NON-COMPACT SYMMETRY GROUP OF A QUANTUM OSCILLATOR

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A quantum oscillator is considered from the viewpoint of the $\mathfrak{L}_3$ Lorentz group which is its symmetry group. The wave functions of the even (odd) oscillator levels form irreducible representations of this group, corresponding to the weights $1/4$ and $3/4$. Some relationships between the wave functions are obtained as a consequence of the symmetry considered. A class of irreducible representations with an arbitrary (and complex) weight of the group $\mathfrak{L}_3$, characterized by condition (A1.3), (semi-infinite representations) is considered in the Appendix. The representations are expressed in terms of complex-variable functions. The principal properties of such representations are discussed. In the general case the representations are realized in infinite-dimensional spaces with an infinite metric of finite rank indefiniteness. Expressions for the matrix elements of the representations are deduced. Two particular representations related to quantum (complex) oscillators are considered separately.

1. Jordan et al.[1] have established, in their paper on generalized field quantization methods (parastatistics), an isomorphism between definite bilinear combinations of creation and annihilation operators of Bose particles and infinitesimal operators of the three-dimensional Lorentz group $\mathfrak{L}_3$ (see also [2]). We analyze here a quantum oscillator from the point of view of the $\mathfrak{L}_3$ group. This group is the symmetry group of the oscillator. Many relations between the wave functions are consequences of this symmetry (and also the symmetry with respect to the rotation group). The analysis presented here is of interest in connection with the group-theoretical approach to certain dynamic problems of quantum mechanics [3-5], and also in connection with investigations of certain quantum-system symmetry groups discussed by Gell-Mann at the Spring School of Physics at Erevan (May, 1965) [6].

In Appendix 1 we construct all the irreducible semi-infinite representations of the $\mathfrak{L}_3$ group. These representations (condition A1.3) are of interest in connection with the problem considered and with the generalized methods of Bose field quantization [1], and also in connection with the group-theoretical approach [7,8] to the study of complex spin [9].

2. The creation and annihilation operators of a one-dimensional operator a^+ and a are expressed in terms of the coordinate x and the momentum p $= -id/dx$ by means of the formulas

$$a^+ = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right), \quad a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right). \tag{1}$$

Let us consider the operators

$$L_0 = \frac{1}{4}(a^+a + aa^+) = \frac{1}{4}\left(x^2 - \frac{d^2}{dx^2}\right),$$
$$L_1 = \frac{i}{4}(aa - a^+a^+) = \frac{i}{4}\left(x\frac{d}{dx} + \frac{d}{dx}x\right),$$
$$L_2 = -\frac{1}{4}(aa + a^+a^+) = -\frac{1}{4}\left(x^2 + \frac{d^2}{dx^2}\right), \tag{2}$$

which (apart from a factor $1/2$) are respectively, the Hamiltonian, the Pfaffian and the Lagrangian of the oscillator [1)] (the dynamic variables are written out in a system in which $\hbar = m = \omega = 1$; m is the oscillator mass and ω is the natural frequency).

The operators (2) satisfy the commutation relations

$$[L_0, L_1] = iL_2, \quad [L_1, L_2] = -iL_0, \quad [L_2, L_0] = iL_1, \tag{3}$$

which hold for infinitesimal operators of the three-dimensional Lorentz group $\mathfrak{L}_3$ (two spatial coordinates and one temporal). Consequently, (2) forms the Lie algebra of the $\mathfrak{L}_3$ group. The operator L_0 is the generator of the rotation transformation in the plane of the two spatial coordinates (1—2) while L_1 and L_2 are the generators of the Lorentzian transformations in the planes (0—2) and (0—1).

[1)] In [10] these operators were considered in connection with four-valued representations of the rotation group R_3.

