VECTOR MESON IN A COULOMB FIELD

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The motion of bosons of spin 0 and 1 in a Coulomb field is treated on the basis of the 16-row Kemmer representation. The integrals of the motion which completely classify the states are determined. The radial and the angular parts in the initial equation can be separated by employing the integrals of the motion and a set of projection operators.

INTRODUCTION

THE Kepler problem for a vector particle was discussed in papers by a number of authors[1-5]. However, it cannot be said that this problem is solved with the same simplicity and elegance as in the case of a spinor particle[5]. In particular, Gunn's paper[3] is devoted to the investigation of the Kepler problem for the vector meson by means of the Tamm-Sakata-Taketani matrix formalism[1]. Utilizing the explicit representation of the spin matrices the author seeks the solution in the form of an expansion in terms of spin-angle functions. For the radial parts a system of four second-order equations is obtained which is then reduced to a system of two second-order equation. An asymptotic investigation of the solutions of these equations was carried out by Case[4].

In our opinion a discussion of the Kepler problem can be carried out considerably more simply in the Kemmer representation. Here there exists a well developed theory of the projection operators[6] which elucidates the structure of the wave function.

1. MATRICES, THE WAVE EQUATION, INTEGRALS OF THE MOTION

Two commuting groups of 16-row Dirac matrices γ_μ and $\bar{\gamma}_\mu$

$$\{\gamma_\mu\gamma_\nu\} - 2\delta_{\mu\nu} = \{\bar{\gamma}_\mu\bar{\gamma}_\nu\} - 2\delta_{\mu\nu} = [\gamma_\mu\bar{\gamma}_\nu] = 0 \tag{1}$$

provide the possibility of constructing two "orthogonal" groups of Kemmer matrices $\beta_\mu^{(\pm)}$

$$\beta_\mu^{(\pm)} = \tfrac{1}{2}(\gamma_\mu \pm \bar{\gamma}_\mu)$$

with the properties

$$\beta_\mu^{(\pm)}\beta_\nu^{(\pm)}\beta_\lambda^{(\pm)} + \beta_\lambda^{(\pm)}\beta_\nu^{(\pm)}\beta_\mu^{(\pm)} = \delta_{\mu\nu}\beta_\lambda^{(\pm)} + \delta_{\nu\lambda}\beta_\mu^{(\pm)},$$

$$\beta_\mu^{(\pm)}\beta_\nu^{(\pm)} + \beta_\nu^{(\mp)}\beta_\mu^{(\mp)} - \delta_{\mu\nu} = \beta_\mu^{(\pm)}\beta_\nu^{(\mp)} + \beta_\nu^{(\pm)}\beta_\mu^{(\mp)} = 0. \tag{2}$$

The matrices $\beta_\mu^{(\pm)}$ are 16-row reducible Kemmer representations which can be decomposed into irreducible ones—a 10-row, a 5-row and a trivial one-row representation. These representations for $\beta_\mu^{(+)}$ can be explicitly picked out by means of the operator

$$\mathscr{P}(10\times 10) = (M^2-16)(M+2)(M-6)/192,$$
$$\mathscr{P}(5\times 5) = M(M+4)(M-2)(M-6)/128,$$
$$\mathscr{P}(1\times 1) = M(M^2-4)(M-4)/384,$$

$$M = \sum_{\mu=1}^{4} R_\mu,\quad R_\mu = \pm(2\beta_\mu^{(\pm)2} - I)\,(\text{without summation}). \tag{3}$$

Thus, the Kemmer equation

$$(i\beta_\mu P_\mu + m)\psi = 0, \qquad P_\mu = p_\mu - eA_\mu, \tag{4}$$

($p_\mu = -i\partial/\partial x_\mu$, A_μ is the potential of the external electromagnetic field) describes the interaction of a pseudoscalar particle—the (5×5) representation—and of a vector particle—the (10×10) representation—with the electromagnetic field.

The wave function ψ contains the noncanonical part $\psi_L = \beta_4^{(-)2}\psi$ which is related to the canonical part $\psi_T = \beta_4^{(+)2}\psi$ $(\beta_4^{(+)2} + \beta_4^{(-)2} = I)$ by the relation

$$\psi_L = (i/m)(\boldsymbol{\beta}^{(+)}, \mathbf{P})\psi_T.$$

Eliminating ψ_L from (4) we obtain

$$i\partial\psi_T/\partial t = H\psi_T, \tag{5}$$

where

$$H = m^{-1}\beta_4^{(+)}((\boldsymbol{\beta}^{(+)}, \mathbf{P})^2 + m^2 + me\beta_4^{(+)}A_0). \tag{6}$$

Equation (5) coincides with the Tamm-Sakata-Taketani equation[4].

