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CONTRIBUTION TO THE NONLINEAR THEORY OF TWO-STREAM INSTABILITY IN THE PRESENCE OF THE ANOMALOUS DOPPLER EFFECT

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The nonlinear problem of exciting one-dimensional transverse waves by a beam of electrons moving in a retarding medium with a velocity greater than the wave phase velocity is considered in the hydrodynamic approximation. Solutions are studied which are presented as waves with fixed wave number and time-varying amplitude and phase. It is shown that for small beam densities the main nonlinear effect limiting the increase in oscillation amplitude is the loss of synchronism between the particles and the field as a result of beam deceleration; this leads to a periodic change in the excitation and absorption of the field by the beam. The maximum oscillation amplitudes are computed and the possibilities of using this effect for cutting off beam instabilities by an external field are discussed.

THE exponential law that holds for the increase in perturbation amplitude with time during the growth of instability in a system consisting of a decelerating medium (plasma) and a beam of charged particles is valid only during the beginning of the process, when perturbation amplitudes are small. For large field amplitudes it is necessary to consider the nonlinear effects of the reaction of the excited oscillations on the beam motion, which limit the growth of the field perturbation amplitude and of the beam velocity.

When there is no phase correlation of the excited oscillations and the excited-field amplitudes are small, the reaction of instability on the distribution function of the beam particles is accounted for by using nonlinear theory [1]. In this article another limiting case is examined—nonlinear oscillations with fixed phase, excited by a beam of charged particles moving along a magnetic field with velocity greater than the wave phase velocity (i.e., when the conditions of the anomalous Doppler effect are satisfied [2,3]).

1. The initial system of equations consists of the nonlinear hydrodynamic equations of motion for beam particles and Maxwell's equations for a field in a medium with dielectric constant $\epsilon = n^2$:

$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}, \mathbf{H}_0 + \mathbf{H}]\right),$$

$$\frac{\partial N}{\partial t} + \operatorname{div} N\mathbf{v} = 0, \qquad \mathbf{j} \equiv eN\mathbf{v},$$

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{rot}\mathbf{H} = \frac{n^2}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}. \quad (1)^*$$

*$[\mathbf{v}, \mathbf{H}_0 + \mathbf{H}] \equiv \mathbf{v}_0 \times [\mathbf{H}_0 + \mathbf{H}]$; rot ≡ curl.

It is easy to show that this system of equations admits of a solution in the form

$$v_x + iv_y = ca(t)e^{i[\Phi+\vartheta(t)]},$$
$$E_x + iE_y = H_0\varepsilon(t)e^{i[\Phi+\psi(t)]}, \quad (2)$$

where $\Phi \equiv kz - \omega t$ and the z axis is directed along the external magnetic field $\mathbf{H}_0$.

In the general case, the system of equations describing the time dependence of the amplitudes and phases of the field and the beam particle velocities is very complex. However, it is greatly simplified for small beam densities when the increment of the excited oscillations is small. It is just this case which is examined below. The amplitude of the transverse beam velocity a(t), its longitudinal velocity $v_{\|} \equiv c\beta$, the phase ϑ, and the field amplitude ψ can be assumed here to vary slowly with time, while the field phase ψ is constant (henceforth we shall let $\psi \equiv 0$).

Substituting (2) into (1) and retaining only the leading terms with respect to beam density, we obtain the following system of equations for a, ϑ, β and ϵ [1)]:

$$\frac{da}{d\tau} = \varepsilon(\beta n - 1)\cos\vartheta, \qquad \frac{d\varepsilon}{d\tau} = \frac{1}{2}q^2 a\cos\vartheta,$$

$$\frac{d\beta}{d\tau} = -n\varepsilon a\cos\vartheta, \qquad \frac{d\vartheta}{d\tau} = 1 - \Omega(\beta n - 1) - \frac{\varepsilon}{a}(\beta n - 1)\sin\vartheta, \quad (3)$$

[1)] We should emphasize that for the space relationship chosen by us the latter is eliminated from (1), so that in deriving (3) from (1) it is not necessary to use the averaging method.

where we introduce the notation

$$\tau \equiv \omega_H t, \quad \omega_H \equiv \frac{|e|H_0}{mc}, \quad q^2 = \frac{4\pi N_0 mc^2}{n^2 H_0^2}, \quad \Omega \equiv \frac{\omega}{\omega_H},$$

and N_0 is the beam density. The condition that the beam density be small reduces here to the requirement $\gamma \ll \Omega$ ($\gamma \equiv q(2\Omega)^{-1/2}$ is the oscillation growth increment in units of ω_H). Terms containing the longitudinal electric field in the equation for β in system (3) are small when $\Omega \ll 1$, which is also assumed to be satisfied.

