ON THE DYNAMICS OF A SUPERFLUID FERMI GAS

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The nonlinear differential equation satisfied by the gap Δ in a superfluid Fermi gas is derived. The equation is valid under the condition that the change of Δ is small over distances of the order of the correlation length v_F/Δ in times of the order $1/\Delta$. The equation is obtained for the case of absolute zero temperature. It is shown that the hydrodynamic equations of an ideal liquid can be derived from the equation.

In superconductivity theory great interest has recently been stimulated by problems associated with the spatial and temporal variations of the gap Δ. These problems include the motion of Abrikosov's vortex lines, the theory of the Josephson effect etc. All these questions in the Bardeen-Cooper-Schrieffer theory should be answerable in principle on the basis of Gor'kov's system of equations[1] [see our Eq. (1)]. However, these equations are quite complicated and unamenable to a general solution. It is therefore important to obtain a differential equation directly for the parameter Δ. Of course, this can be done only when Δ varies sufficiently slowly in space and time (the Ginzburg-Landau equations). In the present work we derive this equation at absolute zero. Even in this case the problem for a real superconductor is complicated by the presence of a crystalline lattice in a magnetic field. We therefore confine ourselves to the derivation of equations for an uncharged superfluid Fermi gas. Aside from its independent methodological interest, the study of this model enables us to elucidate fundamental questions that also apply to the general case. We plan to extend our results to charged systems at finite temperatures.

Gor'kov's equations for a superfluid Fermi system are

$$\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}\right)G(x,x')-i\Delta(x)F^+(x,x')=\delta(x-x'),$$

$$\left(i\frac{\partial}{\partial t}-\frac{\nabla^2}{2m}-2\mu\right)F^+(x,x')+i\Delta^*(x)G(x,x')=0, \quad (1)$$

where G is a Green's function and Δ^* is related to F^+ by

$$\Delta^*(x)=gF^+(x,x) \quad (2)$$

($g < 0$ is the interaction constant). It will be convenient for our subsequent calculations to eliminate G from (1) immediately. We thus obtain the following equation for F^+:

$$\left[\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}\right)\frac{1}{\Delta^*(x)}\left(i\frac{\partial}{\partial t}-\frac{\nabla^2}{2m}-2\mu\right)-\Delta(x)\right]F^+(x,x')=-i\delta(x-x'). \quad (3)$$

As already stated, we shall assume that Δ is a slowly varying function of $\mathbf{r}$ and t. More exactly, it will be assumed that the characteristic dimension $1/k$ of the spatial variation of Δ and the characteristic frequency ω_0 satisfy the inequalities

$$\omega_0\ll\Delta, \qquad k\xi_0=kv_F/\Delta\ll 1, \quad (4)$$

where ξ_0 is the correlation length and v_F is the Fermi velocity of the electrons.

Subject to the conditions in (4), F^+ can be expressed approximately in terms of Δ and its derivatives. By substituting that expression into (2), we shall obtain the desired differential equation for Δ. In order to represent F^+ in terms of Δ and its derivatives we can, in principle, employ the following regular procedure. We represent F^+ by

$$F^+(x,x')=F_0^++F_1^+\frac{\partial\Delta}{\partial t}+F_2^+\frac{\partial\Delta^*}{\partial t}+\mathbf{F}_3^+\nabla\Delta+\ldots,$$

where F_0^+, F_1^+, F_2^+ etc. are functions of $x-x'$, $\Delta(x)$, and $\Delta^*(x)$. By substituting this expansion into (3), differentiating, and collecting the coefficients of the different derivatives of Δ, we can successively determine F_0^+, F_1^+, F_2^+ etc. However, this calculation would be very laborious, and we shall simplify the work considerably by means of a more artificial procedure.