The quadratic form $Q = L_1^2 + L_2^2 - L_0^2$ commutes with all the operators L_i, with $Q \equiv 3/16 = \lambda(1-\lambda)$ for the operators (2). From this we get $\lambda = 1/4$ and $\lambda = 3/4$. Consequently, the operators (2) give the representations of the $\mathfrak{L}_3$ group with weights $1/4$ and $3/4$. It is known [11] that for fixed λ there exists an infinite set of representations for the $\mathfrak{L}_3$ group. All these representations differ from one another in the number ν_0 that enters into the definition of the canonical basis of the representation $f_m^{(\lambda)}$ satisfying the system of equations [11] ($L_\pm = L_1 \pm iL_2$)

$$L_0 f_m^{(\lambda)} = (\nu_0 + m) f_m^{(\lambda)},$$

$$L_+ f_m^{(\lambda)} = -i[\lambda(1-\lambda) + (\nu_0+m)(\nu_0+m+1)]^{1/2} f_{m+1}^{(\lambda)},$$

$$L_- f_m^{(\lambda)} = i[\lambda(1-\lambda) + (\nu_0+m)(\nu_0+m-1)]^{1/2} f_{m-1}^{(\lambda)}, \tag{4}$$

and which can be an arbitrary complex number (for fixed λ).

For the oscillator we have

$$L_+ = \frac{i}{4}\left(-\frac{d^2}{dx^2} + 2x\frac{d}{dx} - x^2 + 1\right),$$
$$L_- = \frac{i}{4}\left(\frac{d^2}{dx^2} + 2x\frac{d}{dx} + x^2 + 1\right). \tag{5}$$

It follows from (4) that if $\lambda = 1/4$ and $\lambda = 3/4$ all the representations (different ν_0) are infinite-dimensional (the number m takes on values at m = 0, ±1, ±2, ...). Corresponding to the oscillator are only those representations for which [2)] $\nu_0 = \lambda$. In this case formulas (4) define infinite-dimensional unitary representations of the group $\mathfrak{L}_3$ (the matrices of the infinitesimal operators L_i are hermitian), which terminate from below at m = 0 (if $\nu_0 = \lambda$, then m takes on only integer positive values). All these representations are irreducible.

If $\nu_0 = \lambda$, the functions of the canonical basis satisfy the equations

$$\frac{1}{4}\left(x^2 - \frac{d^2}{dx^2}\right) f_m^{(1/4)} = \left(m + \frac{1}{4}\right) f_m^{(1/4)},$$
$$\frac{1}{4}\left(x^2 - \frac{d^2}{dx^2}\right) f_m^{(3/4)} = \left(m + \frac{3}{4}\right) f_m^{(3/4)}. \tag{6}$$

The solutions of these equations are the known Hermite functions of even and odd index:

$$f_m^{(1/4)}(x) = \psi_{2m}(x), \qquad f_m^{(3/4)}(x) = \psi_{2m+1}(x),$$

$$\psi_n(x) = (\sqrt{\pi}\, 2^n n!)^{-1/2} e^{-x^2/2} H_n(x), \tag{7}$$

where $H_n(x)$ are Hermite polynomials.

[2)]We consider in Appendix 1, subject to the condition $\nu_0 = \lambda$ all the representations of the Lorentz group L_3 corresponding to arbitrary (and complex) values of the weight λ.

Two infinite systems of the functions $\{\psi_{2m}\}$ and $\{\psi_{2m+1}\}$ form an orthonormal basis in the Hilbert spaces H_+ and H_-. The spaces H_+ and H_- are made up of functions that are even and odd with respect to the substitution $x \to -x$, and whose modulus squared is integrable. The scalar product is specified in H_+ and H_- in the form

$$(f, g) = \int_{-\infty}^{\infty} \bar{f}(x) g(x)\, dx. \tag{8}$$

The complete space of wave functions of the oscillator (state vectors) is the direct sum of the space H_+ and H_-: $H = H_+ \oplus H_-$.

Thus, from the point of view of the Lorentz group $\mathfrak{L}_3$, the entire infinite system of oscillator levels breaks up into two subsystems of even (n = 2m) and odd (n = 2m + 1) levels. The wave functions of these subsystems for irreducible representations of the $\mathfrak{L}_3$ group corresponding to the weights $\lambda = 1/4$ and $\lambda = 3/4$.

Finite transformations of the group $\mathfrak{L}_3$ in H_+ and H_- are specified by operators U represented in the form

$$U = \exp(-i\gamma L_0)\exp(-i\beta L_2)\exp(-i\alpha L_0). \tag{9}$$

In order to cover the entire group $\mathfrak{L}_3$, the parameters α, β, and γ must take on values from the region $0 \le \alpha, \gamma < 2\pi$, and $0 \le \beta < \infty$.