As is seen from system (3), the main source of its nonlinearity is the Lorentz force in the equation for the longitudinal motion of the beam particles. In the linear approximation, ignoring this term, we obtain for $\Omega(\beta_0 n - 1) = +1$ (the anomalous Doppler effect condition)

$$\beta - \beta_0 = \vartheta = \vartheta_0 = 0, \quad \ddot{a} - \gamma^2 a = 0,$$

i.e., an exponential growth of oscillation amplitude with time.

In the nonlinear approximation, as follows from (3), the growth of oscillations is accompanied by the deceleration of the beam, which in turn leads to an increase in phase and a loss of synchronism between the particles and the field. An examination of system (3) with allowance for this effect is considerably facilitated by the presence of the integrals of motion. In fact, from the first three equations of system (3) it is easy to obtain the following relationships, which represent the law of conservation of energy and momentum in the system:

$$n^2 a^2 + (\beta n - 1)^2 = R^2, \quad n^2 \varepsilon^2 + q^2 \beta = n^2 \varepsilon_0^2 + q^2 \beta_0.$$

Using these relationships, with the aid of the change of variables

$$na = R \sin\psi, \quad \beta n - 1 = R\cos\psi, \quad d\xi = \cos\vartheta\, d\tau$$

we get from (3)

$$\frac{d^2\psi}{d\xi^2} - \gamma^2 \sin\psi = 0, \qquad (4a)$$

$$\frac{dX}{d\xi} = 1 - \Omega R\cos\psi - \frac{d\psi}{d\xi}\operatorname{ctg}\psi X, \quad X \equiv \sin\vartheta. \qquad (4b)^*$$

A solution to Eq. (4b) with $X_0 = 0$ can be written in the following form:

$$X(\xi) = \frac{1}{\sin\psi}\int_0^{\xi}(1 - \Omega R\cos\psi)\sin\psi\, d\xi'. \qquad (5)$$

Substituting $\sin\psi$ from (4a) into the integrand of (5) we find the explicit form of X as a function of ψ:

$$X(\psi) = \frac{1}{\gamma^2 \sin\psi}\left\{1 - \Omega R\cos\psi)\frac{d\psi}{d\xi} - (1 - \Omega R\cos\psi_0)\dot{\psi}_0 + \frac{\Omega R}{3\gamma^2}\left[\left(\frac{d\psi}{d\xi}\right)^3 - \dot{\psi}_0^3\right]\right\}, \qquad (6)$$

where the dot indicates time differentiation τ (at $\tau = 0$, $d\psi/d\tau \equiv d\psi/d\xi$ since $\vartheta_0 = 0$).

Substituting the expression for $X(\psi)$ from (6) into the first integral in (4a):

$$\left(\frac{d\psi}{d\tau}\right)^2 = (1 - X^2)\{\dot{\psi}_0^2 + 2\gamma^2(\cos\psi_0 - \cos\psi)\}, \qquad (7)$$

we obtain an equation which is first order in ψ, from which it is clear that $\psi(\tau)$ is periodic.

The most important quantities—the maximum field oscillation amplitudes and beam particle velocities—can be found knowing the turning point of the system: $d\psi/d\tau = 0$. In the simplest particular case when the initial conditions have the form $\psi_0 = 0$ and $\dot{\psi} \neq 0$, the turning points of the system are determined by the equation $X(\psi_m) = 1$, from which we obtain, using the small parameter $\gamma \ll 1$,

$$a_{max} = \beta_0(6\gamma)^{1/2}, \quad \dot{\psi}_0 \ll \gamma^{3/2} \ll 1;$$

$$a_{max} = 2\beta_0(\dot{\psi}_0)^{1/3}, \quad \gamma^{3/2} \ll \dot{\psi}_0 \ll 1. \qquad (8)$$

It is clear from these relationships that, for small perturbation amplitudes, the maximum amplitude of the transverse velocity is independent of the initial conditions, while the amplitude variation period has an order of magnitude $T \sim 1/\gamma$. The maximum energy consumed in the transverse oscillations of the beam particles and excitation of the field is proportional to the increment γ (cf. [4]).

The second limiting case corresponds to the approximation of the specified external field, when the polarization fields created by the beam are small in comparison with the external field. In both cases, the restriction on the amplitude is due to the loss of synchronism (resonance) between the oscillators and field. Since $\dot{\vartheta}(|X| = 1) \neq 0$, this nonlinear effect leads, according to (3)—(6), to a periodic alternation of the processes of buildup and absorption of the field by the beam.

