\eta = -2^{-1/3}$.

Formulas (4.7), (4.8), and (4.9) can be written in a somewhat different form, if we note that in the vicinity of resonance the angle θ can be represented as a function of

$$\alpha = \Omega_e^2 / \omega_{Le}^2, \quad \beta = \omega_0^2 / \omega_{Le}^2$$

and Δ_n. Namely, for the resonant angle we obtain

$$\cos^2\theta_r = n^2\beta(1 + \Delta_n)[1 + \alpha - n^2\beta(1 + \Delta_n)] / \alpha. \qquad (4.10)$$

Thus, because in a magnetic field different frequencies of the external field correspond to different resonant angles, the region of frequencies in which the dispersion equation (3.3) is valid and gives the maximum values of the increments of the parametric resonance broadens. Inasmuch as Δ_n is a small quantity, it can be neglected in expression (4.10) everywhere except for angles close to zero or $\pi/2$. In this case the values of the parameters α and β, at which resonant buildup of the oscillations is possible, are determined from the condition $0 \leq \cos^2\theta_r \leq 1$ and satisfy the inequalities

$$n^2\beta < \min(1, \alpha), \quad \max(1, \alpha) < n^2\beta < 1 + \alpha. \qquad (4.11)$$

From formulas (4.8) and (4.9) we obtain the following expressions for the buildup increments $\gamma = |\mathrm{Im}\,\omega|$:

$$\frac{\gamma^2}{\omega_{Le}^2} = \frac{1}{8}\left[4nJ_n^2 \frac{\omega_{Li}^2}{\omega_{Le}^2} \beta^{1/2} \left(\frac{n^2\beta - \alpha}{2n^2\beta - \alpha - 1}\right)\right]^{2/3} \times (\sqrt{\eta^4 + 8\eta} - \eta^2), \qquad (4.12)$$

$$\frac{\gamma^2}{\omega_{Le}^2} = \frac{1}{8}\left[4nJ_n^2 \frac{\omega_{Li}^2}{\omega_{Le}^2} \beta^{1/2} \left(\frac{n^2\beta - \alpha}{2n^2\beta - \alpha - 1}\right)\right]^{2/3} \times \left(\frac{\sqrt{8|\eta|} - \eta^2}{2}\right) \qquad (4.13)$$

For specified α and β, the buildup increments have maximum values when the arguments of the Bessel function $J_n^2(a_B)$ correspond to its maximum. Therefore, if the resonant angle is not close to zero or $\pi/2$, then γ^2 has a maximum which is given by formula (4.12), in which we substitute $\eta = 1$. Near angles θ_r equal to zero and $\pi/2$, the possible values of η are bounded by the conditions $\cos^2\theta_r > 0$ and $\cos^2\theta_r < 1$, from which we get by means of (4.10)

$$n^2\beta(1 + \Delta_n) < \min(1, \alpha), \quad \max(1, \alpha) < n^2\beta(1 + \Delta_n) < 1 + \alpha. \qquad (4.14)$$

Using the inequality (4.14) we can readily see that near the upper limits of the values of $n^2\beta$, when $n^2\beta \geq \min(1, \alpha)$ or $n^2\beta \geq 1 + \alpha$, corresponding to the start of the resonant buildup of the oscillations at the n-th harmonic of the external frequency, the value of η is negative and thus resonant buildup of the oscillations always begins from the branch (4.9), with the maximum negative value $\eta = -2$, and the corresponding maximum value of the frequency at which the buildup of the oscillations begins is equal to

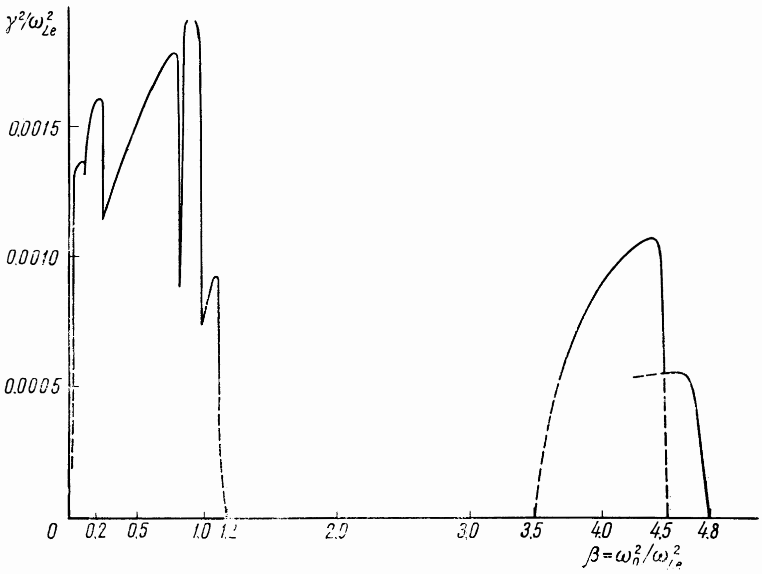
$$\beta_{max} = \alpha + 1 + 2\left[4\frac{\omega_{Li}^2}{\omega_{Le}^2}(\alpha + 1)\max J_1^2\right]^{1/3}$$

or

$$\omega_0^2 = (\omega_{Le}^2 + \Omega_e^2)\left[1 + 2.2\frac{\omega_{Li}^{2/3}\omega_{Le}^{2/3}}{(\omega_{Le}^2 + \Omega_e^2)^{1/3}}\right].$$