Our method will be based essentially on the substitution

$$\Delta(x) = \Delta_0 \exp[i(\mathbf{qr} - \Omega t)] + \Delta_1(x) \tag{5}$$

with the assumption $\Delta_1 \ll \Delta_0$. This condition for Δ_1 permits a relatively easy derivation of a linear differential equation as a basis for a nonlinear equation that will be satisfied by $\Delta(x)$. We must emphasize that with the customary simple substitution

$$\Delta(x) = \Delta_0 + \Delta_1(x),$$

the equation for Δ_1 cannot establish the presence of terms containing products of derivatives in the equation for Δ; such products drop out in the course of linearization. The substitution (5), however, permits the consideration of these terms. We note, to begin with, that this substitution actually corresponds to investigating how small deviations of Δ behave in a moving coordinate system. This becomes clear from a comparison between our equations (5) and (18) for a Galilean transformation of Δ.

We make the substitution

$$\Delta(x) = \tilde{\Delta}(x) \exp[i(\mathbf{qr} - \Omega t)],$$

$$F^+(x, x') = \tilde{F}^+(x, x') \exp\{-{}^1\!/_2 i[\mathbf{q}(\mathbf{r} + \mathbf{r}') - \Omega(t + t')]\}. \tag{6}$$

Inserting this into (3), we obtain

$$\left\{\left[i\left(\frac{\partial}{\partial t} - \frac{i\Omega}{2}\right) + \frac{(\nabla + i\mathbf{q}/2)^2}{2m}\right]\frac{1}{\tilde{\Delta}^*(x)}\left[i\left(\frac{\partial}{\partial t} + \frac{i\Omega}{2}\right) - \frac{(\nabla - i\mathbf{q}/2)^2}{2m} - 2\mu\right] - \tilde{\Delta}(x)\right\}\tilde{F}^+(x, x') = -i\delta(x - x'). \tag{7}$$

If $\tilde{\Delta}(x) = \Delta_0 = \text{const}$, this equation can be solved conventionally by means of a Fourier transformation. Denoting $\tilde{F}^+(x,x')$ as $\tilde{F}_0^+(x - x\ )$ for $\tilde{\Delta}(x) = \Delta_0$, and introducing the Fourier components

$$\tilde{F}_0^+(\mathbf{r}, t) = \int \exp[i(\mathbf{pr} - \omega t)]\, \tilde{F}^+_{0\omega\mathbf{p}} \frac{d^3p\,d\omega}{(2\pi)^4},$$

we obtain

$$\tilde{F}^+_{0\omega\mathbf{p}} = \frac{-i\Delta_0^*}{[(\omega + \Omega/2) - \xi_{\mathbf{p}+\mathbf{q}/2}][(\omega - \Omega/2) + \xi_{\mathbf{p}-\mathbf{q}/2}] - |\Delta_0|^2}. \tag{8}$$

We have here used the notation

$$\xi_\mathbf{p} = \mathbf{p}^2/2m - \mu$$

and the customary substitution $\omega \to \omega + \mu$.

We use

$$\tilde{\Delta}(x) = \Delta_0 + \tilde{\Delta}_1(x), \qquad \tilde{\Delta}_1(x) = \Delta_1(x) \exp[i(\mathbf{qr} - \Omega t)] \tag{9}$$

in accordance with (5), and linearize (7) with respect to $\tilde{\Delta}_1$ and the addition $\tilde{F}_1^+$ to $\tilde{F}^+$:

$$\tilde{F}^+(x, x') = \tilde{F}_0^+(x - x') + \tilde{F}_1^+(x, x').$$

The equation for $\tilde{F}_1^+$ is then

$$\left\{\left[i\left(\frac{\partial}{\partial t} - \frac{i\Omega}{2}\right) + \frac{(\nabla + i\mathbf{q}/2)^2}{2m} + \mu\right]\frac{1}{\tilde{\Delta}_0^*}\left[i\left(\frac{\partial}{\partial t} + \frac{i\Omega}{2}\right) - \frac{(\nabla - i\mathbf{q}/2)^2}{2m} - \mu\right] - \tilde{\Delta}_0\right\}\tilde{F}_1^+ = \left\{-\left[i\left(\frac{\partial}{\partial t} - \frac{i\Omega}{2}\right) + \frac{(\nabla + i\mathbf{q}/2)^2}{2m} + \mu\right]\left(-\frac{\tilde{\Delta}_1^*}{\tilde{\Delta}_0^{*2}}\right)\left[i\left(\frac{\partial}{\partial t} + i\frac{\Omega}{2}\right) - \frac{(\nabla - i\mathbf{q}/2)^2}{2m} - \mu\right] + \tilde{\Delta}_1\right\}\tilde{F}_0^+.$$