In conclusion, we should note that the calculation given above is a good model of the problem of exciting magnetohydrodynamic waves in a dense plasma ($\omega_p^2 \gg \omega_{He}\omega_{Hi}$) by an electron beam, since in this case all of the foregoing requirements ($\beta_0^2 n^2 = \beta_0^2\omega_p^2/\omega_{He}\omega_{Hi} \gg 1$, $\omega \ll \omega_{Hi} \ll \omega_{He}$ etc.) can easily be satisfied.

2. It was shown above that in our case the oscillation process continues until synchronism between oscillators in the beam and the field is disturbed. It is of interest to investigate the possibility of disturbing this synchronism with the aid

*ctg ≡ cot.

of an external field so as to suppress two-stream instabilities.

The possibility of suppressing two-stream instabilities by modulation with external fields and by phase stabilization in the external fields was indicated by Faĭnberg[5] and experimentally observed by Berezin et al.[6] The reason for the suppression of the instabilities in this case is that under our conditions it is the oscillators and not the free particles that interact with the growing field of the instability. Therefore, unless there is resonance between the external field frequency and the natural oscillation frequency, the effectiveness of the interaction of the beam with the field of the developing instability, and the growth increments, should markedly decrease.[6] The possibility of attenuating two-stream instabilities (for Cerenkov instability in longitudinal waves) in strong external fields was demonstrated and examined in detail by Aliev and Silin[7].

We shall consider below the suppression of instability for transverse waves excited by a beam of electrons in the presence of the anomalous Doppler effect. We shall assume that the external field is directed along the beam and that its amplitude is so high that longitudinally polarization fields can be ignored. In this case longitudinal motion of beam particles is completely determined by the external field $E_{\|} = \epsilon_{\|} \cos \Omega_0 \tau$. The equation for the transverse electric field then takes the form:

$$\frac{d^2\varepsilon}{d\xi^2} - \gamma^2[1 - h\sin\Omega_0\tau(\xi)]\,\varepsilon = 0,$$

$$\xi(\tau) = \tau\cos\alpha + 2\sum_{k=1}^{\infty}\cos\left(\alpha + \frac{\pi k}{2}\right) J_k(\alpha)\frac{\sin k\Omega_0\tau}{k\Omega_0},$$

$$\alpha \equiv n\varepsilon_{\|}/\Omega_0^2, \quad h \equiv \alpha\Omega_0. \qquad (9)$$

It is easy to see that $\xi \approx \tau \cos \alpha$ for $\tau \gg 1$.

Let us investigate (9) for two limiting cases.

A. Near the instability threshold when $h \gg 1$, the first term in the square brackets of (9) can be ignored. The resultant equation has no increasing solutions when $\gamma^2 h < 1$ (when $\gamma^2 h > 1$, parametric instabilities are possible). Thus, a sufficiently strong external field disrupts the instability under consideration near its threshold.

B. When $h \ll 1$ (far from the threshold) instability is maintained; however, its increment decreases with increasing external field amplitude, $\gamma_{\text{eff}}^{\|} = \gamma \cos \alpha$.

It is easy to show that a similar result can also be obtained by modulating the external magnetic field. In this case, the quantity h is taken as the modulation coefficient in formula (9). When the beam is modulated by an external transverse wave with frequency Ω_0 and amplitude $n\epsilon_0 \gg \gamma^{3/2}$, the growth increment of the wave with frequency Ω has far from the threshold the order of magnitude

$$\gamma_{\text{eff}}^{\perp} = \gamma\cos\Omega/\Omega_0; \quad \Omega(1 - \beta_0 n) + 1 = -(\Omega/\Omega_0)(n\varepsilon_0)^{2/3},$$

$$(n\varepsilon_0)^{2/3} \ll 1, \quad \Omega_0 \lesssim \Omega \ll 1.$$

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[1] A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, UFN **73**, 701 (1961), Soviet Phys. Uspekhi **4**, 332 (1961). V. D. Shapiro, Izv. Vuzov, Radiofizika **4**, No. 5 (1961).

[2] V. V. Zheleznyakov, Izv. vuzov, Radiofizika **2**, 14 (1959).

[3] K. N. Stepanov and A. V. Kitsenko, ZhTF **31**, 167 (1961), Soviet Phys. Tech. Phys. **6**, 120 (1961).

[4] V. D. Shapiro and V. I. Shevchenko, JETP **42**, 1515 (1962), Soviet Phys. JETP **15**, 1053 (1962).

[5] Ya. B. Faĭnberg, Atomnaya énergiya 11, 313, 1961.

[6] A. K. Berezin et al., Col. Mezhdynarodnaya konferentsiya po uskoritelyam (International Conference on Accelerators), Dubna, 1963, Atomizdat, 1964, page 1023.

[7] Yu. M. Aliev and V. P. Silin, JETP **48**, 901 (1965), Soviet Phys. JETP **21**, 601 (1965).

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