The resonant angle is in this case close to $\pi/2$. When $\eta > 0$ (which is possible only when $n^2\beta < \min(1, \alpha)$ or $n^2\beta < 1 + \alpha$), buildup of oscillation in the mode (4.8) is possible. The maximum increment for this branch is reached at $\eta = 1$ and exceeds the maximum increment (reached when $\eta = -2^{-1/3}$) for mode (4.9).

The figure shows the dependence of the maximum increment of the oscillation buildup on the frequency of the external electric field for the case when $\alpha = \Omega_e^2/\omega_{Le}^2 = 3.5$. In the region of values $3.5 < \beta < 4.8$ resonance occurs at the frequency $(\omega_{re}^{+})^2$, and for $4.5 < \beta < 4.8$ the increment is described by formula (4.13), while in the region $3.5 < \beta < 4.5$ it is described by formula (4.12) with $n = 1$. With decreasing frequency of the external electric field, resonances become possible at the



Maximum value of the increment of the instability $\gamma = |\mathrm{Im}\,\omega|$ as a function of the frequency of the external high frequency electric field at $\Omega_e^2/\omega_{Le}^2 = 3.5$; ω_0 is the frequency of the external high frequency field.

overtones of the external frequencies, when $n^2\beta \approx (\omega_{re}^+)^2$, and also resonances at the branch $(\omega_{re}^-)^2$. For the given values of the parameters this takes place in the region $\beta < 1.2$. In the vicinity of the points $\beta = (1+\alpha)/(n^2+m^2)$, double resonance is possible, at which

$$n^2\beta \approx (\omega_{re}^+)^2, \quad m^2\beta \approx (\omega_{re}^-)^2,$$

where n and m are integers satisfying the conditions $n^2/m^2 > \alpha > 1$ (or $m^2/n^2 < \alpha < 1$). The build-up increments under these conditions are obtained by solving a cubic equation with respect to ω^2, the form of which is given in Appendix II.

APPENDIX I

If the external high-frequency field is of the form

$$\mathbf{E}(t) = \sum_r \mathbf{E}_r \sin(\omega_0 t + \delta_r),$$

then it is necessary to use throughout in the arguments of the Bessel functions the quantity a_B, defined by

$$a_B^2 = \Big\{\sum_r \cos\delta_r \Big[\frac{k_z E_{rz}}{\omega_0^2}\Big(\frac{e}{m} - \frac{e_i}{m_i}\Big) + (k_x E_{rx} + k_y E_{ry}) \times \Big(\frac{e}{m(\omega_0^2 - \Omega_e^2)} - \frac{e_i}{m_i(\omega_0^2 - \Omega_i^2)}\Big)\Big] + \sum_r \sin\delta_r \times \Big[\frac{(E_{rx}k_y - E_{ry}k_x)}{\omega_0^2}\Big(\frac{e\Omega_e}{m(\Omega_e^2 - \omega_0^2)} - \frac{e_i\Omega_i}{m_i(\Omega_i^2 - \omega_0^2)}\Big)\Big]\Big\}^2 + \Big\{\sum_r \sin\delta_r \Big[\frac{k_z E_{rz}}{\omega_0^2}\Big(\frac{e}{m} - \frac{e_i}{m_i}\Big) + (k_x E_{rx} + k_y E_{ry}) \times \Big(\frac{e}{m(\omega_0^2 - \Omega_e^2)} - \frac{e_i}{m_i(\omega_0^2 - \Omega_i^2)}\Big)\Big] - \sum_r \cos\delta_r \Big[\frac{(E_{rx}k_y - E_{ry}k_x)}{\omega_0} \times \Big(\frac{e\Omega_e}{m(\Omega_e^2 - \omega_0^2)} - \frac{e_i\Omega_i}{m_i(\Omega_i^2 - \omega_0^2)}\Big)\Big]\Big\}^2.$$

In particular, if the field $\mathbf{E}(t)$ is a circularly polarized wave propagating along the magnetic field, we have for a wave with a polarization vector rotating counterclockwise

$$a_B^2 = \frac{E_0^2(k_x^2 + k_y^2)}{2\omega_0^2}\Big(\frac{e}{m(\Omega_e - \omega_0)} - \frac{e_i}{m_i(\Omega_i - \omega_0)}\Big)^2$$

For a wave with polarization vector rotating clockwise,

$$a_B^2 = \frac{E_0^2(k_x^2 + k_y^2)}{2\omega_0^2}\Big(\frac{e}{m(\Omega_e + \omega_0)} - \frac{e_i}{m_i(\Omega_i + \omega_0)}\Big)^2$$

where E_0 is the absolute value of the electric field intensity vector.