To solve this equation we represent $\tilde{\Delta}_1$ and $\tilde{\Delta}_1^*$ as Fourier integrals:

$$\tilde{\Delta}_1(x) = \int \tilde{\Delta}_1(\omega_0\mathbf{k}) \exp[i(\mathbf{kr} - \omega_0 t)] \frac{d^3k\,d\omega_0}{(2\pi)^4},$$

$$\tilde{\Delta}_1^*(x) = \int \tilde{\Delta}_1^*(\omega_0\mathbf{k}) \exp[i(\mathbf{kr} - \omega_0 t)] \frac{d^3k\,d\omega_0}{(2\pi)^4}.$$

It is then easy to derive an expression for the Fourier components of $\tilde{F}_1^+$ with respect to $\mathbf{r} - \mathbf{r}'$ and $t - t'$:

$$\tilde{F}^+_{1\omega p}(\mathbf{r}, t) = \int Q \exp[i(\mathbf{kr} - \omega_0 t)] \frac{d^3k\,d\omega_0}{(2\pi)^4},$$

where

$$Q = -i[\tilde{\Delta}_0^{*2}\tilde{\Delta}_1(\omega_0\mathbf{k}) + \tilde{\Delta}_1^+(\omega_0\mathbf{k})(\omega + \Omega/2 - \xi_{\mathbf{p}+\mathbf{q}/2}) \times (\omega - \omega_0 + \Omega/2 + \xi_{\mathbf{p}-\mathbf{k}-\mathbf{q}/2})]\{[(\omega + \Omega/2 - \xi_{\mathbf{p}+\mathbf{q}/2}) \times (\omega - \Omega/2 + \xi_{\mathbf{p}-\mathbf{q}/2}) - |\Delta_0|^2][(\omega - \omega_0 + \Omega/2 - \xi_{\mathbf{p}-\mathbf{k}+\mathbf{q}/2}) \times (\omega - \omega_0 - \Omega/2 + \xi_{\mathbf{p}-\mathbf{k}-\mathbf{q}/2}) - |\Delta_0|^2]\}^{-1}. \tag{10}$$

The equations for $\tilde{\Delta}_0$ and $\tilde{\Delta}_1$ are

$$\tilde{\Delta}_0^* = g\int \tilde{F}^+_{0\omega\mathbf{p}} \frac{d^3p\,d\omega}{(2\pi)^4}, \tag{11}$$

$$\tilde{\Delta}_1^* = g\int \tilde{F}^+_{1\omega\mathbf{p}} \frac{d^3p\,d\omega}{(2\pi)^4}. \tag{12}$$

It is easily shown that, to small terms of the order of $(q^2/4m - \Omega)/\mu$, Eq. (11) is reduced to the ordinary equation

$$^1\!/_2\pi^{-2} g m p_0 \ln(2\tilde{\omega}/|\Delta_0|) = -1,$$

where $\tilde{\omega}$ is the cutoff frequency, which is of the same order of magnitude as μ in the Fermi gas model. Equation (12) can be rewritten as

$$\tilde{\Delta}_1^+(\omega_0\mathbf{k}) = \int Q \frac{d^3p\,d\omega}{(2\pi)^4}. \tag{13}$$

This last equation is the Fourier transformation of the desired equation for Δ^*, linearized with respect to $\tilde{\Delta}_1^*$.

As a further simplification we shall assume that the desired equation for Δ^* can be derived through the variational principle, by equating to zero the derivative of some real functional with respect to Δ. This is obviously true for the static case, since $\Delta(\mathbf{r})$ must minimize the free energy of the system. The assumption is also reasonable in the general case, because the variational principle ensures energy and momentum conservation in a Galilean invariant system. We shall find that the variational principle, unassisted by any additional considerations, can be used to derive a unique nonlinear equation for Δ^* from the linearized equation (13).