The case of spin 0. Separating from (5) by the operator $\mathscr{P}(5\times 5)$ the 5-row representation and utilizing the relations

$$\mathscr{P}(5\times 5)\beta_4^{(+)2}\beta_i^{(\pm)}\beta_k^{(\pm)} = \tfrac{1}{2}(I \pm R_0)\delta_{ik}\mathscr{P}(5\times 5)\beta_4^{(+)2}, \tag{7}$$

where $R_0 = R_1R_2R_3$, $R_0^2 = I$, we obtain for the stationary states

$$(1/2(I+R_0)\mathbf{p}^2 + m^2 - m\beta_4^{(+)}f)\Phi = 0,$$
$$\Phi = \mathcal{P}(5\times 5)\psi_T, \qquad f = E - eA_0. \tag{8}$$

Equation (8) is easily reduced to the Klein-Gordon equation for the function $\Phi_1 = 1/2\,(I+R_0)\Phi$:

$$(\mathbf{p}^2 + m^2 - f^2)\Phi_1 = 0,$$
$$\Phi_2 = 1/2(I-R_0)\Phi = m^{-1}\beta_4^{(+)}f\Phi_1. \tag{9}$$

The solutions of (9) for a Coulomb field are investigated in [7].

The case of spin 1. For a vector particle utilizing $\mathcal{P}(10\times 10)$ we obtain

$$i\partial(\mathcal{P}_0\psi)/\partial t = m^{-1}\beta_4^{(+)}((\beta^{(+)}, \mathbf{P})^2 + m^2 + me\beta_4^{(+)}A_0)\mathcal{P}_0\psi, \tag{10}$$

where

$$\mathcal{P}_0 = \mathcal{P}_0^2 = \mathcal{P}(10\times 10)\beta_4^{(+)2} = 1/4(3 - R_0(M - R_4)). \tag{11}$$

The operator $\mathcal{P}_0$ provides for the wave function ψ only the canonical components associated with spin 1.

For the stationary states in a Coulomb field ($\mathbf{A} = 0$, $A_0 = Ze/x$), Eq. (10) reduces to the form

$$((\beta^{(+)}, \mathbf{p})^2 + m^2 - m\beta_4^{(+)}f)\mathcal{P}_0\psi = 0. \tag{12}$$

Compatible integrals of motion will be

$$\mathbf{J}^2,\quad J_z,\quad \Lambda = K(K+I);$$
$$\mathbf{J} = (\mathbf{L}+\mathbf{S})\mathcal{P}_0, \qquad K = (\mathbf{SL}+I)\mathcal{P}_0,$$
$$\mathbf{L} = [\mathbf{xp}], \qquad \mathbf{S} = -i[\beta^{(\pm)}, \beta^{(\pm)}]. \tag{14*}$$

The wave function $\mathcal{P}_0\psi$ from (12) can be chosen as the eigenfunction of the integrals (13) with the eigenvalues $j(j+1)$, m, and λ respectively.

In order to find the eigenvalues of the operator K we utilize the identity

$$K(K(K+I) - \mathbf{J}^2) = 0,$$

from which it follows that $k = j,\ -j-1,\ 0$. The last value can occur for any $j \neq 0$, and this is related to the triangle relation for angular momenta; $\lambda = j(j+1)$ or zero (for arbitrary $j \neq 0$).

2. CLASSIFICATION OF STATES

We examine in greater detail the properties of the spin-angle functions. The spin functions are defined by the equations

$$\mathbf{S}^2\chi_{s\mu} = s(s+1)\chi_{s\mu}, \qquad S_z\chi_{s\mu} = \mu\chi_{s\mu} \tag{15}$$

for a given s, $-s \leq \mu \leq +s$. In the 10-row representation the spin functions contain three times the irreducible representation of the rotation group D_1 and one representation D_0. The function $\mathcal{P}_0\psi$ contains only two representations D_1 which are separated by the operators $1/2\,(I+R_0)$. We shall henceforth be interested only in these irreducible representations

$$\chi_{1\mu}^{(\pm)} = 1/2(I \pm R_0)\mathcal{P}_0\chi_{1\mu}. \tag{16}$$

The spin-angle functions have the form (to reduce the amount of writing we omit the indices m and 1)

$$\Psi_{lj}^{(\pm)} \equiv \Psi_{l1jm}^{(\pm)} = \sum_\mu (l1m-\mu\mu\,|\,jm)\,Y_{lm-\mu}\chi_{1\mu}^{(\pm)}. \tag{17}$$