The representations considered by us are four-valued. Indeed, inasmuch as the spectrum of the operator L_0 is made up by the sequence of numbers $m + 1/4$ for $\lambda = 1/4$ or $m + 3/4$ for $\lambda = 3/4$, we get

$$\exp(-2\pi ikL_0)\psi_{2m} = (-i)^k\psi_{2m},$$
$$\exp(-2\pi ikL_0)\psi_{2m+1} = i^k\psi_{2m+1}.$$

Only after four circuits (k = 4) do we return to the initial function. Therefore, the parameters α, β and γ (in 9) take on values from the region $0 \le \alpha, \gamma < 8\pi$, $0 \le \beta < \infty$.

Under the transformations (9) the infinitesimal operators (2) can be linearly transformed through one another

$$UL_iU^{-1} = w_{ij}L_j \tag{10}$$

with a transformation matrix

$$w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \operatorname{ch}\beta & -\operatorname{sh}\beta & 0 \\ -\operatorname{sh}\beta & \operatorname{ch}\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{11*}$$

The operators of the coordinate x and of the momentum $p = -id/dx$ are in this case subjected to the linear transformation

*ch ≡ cosh, sh ≡ sinh.

$$UxU^{-1} = ax + bp, \qquad UpU^{-1} = cx + dp \tag{12}$$

with real matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \\ -\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} \operatorname{ch}\frac{\beta}{2} & -\operatorname{sh}\frac{\beta}{2} \\ -\operatorname{sh}\frac{\beta}{2} & \operatorname{ch}\frac{\beta}{2} \end{pmatrix}$$

$$\times \begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}, \qquad ad - bc = 1. \tag{13}$$

Formulas (4) lead to the following recurrence relation between the Hermite functions

$$\psi''_{2m} - 2x\psi'_{2m} + (x^2 - 1)\psi_{2m} = 4[(m+1)(m+{}^1/_2)]^{1/2}\psi_{2m+2},$$
$$\psi''_{2m} + 2x\psi'_{2m} + (x^2+1)\psi_{2m} = 4[m(m-{}^1/_2)]^{1/2}\psi_{2m-2} \tag{14}$$

and analogous formulas for the Hermite functions for odd index.

Further, it follows from the transformation formulas (10) that the rotated Hermite functions $\psi_n(\sqrt{i}\,x)$ $(i = \sqrt{-1})$, which are the eigenfunctions of the operator L_2, are obtained from the power functions x^n (eigenfunctions of the operator L_1) by rotation through $\pi/2$ in the (1—2) plane:

$$\psi_n(\sqrt{i}\,x) \sim \exp\left(-i\frac{\pi}{2}L_0\right)x^n. \tag{15}$$

Other relations (in particular, the connection between the Hermite functions $\psi_n(x)$ with the rotated functions $\psi_n(\sqrt{i}\,x)$ follow from an analysis of the operator from the point of view of the rotation group R_3).

APPENDIX 1

1. A study of the representations of the Lorentz group $\mathfrak{L}_3$ (non-compact group) is of interest primarily in connection with the problem under consideration and the generalized methods of quantization of the Bose fields[1] (the so-called Bose parastatistics). Namely, to each irreducible representation of the group $\mathfrak{L}_3$ there corresponds a definite Bose parastatistic. It is necessary here to impose on the representation the condition (A1.3) (semi-infinite representations), which follows from the requirement that the number of particles be positive[3)]. In this case we arrive at unitary representations (real $\lambda > 0$) and at representations that are unitary in an indefinite metric (all other values of λ).

The representations of the group $\mathfrak{L}_3$ with arbitrary weight λ are also of interest in connection with the complex angular momentum[7].

The author has considered[7,8] the representations of the rotation group R_3 and the Lorentz group $\mathfrak{L}_4$ with complex spin. It is known[12,13] that the $\mathfrak{L}_4$ group has three stationary subgroups: the rotation group R_3, the three-dimensional Lorentz group $\mathfrak{L}_3$, and the cone group, corresponding respectively to time-like, space-like, and isotropic vectors. Since the group $\mathfrak{L}_3$ is also connected with the classification of the spin states of quantum systems, it is of interest to study the representations of the $\mathfrak{L}_3$ group with complex spin.