We first obtain an equation that is limited to second order derivatives with respect to $\mathbf{r}$ and t. Expanding Q with respect to $\mathbf{k}$, $\mathbf{q}$, ω_0, and Ω, then integrating and confining ourselves to the leading terms in $|\Delta_0|^{-2}$, we obtain

$$\tilde{\Delta}_1^+(\omega_0\mathbf{k}) - \frac{gmp_0}{2\pi^2}\left\{\left[-\ln\frac{2\tilde{\omega}}{|\Delta_0|} + \frac{1}{3|\Delta_0|^2}\right.\right.$$
$$\left.\times\left(-\frac{\omega_0^2}{2} + \frac{v_F^2k^2}{6}\right)\right]\tilde{\Delta}_1^+(\omega_0\mathbf{k})$$
$$\left. + \frac{1}{6|\Delta_0|^2}\left[\frac{\omega_0^2}{2} - \frac{v_F^2k^2}{6}\right]\tilde{\Delta}_1(\omega_0\mathbf{k})\right\} = 0. \tag{14}$$

The functional that will be varied with respect to Δ for the purpose of obtaining a second order equation is

$$S = \int d^4x\left\{\Phi(|\Delta|^2) + g\frac{mp_0}{2\pi^2}\left[|\Delta|^{2n_1}\left\{\alpha_1\left(\frac{\partial\Delta^*}{\partial t}\right)^2\Delta^2\right.\right.\right.$$
$$\left. + \frac{\beta_1}{2}|\Delta|^2\frac{\partial\Delta^*}{\partial t}\frac{\partial\Delta}{\partial t}\right\} - \frac{v_F^2}{3}|\Delta|^{2n_2}\left\{\alpha_2\Delta^2(\nabla\Delta^*)^2\right.$$
$$\left.\left.\left. + \frac{\beta_2}{2}|\Delta|^2(\nabla\Delta^*)(\nabla\Delta)\right\} + \text{c. c.}\right]\right\}, \tag{15}$$

where c.c. denotes the complex conjugate expression; Φ is an unknown function of $|\Delta|^2$; α_1, α_2, β_1, β_2, n_1, and n_2 are unknown coefficients. In writing (15) we took into consideration that S must be invariant under the transformations

$$\Delta \to \Delta e^{i\alpha}, \quad \Delta^* \to \Delta^* e^{-i\alpha}$$

[an invariance possessed by the initial equation (3)], and also that (14) contains no terms that are linear in ω_0 or Ω.[1)]

We now perform the variation of (15) with respect to Δ, linearization of the result with respect to $\tilde{\Delta}_1$ in accordance with (9), and a Fourier transformation. It is easily verified that the expression thus derived will agree with (14) only if

$$\Phi(|\Delta|^2) = \left(1 + g\frac{mp_0}{2\pi^2}\ln\frac{2\tilde{\omega}}{|\Delta|}\right)|\Delta|^2,$$

$$n_1 = n_2 = -2, \quad \alpha_1 = \alpha_2 = {}^1/_{24}, \quad \beta_1 = \beta_2 = -{}^1/_6.$$

Consequently, the functional (15) becomes

$$S = \int d^4x\left\{\left(1 + g\frac{mp_0}{2\pi^2}\ln\frac{2\tilde{\omega}}{|\Delta|}\right)|\Delta|^2\right.$$
$$+ g\frac{mp_0}{12\pi^2}\left[\left\{\frac{1}{4}\left(\frac{\partial\varphi^*}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\varphi^*}{\partial t}\right)\left(\frac{\partial\varphi}{\partial t}\right)\right\}\right.$$
$$\left.\left. - \frac{v_F^2}{3}\left\{\frac{1}{4}(\nabla\varphi^*)^2 - \frac{1}{2}(\nabla\varphi^*)(\nabla\varphi)\right\} + \text{c.c.}\right]\right\},$$

where we have used the notation

$$\varphi = \ln\Delta.$$

The desired second order equation is then, finally,

$$\left(1 + g\frac{mp_0}{2\pi^2}\ln\frac{2\tilde{\omega}}{|\Delta|}\right)|\Delta|^2 = g\frac{mp_0}{12\pi^2}\left\{\left[\frac{\partial^2\varphi^*}{\partial t^2} - \frac{1}{2}\frac{\partial^2\varphi}{\partial t^2}\right]\right.$$
$$\left. - \frac{v_F^2}{3}\left[\nabla^2\varphi^* - \frac{1}{2}\nabla^2\varphi\right]\right\}. \tag{16}$$