The column matrices $\psi_{lj}^{(\pm)}$ have only three nonvanishing components in the region defined by the operators $1/2\,(I+R_0)\mathcal{P}_0$. From the relation for the angular momenta [6]

$$\mathbf{J}^2 = (\mathbf{L}^2 + 2K)\mathcal{P}_0 \tag{18}$$

it follows that the eigenvalues are $k = 1/2(j(j+1) - l(l+1))$. The eigenvalues λ and l are equal to:

$$\lambda = \begin{cases} 0, & l = j \neq 0, \\ j(j+1), & l = j \pm 1. \end{cases} \tag{19}$$

Thus, for $\lambda = 0$ and for fixed j and m there exists only one function $\Psi_{jj}^{(\pm)}$, while for $\lambda = j(j+1)$ there exist two functions: $\Psi_{j-1,j}^{(\pm)}$ and $\Psi_{j+1,j}^{(\pm)}$.

We examine the properties of these functions. From the relations

$$(K + 1/2(I+R_0))\beta_x^{(+)2}\mathcal{P}_0 = \beta_x^{(-)2}(K + 1/2(I-R_0))\mathcal{P}_0, \tag{20}$$

$$(K + 1/2(I-R_0))\beta_x^{(-)2}\mathcal{P}_0 = \beta_x^{(+)2}(K + 1/2(I+R_0))\mathcal{P}_0 \tag{21}$$

we obtain

$$[\Lambda, \beta_x^{(\pm)2}] = 0. \tag{22}$$

Moreover, since $\beta_x^{(\pm)2}$ commute with $\mathbf{J}^2$ and J_z, together with $\Psi_{lj}^{(\pm)}$ the functions $\beta_x^{(+)2}\Psi_{lj}^{(\pm)}$ and $\beta_x^{(-)2}\Psi_{lj}^{(\pm)}$ are also eigenfunctions of the operators $\mathbf{J}^2$, J_z and Λ with the same eigenvalues j, m and λ.

a) The case $\lambda = 0$:

$$\beta_x^{(\pm)2}\Psi_{jj}^{(+)} = a^{(\pm)}\Psi_{jj}^{(+)}. \tag{23}$$

Taking (20) into account we find that $a^{(+)} = 0$, $a^{(-)} = 1$. In a similar manner one can also obtain relations for $\Psi_{jj}^{(-)}$. Thus, we have

$$\beta_x^{(+)2}\Psi_{jj}^{(+)} = \beta_x^{(-)2}\Psi_{jj}^{(-)} = 0, \qquad \beta_x^{(\pm)2}\Psi_{jj}^{(\mp)} = \Psi_{jj}^{(\mp)}. \tag{24}$$

*$[\mathbf{xp}] = \mathbf{x}\times\mathbf{p}$.

As noted previously, the functions $\Psi_{lj}^{(\pm)}$ represent three-dimensional vectors. The geometrical meaning of the operators $(\beta^{(+)}\mathbf{n})^2$ and $(\beta^{(-)}\mathbf{n})^2$ ($\mathbf{n}$ is a classical unit vector) is that they separate from Ψ respectively the parts which are transverse and longitudinal with respect to the vector $\mathbf{n}$[6]; from equations (24) it follows that $\Psi_{jj}^{(+)}$ contains only the vector parallel to $\mathbf{x}$, while $\Psi_{jj}^{(-)}$ contains only the vector perpendicular to $\mathbf{x}$.

b) The case $\lambda = j(j+1)$. In this case we can set

$$\beta_x^{(+)2}\Psi_{j-1,j}^{(+)} = a\Psi_{j-1,j}^{(+)} + b\Psi_{j+1,j}^{(+)},$$

$$\beta_x^{(+)2}\Psi_{j+1,j}^{(+)} = c\Psi_{j-1,j}^{(+)} + d\Psi_{j+1,j}^{(+)}. \quad (25)$$