We shall construct here all the representations (subject to condition (A1.3)) in the realization by means of functions of complex variable. Two of these representations (with weights $1/4$ and $3/4$) are connected with the one-dimensional quantum oscillator (these representations are considered separately in Appendix 2). Many representations of the group $\mathfrak{L}_3$ (single- and double-valued) were considered by Bargmann[11].

2. The representation of infinitesimal operators. Since the group $\mathfrak{L}_3$ (like a cyclic group) is infinitely connected[11] (unlike the groups R_3 and $\mathfrak{L}_4$, which are doubly connected), there exist for it multiply-valued continuous representations. We shall show below that all the representations are continuous (unlike the group R_3 for which only the finite-dimensional representations are continuous[14]).

Any irreducible representation of the $\mathfrak{L}_3$ group is defined by three infinitesimal operators L_0, L_1 and L_2 satisfying the commutation relations

$$[L_0, L_1] = iL_2, \quad [L_1, L_2] = -iL_0, \quad [L_2, L_0] = iL_1. \tag{A1.1}$$

The quadratic form $Q = L_1^2 + L_2^2 - L_0^2$ commutes with all the L_i and assumes for the irreducible representation of weight λ a value $Q = \lambda(1-\lambda)$. In the canonical basis $f_\nu^{(\lambda)}$ the operators L_i are expressed by the formulas ($L_\pm = L_1 \pm iL_2$):

$$L_0 f_\nu^{(\lambda)} = \nu f_\nu^{(\lambda)}, \quad L_+ f_\nu^{(\lambda)} = \alpha_{\nu+1}^{(\lambda)} f_{\nu+1}^{(\lambda)}, \quad L_- f_\nu^{(\lambda)} = \alpha_\nu^{(\lambda)} f_{\nu-1}^{(\lambda)}, \tag{A1.2}$$

where $\alpha_\nu^{(\lambda)} = [\lambda(1-\lambda) + \nu(\nu-1)]^{1/2}$.[11] As in the case of the rotation group R_3 for fixed (complex) λ, we confine ourselves to only one representation, in which ν takes on the values

$$\nu = n + \lambda \tag{A1.3}$$

(n is integer). It follows from (A1.2) that such

[3)] A similar condition was assumed in the analysis of the representation of the rotation group[7]. This limitation was connected there with the unitarity of the corresponding representations.

representations are cut off from below at n = 0, so that n assumes only positive integer values. The multiplicity of these representations, determined by the spectrum of the operator L_0, is thus connected with the weight λ itself (for rational λ = l/s, where l and s are mutually prime, the representation is s-valued, and if λ is irrational and complex, the representations are infinitely-valued). Bargmann [11] considered only single-valued and doubly-valued representations. In this case ν takes on only integer or half-integer values independently of λ.

When λ = m/2 (m = positive integer) formulas (A1.2) give finite-dimensional representations (in this case the representation is cut off at n = 0 and n = m; such representations are realized in spaces with dimensionality m + 1). For all other values of λ, the representations are infinite-dimensional. When λ > 0, formulas (A1.2) define unitary representations.

3. The representation as a whole is specified by operators T, which in Euler variables are written in the form

$$T(L) = \exp(-i\gamma L_0)\exp(-i\beta L_2)\exp(-i\alpha L_0). \quad (A1.4)$$

For s-valued representations, the parameters α, β and γ take on values from the region

$$0 \leqslant \alpha,\ \gamma < 2\pi s, \quad 0 \leqslant \beta < \infty. \quad (A1.5)$$

From the commutation relations (A1.1) it follows that

$$TL_iT^{-1} = L_{ij}L_j, \quad (A1.6)$$

where the matrix is

$$L_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \operatorname{ch}\beta & -\operatorname{sh}\beta & 0 \\ -\operatorname{sh}\beta & \operatorname{ch}\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}. \quad (A1.7)$$

We now construct in explicit form the representation of weight λ in the realization by means of the functions of the complex variable z = x + iy.