When its complex conjugate is subtracted from (16), we obtain an earlier equation on the basis of a linear approximation by Ambegaokar and Kadanoff.[2] When linearized this equation describes sound whose velocity has the ordinary relationship to the compressibility of the gas. (For an ideal Fermi gas $dp/d\rho = v_F^2/3$.) The existence of such excitations was first observed by Bogolyubov,[3] Galitskii,[4] and Anderson.[5]

Let us now examine Eq. (16) more thoroughly. The left-hand side is obviously not affected by the substitution

$$\Delta(\mathbf{r}, t) \to \Delta(\mathbf{r}, t)\exp[i(\mathbf{kr} - \omega_0 t)]. \tag{17}$$

This fact is closely associated with the Galilean invariance of Gor'kov's equations,[2)] i.e., if they

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1)However, such terms appear in the next approximation with respect to $1/|\Delta_0|^2$. This leads to the added expression

$$\frac{1}{4\mu}\ln\left(\frac{\Delta_0}{\Delta}\right)\Delta\left[i\frac{\partial}{\partial t} - \frac{1}{4m}\nabla^2\right]\Delta^*$$

on the left-hand side of (20), and the corresponding expression on the left-hand side of (16). This term ensures a change of $|\Delta|$ when the chemical potential is changed, but is unimportant for our further discussion.

2)The important role of Galilean invariance in the subject of discussion here was brought to the attention of the authors by Dr. Nozieres, to whom we are deeply indebted for this important comment.

are satisfied by $\Delta(\mathbf{r}, t)$ they will also be satisfied by

$$\Delta(\mathbf{r} - \mathbf{V}t, t) \exp\left[i(2m\mathbf{V}\mathbf{r} - mV^2 t)\right]. \tag{18}$$

The transformation (18) can be verified directly. It follows, of course, from the general formula for Galilean transformations of wave functions, which is also valid for ψ operators in the second quantization representation.[6] Since Eq. (16) has no terms of the order $\partial\varphi/\partial t$, the leading terms in connection with the transformation (18) are clearly $\nabla^2\varphi$ and $\nabla^2\varphi^*$, which must themselves be invariant under this transformation. This result is possible only if they are invariant under the more general transformation (17).

If Δ is replaced by $\Delta_0 e^{i\vartheta}$ in (16), then Δ_0 will satisfy the same equation as Δ. Then in our model the asymptotic expansion of the parameter Δ at large distances from a vortical filament will not contain terms $\sim(\xi_0/r)^2$ that are present at finite temperatures. (The contrary assertion following from the work of Rappoport and Kryloveтskiĭ[7] is a result of the insufficiently accurate approximation used by these authors.) The difference at low temperatures results from the difference between the velocities of normal and superfluid motion $(\mathbf{v}_n - \mathbf{v}_s)$ around a vertex line; $|\Delta|$ is then reduced far from the line. However, at absolute zero the existence of a superfluid velocity $\mathbf{v}_s$ cannot alter $|\Delta|$ because of Galilean invariance.

It can be seen from the foregoing that Eq. (16) is insufficiently accurate to represent the motion of the system, at the very least because of the fact that in virtue of (17) it possesses very many more solutions than are possible in reality. To obtain a complete equation we must take higher order terms, $\partial\nabla^2/\partial t$ and ∇^4, into account. In the Galilean transformation (18) and also in the problem concerning oscillations of a vortex line these terms are of the same order as the already written terms of the order $\partial^2/\partial t^2$. We note that all terms of the order ∇^4, which in themselves are invariant under (17), should be dropped because they introduce nothing new and their inclusion would amount to excessive accuracy.