Taking into account (20) and the fact that $(\beta_x^{(+)2})^2 = \beta_x^{(+)2}$ we obtain

$$a = j/(2j+1),$$
$$d = (j+1)/(2j+1), \qquad ad - bc = 0.$$

The last relation shows that the functions $\beta_x^{(+)2}\Psi_{j-1,j}^{(+)}$ and $\beta_x^{(+)2}\Psi_{j+1,j}^{(+)}$ are linearly independent. The phases of $\Psi_{j-1,j}^{(+)}$ and $\Psi_{j+1,j}^{(+)}$ can be so chosen as to have

$$b = c = (2j+1)^{-1}[j(j+1)]^{1/2}.$$

Thus, we obtain

$$\beta_x^{(+)2}\Psi_{j-1,j}^{(+)} = [j/(j+1)]^{1/2}\beta_x^{(+)2}\Psi_{j+1,j}^{(+)} \quad (26)$$
$$= [j/(2j+1)]\Psi_{j-1,j}^{(+)} + (2j+1)^{-1}[j(j+1)]^{1/2}\Psi_{j+1,j}^{(+)},$$
$$\beta_x^{(-)2}\Psi_{j-1,j}^{(+)} = -[(j+1)/j]^{1/2}\beta_x^{(-)2}\Psi_{j+1,j}^{(+)}$$
$$= [(j+1)/(2j+1)]\Psi_{j-1,j}^{(+)}$$
$$-(2j+1)^{-1}[j(j+1)]^{1/2}\Psi_{j+1,j}^{(+)}.$$

We state the analogous relations for $\Psi_{lj}^{(-)}$.

$$\beta_x^{(+)2}\Psi_{j-1,j}^{(-)} = [(j+1)/j]^{1/2}\beta_x^{(+)2}\Psi_{j+1,j}^{(-)}$$
$$= [(j+1)/(2j+1)]\Psi_{j-1,j}^{(-)}$$
$$+(2j+1)^{-1}[j(j+1)]^{1/2}\Psi_{j+1,j}^{(-)},$$
$$\beta_x^{(-)2}\Psi_{j-1,j}^{(-)} = -[j/(j+1)]^{1/2}\beta_x^{(-)2}\Psi_{j+1,j}^{(-)}$$
$$= [j/(2j+1)]\Psi_{j-1,j}^{(-)} - (2j+1)^{-1}[j(j+1)]^{1/2}\Psi_{j+1,j}^{(-)}. \quad (27)$$

3. SEPARATION OF THE VARIABLES IN THE WAVE EQUATION

a) The case $\lambda = 0$. We apply to (12) the operator

$$(D_+ + D_-)\mathscr{P}_0 = \mathbf{J}^2 - \Lambda, \quad (28)$$

where

$$D_\pm = (\beta^{(\pm)}\mathbf{x})(\beta^{(\mp)}\mathbf{p})(\beta^{(\mp)}\mathbf{x})(\beta^{(\pm)}\mathbf{p}); \quad (29)$$

taking into account the orthogonality of the matrices $\beta^{(+)}$ and $\beta^{(-)}$ and the relations

$$(\mathbf{J}^2 - \Lambda)\mathscr{P}_0\psi = j(j+1)\mathscr{P}_0\psi,$$
$$\mathscr{P}_0 D_\pm = D_\pm\mathscr{P}_0 = {}^1/_2(I \pm R_0)D_\pm\mathscr{P}_0, \quad (30)$$

we obtain

$$({}^1/_2(I - R_0)\mathbf{p}^2 + m^2 - m\beta_4^{(+)}f)\mathscr{P}_0\psi = 0. \quad (31)$$

Eliminating the matrix R_0 in exactly the same manner as in (8) we obtain

$$\Phi_1 = m^{-1}f\beta_4^{(+)}\Phi_2, \qquad (\mathbf{p}^2 - f^2 + m^2)\Phi_2 = 0, \quad (32)$$

where

$$\Phi_{1,2} = {}^1/_2(I \pm R_0)\mathscr{P}_0\psi;$$

the complete solution for $\lambda = 0$ will be given by

$$\mathscr{P}_0\psi = (I + m^{-1}f\beta_4^{(+)})R_j(x)\Psi_{jj}^{(-)}. \quad (33)$$

Thus, the vector meson in the state with $\lambda = 0$ (the spin-orbit interaction is absent) in a Coulomb field behaves in almost the same manner as a scalar particle with $l = j$, but which has no s-state.

b) The case $\lambda = j(j+1)$. In this case the spin-orbit interaction is very strong, and states with parallel and anti-parallel combinations of angular momenta are mixed. As has been pointed out earlier, the functions $\beta_x^{(\pm)2}\Psi_{j-1,j}$ and $\beta_x^{(\pm)2}\Psi_{j+1,j}$ differ only by a numerical factor, and one can hope that (12) can be simplified by the separation of the wave function by the operators $\beta_x^{(\pm)2}$. This leads us to the system

$$((-i\mathbf{xp} - (3 + R_0)/2)(K + {}^1/_2(I + R_0)) - m\beta_4^{(+)}fx^2)\psi_1$$
$$+ (m^2x^2 + j(j+1))\psi_2 = 0,$$
$$((\mathbf{xp})^2 - i\mathbf{xp} + I + R_0 + m^2x^2)\psi_1 + ((i\mathbf{xp} - (I + R_0)/2)(K$$
$$+ {}^1/_2(I - R_0)) - m\beta_4^{(+)}fx^2)\psi_2 = 0,$$
$$\psi_{1,2} = \beta_x^{(\pm)2}\mathscr{P}_0\psi. \quad (34)$$