It is known first that the bilinear transformations

$$z \to z' = \sigma z = \frac{az + \bar{b}}{bz + \bar{a}} \quad (A1.8)$$

with matrix

$$\sigma = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} \begin{pmatrix} \operatorname{ch}\frac{\beta}{2} & -\operatorname{sh}\frac{\beta}{2} \\ -\operatorname{sh}\frac{\beta}{2} & \operatorname{ch}\frac{\beta}{2} \end{pmatrix} \times \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}, \quad a\bar{a} - b\bar{b} = 1 \quad (A1.9)$$

realize conformal mapping of the interior of the unit circle |z| < 1 on itself [4]. Let us consider functions that are elliptical in the circle |z| < 1. We shall specify the representation of the weight λ in the class of such functions by means of the formula

$$T(L)f(z) = (bz + \bar{a})^{-2\lambda} f(\sigma z), \quad (A1.10)$$

which generalizes the corresponding formula in the paper by Bargmann [11]. In (10) λ is any complex (fixed) number. Formula (A1.10) determines the analytic transformation of functions f(z) that are analytic in the circle |z| < 1.

The infinitesimal operators which correspond to transformations (A1.10) take the form

$$L_0 = z\frac{d}{dz} + \lambda, \quad L_+ = z^2\frac{d}{dz} + 2\lambda z, \quad L_- = \frac{d}{dz}. \quad (A1.11)$$

They satisfy the commutation relations (A1.1), with Q ≡ λ(1 − λ).

The functions $f_\nu^{(\lambda)}$ of the canonical basis of the representation of weight λ satisfy equations (A1.2) and take the form

$$f_\nu^{(\lambda)}(z) = A_\nu^{(\lambda)} z^n, \quad A_\nu^{(\lambda)} = \left(\frac{\Gamma(n + 2\lambda)}{n!\,\Gamma(2\lambda)}\right)^{1/2} = (C_n^{n+2\lambda-1})^{1/2}. \quad (A1.12)$$

4. Scalar product. When λ > 0 the representations in question are unitary. In this case there exists an invariant scalar product

$$(f, g)_\lambda = \int \bar{f}(z) g(z) \omega_\lambda(z)\, dz, \quad dz = dx\,dy, \quad (A1.13)$$

where the integration is carried out over the region |z| < 1 (unit circle).

The weight function ω_λ is determined from the condition that the form (A1.13) be invariant to the transformations (A1.10), viz., $(Tf, Tg)_\lambda = (f, g)_\lambda$, and is equal to

$$\omega_\lambda(z) = \frac{2\lambda - 1}{\pi}(1 - |z|^2)^{2(\lambda-1)}. \quad (A1.14)$$

The form (A1.13) is a Hermitian positive definite bilinear form. The basis functions (A1.12) are orthogonal in the scalar product (A1.13):

$$(f_\mu^{(\lambda)}, f_\nu^{(\lambda)})_\lambda = \delta_{\mu\nu}. \quad (A1.15)$$

The representations with λ > 0 are realized in Hil-

[4]The matrices (A1.9) are unitary in the indefinite metric:

$$\tilde{\sigma} = \sigma_3\sigma^+\sigma_3^{-1} = \sigma^{-1}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

bert spaces H_λ with scalar products (A1.13). H_λ is made up of the functions

$$f = \sum_n f_n z^n, \quad g = \sum_n g_n z^n,$$

for which

$$\|f\|_\lambda^2 = \sum_n |f_n|^2 (C_n^{n+2\lambda-1})^{-1} < \infty;$$

$$(f, g)_\lambda = \sum_n \bar{f}_n g_n (C_n^{n+2\lambda-1})^{-1}. \qquad \text{(A1.16)}$$

The invariant scalar product can be defined also for representations with arbitrary complex weight λ. In this case it is specified by a regularized bilinear form (if Re $\lambda < 0$) [5]

$$(f^{(\lambda)}, g^{(\lambda)})_\lambda = \text{reg.} \int \bar{f}^{(\bar{\lambda})}(z) g^{(\lambda)}(z) \omega_\lambda(z) dz \qquad \text{(A1.17)}$$

with regularization at the singular points $|z|^2 = 1$ (of the pole type) of the weight function (A1.14) (λ is complex).