In deriving the complete equation we proceed as for the derivation of (16). After a tedious calculation Eq. (13) has the form, to terms of the order $k^2\omega_0$ and k^4,

$$\{\ldots\} = g\frac{mp_0}{2\pi^2}\Bigg\{\frac{1}{|\Delta_0|^2}\Bigg[\frac{\omega_0}{3}\frac{\mathbf{kq}}{2m} + \frac{2\Omega - \omega_0}{3}\frac{k^2}{8m} - \frac{1}{3}\frac{(\mathbf{kq})^2}{8m^2} - \frac{1}{3}\frac{k^4 + k^2(\mathbf{kq}) - k^2q^2}{16m^2}\Bigg]\tilde{\Delta}_1^+(\omega_0\mathbf{k})$$
$$+ \frac{\Delta_0^{*2}}{|\Delta_0|^4}\Bigg[-\frac{\omega_0}{6}\frac{\mathbf{kq}}{2m} - \frac{\Omega}{3}\frac{k^2}{8m} + \frac{1}{6}\frac{(\mathbf{kq})^2}{8m^2} + \frac{1}{6}\frac{k^4 + k^2q^2}{16m^2}\Bigg]\tilde{\Delta}_1(\omega_0\mathbf{k})\Bigg\} \tag{19}$$

where $\{\ldots\}$ designates the left-hand side of (14). Terms of the order $\mu^2 k^4/|\Delta_0|^4$ have been dropped from (19). Although these terms are formally large by comparison with $k^4/|\Delta_0|^2$, it is easily comprehended that because they possess excessive invariance under (17) they contribute nothing new by comparison with the terms containing ∇^2 in (16). We must now formulate the general variational principle of suitable order and select the coefficients in the functional that will make the linear equation agree with (19). When selecting the functional S we must take into account that it should be invariant under time reversal:

$$t \to -t, \quad \Delta \to \Delta^*,$$

which leaves (2) and (3) unchanged. We shall not present all these laborious calculations nor write out the expression for S. The nonlinear equation for Δ^* becomes finally (recalling that $\varphi = \ln\Delta$)

$$|\Delta|^2 \ln\frac{\Delta_0}{\Delta} = \frac{1}{6}\left[\frac{\partial^2\varphi^*}{\partial t^2} - \frac{1}{2}\frac{\partial^2\varphi}{\partial t^2}\right] - \frac{v_F^2}{3}\frac{1}{6}\left[\nabla^2\varphi^* - \frac{1}{2}\nabla^2\varphi\right]$$
$$- \frac{i}{24m}\Bigg\{-\nabla^2\frac{\partial\varphi^*}{\partial t} + \frac{\partial}{\partial t}\left[(\nabla\varphi)^*(\nabla\varphi) - \frac{1}{2}(\nabla\varphi^*)^2\right]$$
$$+ \operatorname{div}\left[\frac{\partial\varphi}{\partial t}\nabla\varphi^* + \frac{\partial\varphi^*}{\partial t}\nabla\varphi - \frac{\partial\varphi^*}{\partial t}\nabla\varphi^*\right]\Bigg\}$$
$$+ \frac{1}{96m^2}\Big\{-\operatorname{div}\left[(\nabla\varphi)^2\nabla\varphi + (\nabla\varphi^*)^2\nabla\varphi^* + (\nabla\varphi)^2\nabla\varphi^*\right.$$
$$\left. + 2(\nabla\varphi^*\cdot\nabla\varphi)\nabla\varphi\right] + \operatorname{div}(\nabla^2\varphi^*\cdot\nabla\varphi) - \frac{1}{2}\nabla^2(\nabla\varphi^*)^2\Big\}. \tag{20}$$