Eliminating ψ_2 from this system we obtain the equation

$$(D_j + R_0F_j + \varphi_{1j}(K + {}^1/_2(I - R_0))\beta_4^{(+)})\psi_1 = 0, \quad (35)$$

$$D_j = (\mathbf{xp})^2 - i\mathbf{xp} + I + j(j+1) + (m^2 - f^2)x^2$$
$$-2(m^2x^2 + j(j+1))^{-1}j(j+1)(i\mathbf{xp} + {}^3/_2),$$
$$F_j = (m^2x^2 + j(j+1))^{-1}m^2x^2,$$
$$\varphi_{1j} = m^{-1}f'x - 2(m^2x^2 + j(j+1))^{-1}mfx^2. \quad (36)$$

By means of the projection operators ${}^1/_2(I \pm R_0)$

Eq. (35) can again be reduced to the system ($\psi^{(\pm)} = \frac{1}{2}(I \pm R_0)\psi$):

$$(D_j + F_j)\psi_1^{(+)} + \varphi_{1j} K\beta_4^{(+)}\psi_1^{(-)} = 0,$$

$$(D_j - F_j)\psi_1^{(-)} + \varphi_{1j}(K + I)\beta_4^{(+)}\psi_1^{(+)} = 0. \qquad (37)$$

For the state with j = 0, taking into account (26) and (27), we obtain $\psi_1^{(-)} = 0$, while for $\psi_1^{(+)}$ we obtain the equation

$$\left(\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial}{\partial x}\right) - \frac{2}{x} + f^2 - m^2\right)\psi_1^{(+)} = 0,$$

$$\psi_1^{(+)} = \Phi(x)\Psi_{10}^{(+)}; \qquad (38)$$

this is the Klein-Gordon equation with $l = 1$.

For $j \neq 0$ the system (37) can be put into a more symmetric form by introducing the new functions $\Phi_1 = [j(j+1)]^{1/2}\psi_1^{(+)}$ and $\Phi_2 = K\beta_4^{(+)}\psi_1^{(-)}$:

$$(D_j + F_j)\Phi_1 + \varphi_j\Phi_2 = 0, \qquad (D_j - F_j)\Phi_2 + \varphi_j\Phi_1 = 0,$$

$$\varphi_j = [j(j+1)]^{1/2}\varphi_{1j}. \qquad (39)$$

Here the angle and the radial parts are separated:

$$\Phi_{1,2} = R_{1,2}[(2j+1)/j]^{1/2}\beta_x^{(+)2}\Psi_{j-1,\,j},$$

while the factor $[(2j+1)/j]^{1/2}$ has been introduced for normalization. The equations for the radial functions have the form

$$(D_j + F_j)R_1 + \varphi_j R_2 = 0, \quad (D_j - F_j)R_2 + \varphi_j R_1 = 0. \quad (40)$$

We determine the form of the radial functions near the origin. For $x \to 0$ ($G = x(R_2 - R_1)$, $H = x(R_2 - R_1)$):

$$G \to x^{-1/4}\exp(\pm\alpha x^{-1/2}), \quad H \to Cx^{-1/4}x^3\exp(\pm\alpha x^{-1/2}), \qquad (41)$$

$$H \to x^{-1/4}\exp(\pm i\alpha x^{-1/2}), \qquad G \to Cx^{-1/4}x^3\exp(\pm i\alpha x^{-1/2}),$$

$$\alpha = 2(Ze^2/m)^{1/2}(j(j+1))^{1/4}. \qquad (41a)$$

The functions (41) with $+\alpha$ are not suitable in view of their strong divergence at zero; the functions (41) with $-\alpha$ and (41a) are quadratically integrable at the origin. The solution is a linear combination of three particular solutions. The requirement that functions belonging to different values of the energy should be orthogonal removes one constant[4]. It is possible that the condition that the wave function should vanish at infinity will remove the last constant.

Thus, the number of boundary conditions is greater than in the analogous problem for a spinor particle.

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