For the basis functions (A1.12), the form (A1.17) takes on the values

$$(f_\mu^{(\lambda)}, f_\nu^{(\lambda)})_\lambda = \delta_{\mu\nu} \eta_\nu^{(\lambda)},$$

$$\eta_\nu^{(\lambda)} = \overline{(\Gamma(\bar{\nu}+\bar{\lambda})/\Gamma(2\bar{\lambda}))^{1/2}} / (\Gamma(\nu+\lambda)/\Gamma(2\lambda))^{1/2}. \qquad \text{(A1.18)}$$

$\eta_\nu^{(\lambda)} = 1$ when Re $\lambda > 0$, $\eta_\nu^{(\lambda)} = (-1)^n$ when Re $\lambda < 0$ and $n < -[2 \text{ Re } \lambda]$ ([a] is the integer part of the number a), and $\eta_\nu^{(\lambda)} = \mp 1$ when $n > -[2 \text{ Re } \lambda]$, depending on whether $-[2 \text{ Re } \lambda]$ is even or odd. Thus, for Re $\lambda < 0$ the scalar product is a hermitian indefinite bilinear form. The rank of the indefiniteness is $-[2 \text{ Re } \lambda] + 1$. In particular, all the finite-dimensional representations of the group $\mathfrak{L}_3$ ($\lambda = -m/2$) are realized in finite-dimensional spaces with indefinite metric. We note also that representations with indefinite metric are unitary in their own indefinite metric.

5. The matrix elements of the representation are obtained by expansion of the shifted function $T(L) f_\nu^{(\lambda)}(z)$ in terms of the basis functions (A1.12):

$$T(L) f_\nu^{(\lambda)}(z) = \sum_\mu T_{\mu\nu}^{(\lambda)}(L) f_\mu^{(\lambda)}(z). \qquad \text{(A1.19)}$$

We have

$$T_{\mu\nu}^{(\lambda)}(L) = e^{-i\mu\gamma} t_{\mu\nu}^{(\lambda)}(\beta) e^{-i\nu\alpha}, \quad \mu = m + \lambda, \quad \nu = n + \lambda, \qquad \text{(A1.20)}$$

$$t_{\mu\nu}^{(\lambda)}(\beta) = (-1)^{-2\lambda-\varkappa} \left[\frac{k!\,\Gamma(\rho-\varkappa)/\Gamma(2\lambda)}{\Gamma(\rho-\varkappa-2\lambda)\Gamma(k+2\lambda)/\Gamma(2\lambda)} \right]^{1/2}$$

$$\times (1-x)^{\rho/2}(1+x)^{\varkappa/2} P_k^{(\rho,\varkappa)}(x), \qquad \text{(A1.21)}$$

where $x = \cosh \beta$, $P_k^{(\rho,\kappa)}(x)$ are Jacobi polynomials, and

$$\rho = |\mu - \nu|, \quad \varkappa = -(\mu+\nu), \quad k = -\lambda - (\rho+\varkappa)/2.$$

The matrix elements $T_{\mu\nu}^{(\lambda)}(L)$ satisfy the equation [6]

$$\left\{ \frac{\partial^2}{\partial\beta^2} + \text{cth}\,\beta \frac{\partial}{\partial\beta} + \frac{1}{\text{sh}^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\gamma^2} - 2\text{ch}\,\beta \frac{\partial^2}{\partial\alpha\partial\gamma} \right) + \lambda(1-\lambda) \right\} T_{\mu\nu}^{(\lambda)}(L) = 0. \qquad \text{(A1.22)*}$$

We note that no orthogonality relations exist for the representations considered here (with condition (A1.3)) so that the matrix elements $T^{(\lambda)}$ and $T^{(\lambda')}$ are not orthogonal to each other.

APPENDIX 2

We consider here one particular realization of the representations of the group $\mathfrak{L}_3$ with weights $1/4$ and $3/4$ by means of analytic (entire) functions in the whole z plane (and not in the circle $|z| < 1$ as in Appendix 1). These representations are closely related to the one-dimensional (complex) oscillator and differ from the representations considered in the main text of the article.