Equation (20) is the desired equation describing the dynamic properties of a superfluid Fermi system. Its Galilean invariance can be proved directly by substituting (18). When the complex conjugate of (20) is subtracted from it, we obtain a conservation law having the significance of a continuity equation for the number of particles:

$$\frac{\partial\delta\rho}{\partial t} + \operatorname{div}\mathbf{j} = 0, \tag{21}$$

where

$$\delta\rho = A\Bigg\{\frac{i}{4}\frac{\partial(\varphi^* - \varphi)}{\partial t} + \frac{1}{24m}\left[(\nabla\varphi^*)(\nabla\varphi) - \frac{1}{2}(\nabla\varphi^*)^2 + \text{c.c.}\right]\Bigg\}, \tag{22}$$

$$\mathbf{j} = A\Big\{-i\frac{v_F^2}{3}\frac{1}{4}\nabla(\varphi^* - \varphi)$$

$$+\frac{1}{24m}\left[\frac{\partial\varphi}{\partial t}\nabla\varphi^*-\frac{\partial\varphi^*}{\partial t}\nabla\varphi^*+\text{c.c.}\right]$$
$$-\frac{i}{96m^2}\left[\nabla\varphi^*(\nabla\varphi)^2+2\nabla\varphi(\nabla\varphi^*\cdot\nabla\varphi)+\frac{1}{2}\nabla(\nabla\varphi^*)^2\right.$$
$$\left.-\nabla\varphi\nabla^2\varphi^*-\text{c.c.}\right]-\frac{1}{24m}\nabla\frac{\partial(\varphi^*+\varphi)}{\partial t}\Big\}. \quad (23)$$

The expressions for ρ and **j** were obtained by Stephen and Suhl[8] up to first order derivatives.

The undetermined coefficient A can be determined most simply from a linear approximation, by calculating the change of density brought about through an addition δG to the Green's function. It is easily seen that

$$\delta\rho(\omega_0\mathbf{k})=-2i[\delta G(x,x)]_{\omega_0\mathbf{k}}$$
$$=\frac{mp_0}{\pi^2}\frac{\omega_0}{4|\Delta_0|^2}[\Delta_0\Delta_1^+(\omega_0\mathbf{k})-\Delta_0^*\Delta_1(\omega_0\mathbf{k})]. \quad (24)$$

Comparing (24) with (22), we obtain

$$A=mp_0/\pi^2. \quad (25)$$

We note that in a linear approximation (22) and (25) yield the formula

$$\delta\mu=\frac{i}{4}\frac{\partial(\varphi^*-\varphi)}{\partial t}$$

that has been given in [9].

It follows from (22), (23), and (25) that if we neglect the derivatives of $|\Delta|$ and the terms in (23) containing second derivatives these equations can be written as

$$-\frac{v_F^2}{3}\frac{\delta\rho}{\rho}=\frac{i}{4m}\frac{\partial(\varphi^*-\varphi)}{\partial t}+\frac{v^2}{2},$$
$$\mathbf{j}=(\rho+\delta\rho)\mathbf{v},\quad \mathbf{v}=\frac{i}{4m}\nabla(\varphi^*-\varphi). \quad (26)$$

If we now apply the gradient operator to the first equation of (26) and use the relationship between v_F and compressibility, this equation reduces to

$$\frac{\partial\mathbf{v}}{\partial t}+\nabla\frac{v^2}{2}=-\frac{\nabla p}{\rho}. \quad (27)$$

This is simply Euler's equation for irrotational flow of an ideal liquid. Equations (21) and (27) comprise the complete system of hydrodynamic equations, which thus follow from our basic equation (20). The neglect of terms containing derivatives of $|\Delta|$ and **v** in (22) and (23) signifies the neglect of quantum corrections to hydrodynamic equations. Equation (22) for the variation of density is equivalent to Bernoulli's hydrodynamic equation.

The fact that the equations of hydrodynamics can be derived from (20) means, specifically, that the dispersion of oscillations of a vortex line in a superfluid Fermi gas has the usual hydrodynamic form

$$\omega=\frac{k^2}{2m}\ln\frac{1}{kp_0}. \quad (28)$$

This has been derived with complete rigor, the only essential requirement being a limitation to distances from the vortical axis that are large enough to justify the approximate equation (20).

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