APPENDIX II

In the plasma situated in a magnetic field there exist two electronic resonant frequencies $(\omega_{re}^\pm)^2$ (4.2). Because of this the possibility of double resonance exists, in which one of the overtones of the external frequency is close, say, to ω_{re}^+, and the other overtone is close to ω_{re}^-. Let

$$\Delta_n = (\omega_{re}^+/n\omega_0)^2 - 1, \quad \Delta_m = (\omega_{re}^-/n\omega_0)^2 - 1,$$

$$|\Delta_n|, |\Delta_m| \ll 1.$$

Then, recognizing that in this case the large quan-

ν	$\Phi_\nu(1)$	$\Phi_\nu(2)$	$\Phi_\nu(5)$	ν	$\Phi_\nu(1)$	$\Phi_\nu(2)$	$\Phi_\nu(5)$
$^1/_6$	0.573	0.0293	0.0238	$^5/_3$	1.131	0,508	0.154
$^1/_4$	0.558	0.00159	0.0136	$^7/_4$	1.081	0.207	0.103
$^1/_3$	0.537	—0,0405	—0.00161	$^{11}/_6$	0.996	—0.334	0.0488
$^1/_2$	0.454	—0.190	—0.0542	$^{13}/_6$	1.261	2.541	—0.121
$^2/_3$	0.271	—0.515	—0.166	$^9/_4$	1.190	2.019	—0.204
$^3/_4$	0.0833	—0,850	—0.278	$^7/_3$	1.147	1.732	—0.295
$^5/_6$	—0.305	—1.515	—0.496	$^5/_2$	1.115	1.453	—0.554
$^7/_6$	2.037	2,427	0.232	$^8/_3$	1,056	1.269	—1.017
$^5/_4$	1.644	1.733	0.525	$^{11}/_4$	1.246	1.171	—1,466
$^4/_3$	1.499	1.363	0.401	$^{17}/_6$	1.089	1.023	—2.292
$^3/_2$	1.249	0.914	0.258				

tities are not only $R_e^{(n)}$ and $R_e^{(-n)}$, but also $R_e^{(m)}$ and $R_e^{(-m)}$, we can obtain from (3.5) an equation for determining the oscillation spectrum. (As before, we disregard the case of exact resonance, when the frequencies become close to ω_{ri}.) Namely, we obtain

$$\omega^6 - {}^1/_4\omega_0^2(n^2\Delta_n^2 + m^2\Delta_m^2)\,\omega^{(4)} + {}^1/_2\omega_0\omega^2(-g_n\Delta_n - g_m\Delta_m + {}^1/_8 n^2m^2\omega_0^3\Delta_n^2\Delta_m^2) + {}^1/_8 nm\omega_0^3\Delta_n\Delta_m(\Delta_n g_m + \Delta_m g_n) = 0. \quad (A.1)$$

The values of g_n are given by (4.7′).

In the case of simple resonance, we have $\Delta_m \gtrsim 1$ and Eq. (A.1) can be approximately represented in the form

$$(\omega^2 - {}^1/_4 m^2\omega_0^2\Delta_m^2)(\omega^4 - {}^1/_4 n^2\omega_0^2\Delta_n^2\omega^2 - {}^1/_2 n\omega_0\Delta_n g_n) = 0. \quad (A.2)$$

The vanishing of the second factor leads to an equation whose solutions are the previously-obtained formulas for the oscillation frequencies for resonance at the overtone of the external frequency (see (4.8)).

The numerical solutions of Eq. (A.1) in the region $\Delta_m \ll 1$ are shown in the figure.

APPENDIX III

We present below a table of values of the function

$$\Phi_\nu(z) = \frac{\pi\nu}{\sin\pi\nu} J_\nu(z) J_{-\nu}(z), \qquad \nu^2 \neq 1, 4, 9, \ldots,$$

which determines the increment of the buildup of the oscillations in the nonresonant case (see formulas (4.4) and (4.5)). For $z \ll 1$ the following expansion is valid

$$\Phi_\nu(z) \approx 1 - \frac{1}{2(1-\nu^2)} z^2 + \frac{3z^4}{4(1-\nu^2)(4-\nu^2)} - \ldots$$

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