We consider the three operators

$$L_0 = \frac{1}{4}(zd + dz), \quad L_1 = \frac{1}{4}(d^2 + z^2),$$

$$L_2 = \frac{i}{4}(d^2 - z^2), \quad d \equiv \frac{d}{dz}, \qquad \text{(A2.1)}$$

which are respectively the Pfaffian, Lagrangian, and Hamiltonian of the one-dimensional (complex) oscillator. The operators (A2.1) satisfy the commutation relations (A1.1), with $L_+ = z^2/2$, $L_- = d^2/2$ and $Q \equiv 3/16$, so that $\lambda = 1/4$ and $\lambda = 3/4$. Under the transformations T, the operators z and d are transformed in accordance with the formulas

$$TzT^{-1} = az + bd, \quad TdT^{-1} = \bar{b}z + \bar{a}d \qquad \text{(A2.2)}$$

with matrix (A1.9).

Solutions of (A1.2) with $\lambda = 1/4$ and $\lambda = 3/4$ are respectively the functions

$$f_n^{(1/4)} = \frac{z^{2n}}{\sqrt{(2n)!}}, \quad f_n^{(3/4)} = \frac{z^{2n+1}}{\sqrt{(2n+1)!}}. \qquad \text{(A2.3)}$$

[5] Such a scalar product for the representations of the rotation group was introduced earlier in [7].

[6] Many formulas of this section can be obtained from the corresponding formulas for the rotation group representations [7] with the substitution $\beta \to i\beta$.

*cth ≡ coth.

In the realization under consideration the representations are given by the formula

$$T(L)f(z) = Tf(z)T^{-1}(T\cdot 1) = f(TzT^{-1})\varkappa(z;L)$$
$$= f(az+bd)\varkappa(z;L), \qquad (A2.4)$$

where $\kappa(z;\ L) = T\cdot 1$ and can be readily obtained:

$$\varkappa(z;L) = e^{-i(\alpha+\gamma)/4}\left(\mathrm{ch}\,\frac{\beta}{2}\right)^{-1/2}\exp\left(-\frac{1}{2}z^2e^{-i\gamma}\,\mathrm{th}\,\frac{\beta}{2}\right). \qquad (A2.5)^*$$

The functions (A2.3) form orthogonal systems in the scalar product ($dz = dxdy$):

$$(f_m^{(\lambda)}, f_n^{(\lambda)}) = \int \bar{f}_m^{(\lambda)}(z) f_n^{(\lambda)}(z)\rho(z)\,dz = \delta_{m,n} \qquad (A2.6)$$

with a weight function

$$\rho(z) = \pi^{-1}\exp(-|z|^2), \qquad (A2.7)$$

which is determined from the condition for the invariance of the form (A2.6) to the transformation (A2.4) (with $(f^{(1/4)}, f^{(3/4)}) = 0$). In (A2.6) the integration is over the entire complex z plane.

Representations with $\lambda = 1/4$ and $\lambda = 3/4$ are realized in the class of entire functions

$$f^{(1/4)}(z) = \sum_n f_n z^{2n}$$

(or

$$f^{(3/4)}(z) = \sum_n f_n z^{2n+1}),$$

which are even or odd with respect to the substitution $z \rightarrow -z$, for which

$$\|f\|^2 = (f,f) = \sum_n |f_n|^2(2n)! < \infty$$

(or $\sum_n |f_n|^2(2n+1)! < \infty$). These classes of function form Hilbert spaces with scalar product (A2.6).

The matrix elements of the representations considered here are expressed by formulas (A1.20) and (A1.21) in which it is necessary to put $\lambda = 1/4$ or $\lambda = 3/4$.

*th ≡ tanh.

Note added in proof: (November 4, 1965). Recently, Barut et al.[15] also considered the question of the noncompact symmetry group (dynamic group) of an oscillator. In these papers, the oscillator is set in correspondence with the representation $D_{1/2}$ of the Lorentz group $\mathscr{L}_3$. It is interesting to note that the oscillator can also be related to the representation of a two-parameter group, the infinitesimal operators of which I_1 and I_2 satisfy the commutation relations $[I_1, I_2] = I_2$. In our case we must put $I_1 = H$, $I_2 = a$ (H is the Hamiltonian of the oscillator and a the annihilation operator). Then the entire level system forms one representation of this group. But this group is not semi-simple, since the tensor $g_{ik} = C^l_{im}C^m_{lk} = \begin{pmatrix}0&0\\1&0\end{pmatrix}$ (C^l_{ik} are the structure constants) and has no inverse[16].

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