MAGNITUDE OF THE CRITICAL JOSEPHSON TUNNEL CURRENT

I. O. KULIK

Physico-technical Institute of Low Temperatures, Academy of Sciences, Ukrainian S.S.R.

Submitted to JETP editor April 26, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **49**, 1211-1214 (October, 1965)

It is shown that the maximal magnitude of the superconducting tunnel current for zero potential difference is always less than the current in the normal state for a bias equal to $\pi\Delta/2$ (without taking into account effects associated with limitation of the Josephson current by its own magnetic field).

RECENT investigations have confirmed Josephson's prediction[1] about the existence of an undamped current at zero potential difference between two superconductors separated by a thin dielectric barrier. From the simple theory of this phenomenon developed in, besides in the article of Josephson, also in the articles by Anderson[2] and by Ambegaokar and Baratoff,[3] it follows that the magnitude of the critical Josephson current is given by (at T = 0)

$$J_S = \frac{\pi}{2}\Delta\frac{1}{R_{NN}}, \tag{1}$$

where R_{NN} is the resistance of the tunnel junction in the normal state and Δ is the gap. This formula pertains to the case of superconductors with identical gaps: $\Delta_1 = \Delta_2 = \Delta$. A similar dependence also exists for $\Delta_1 \neq \Delta_2$ (see[2,3]).

In all cases experiments on superconducting tunneling[4-7] give for the maximal Josephson current a value J_{max} which is smaller than J_S. The present note is devoted to an analysis of some of the reasons why this may occur. As will be shown below, a more exact theory leads to the inequality $J_{max} < J_S$. Here we assume that the transverse dimensions of the tunnel junction are small enough that one can neglect the effects of finite "depth of penetration" of the superconducting tunnel current, considered by Ferrel and Prange.[8]

The authors of[2,3] start from a tunneling transition Hamiltonian of the form

$$H_T = \sum_{kq} T_{kq} a_{k\alpha}^+ a_{q\alpha} + \text{h.c.}, \tag{2}$$

where $a_{k\alpha}^+$ and $a_{k\alpha}$ are creation and annihilation operators for an electron with momentum **k** and spin projection α ($\alpha = \pm 1/2$) in the left metal; $a_{q\alpha}^+$ and $a_{q\alpha}$ are the corresponding operators in the right metal (we shall use the same symbols for the operators which create electrons in the left and right metals, distinguishing them only by the indices **k** and **q** for the momentum, which obviously cannot lead to any misunderstanding).

The Hamiltonian (2) describes tunneling transitions through the barrier with conservation of spin. In the present article, we shall start from a Hamiltonian of more general form:

$$H_T = V_n + V_s, \tag{3}$$

$$V_n = \sum_{kq} T_{kq}{}^n a_{k\alpha}{}^+ a_{q\alpha} + \text{h.c.}, \tag{4}$$

$$V_s = \sum_{kq} T_{kq}{}^s a_{k\alpha}{}^+ a_{-q\bar{\alpha}} + \text{h.c.} \tag{5}$$

where V_n describes transitions without change of spin, V_s describes transitions accompanied by a change of the electron's spin projection; T^n_{kq} and T^s_{kq} are the corresponding matrix elements of the transition.

Processes described by the Hamiltonian V_s may occur with appreciable probability in the presence of, for example, paramagnetic inclusions (impurities) in the dielectric layer of the tunnel junction. In actual tunneling experiments, in which a thin (~ 10 Å) layer of the oxide of one of the metals plays the role of the dielectric, atoms (molecules) of oxygen, not entering into the reaction, may play the role of such paramagnetic impurities, or molecules of the oxide themselves if they are paramagnetic. In addition, even if such magnetic centers are not present, spin reversal during the transition may be realized as a result of spin-orbit coupling.

Since the detailed structure of the tunneling barrier is unknown, and H_T is only a phenomenological Hamiltonian, it is of interest to consider a similar Hamiltonian of the more general form (3).

It is easy to verify that the relations

$$T^{n*}_{\mathbf{kq}} = T^n_{-\mathbf{k},-\mathbf{q}}, \quad T^{s*}_{\mathbf{kq}} = T^s_{-\mathbf{k},-\mathbf{q}}. \tag{6}$$

follow from the requirement of invariance of the tunneling Hamiltonian (3) under time reversal (see, for example,[9]).

Using expressions (6), V_n and V_S may be rewritten in the form (compare with[2])

$$V_n = \sum_{kq} T_{kq}{}^n (a_{k\uparrow}^+ a_{q\uparrow} + a_{-q\downarrow}^+ a_{-k\downarrow}) + \text{h.c.}\,, \qquad (7)$$

$$V_s = \sum_{kq} T_{kq}{}^s (a_{k\uparrow}^+ a_{-q\downarrow} + a_{q\uparrow}{}^+ a_{-k\downarrow}) + \text{h.c.} \qquad (8)$$

Going on to the calculation of the Josephson effect, we find the correction ΔE to the energy of two superconductors caused by the interaction $H_T = V_n + V_S$. Repeating the calculation of Anderson,[2] we obtain the following expressions in second-order perturbation theory (for $T = 0$):

$$\Delta E = \Delta E_n + \Delta E_s; \qquad (9)$$

$$\Delta E_n = -2 \sum_{kq} |T_{kq}{}^n|^2 \frac{|u_k v_q + u_q v_k|^2}{\varepsilon_k + \varepsilon_q}, \qquad (10)$$

$$\Delta E_s = -2 \sum_{kq} |T_{kq}{}^s|^2 \frac{|u_k v_q - u_q v_k|^2}{\varepsilon_k + \varepsilon_q}, \qquad (11)$$

where u_k, v_k and u_q, v_q are the parameters of Bogolyubov's canonical transformation for the left and right superconductors, and ϵ_k and ϵ_q are the corresponding energies of the quasi-particles:

$$\varepsilon_k = (|\Delta_k|^2 + \xi_k{}^2)^{1/2}, \quad \varepsilon_q = (|\Delta_q|^2 + \xi_q{}^2)^{1/2}.$$

Here one should regard u_k, v_k, u_q, v_q and the gaps Δ_k, Δ_q as complex quantities, proportional to certain phase factors $e^{i\varphi_1}$ and $e^{i\varphi_2}$ (for the left and right metals, respectively).

On the basis of Eqs. (10) and (11), the term in the total energy depending on the relative phase difference $\varphi_1 - \varphi_2$ has the form

$$\Delta \tilde{E} = -2 \sum_{kq} (|T_{kq}{}^n|^2 - |T_{kq}{}^s|^2)\, 2\,\mathrm{Re}\, \frac{u_k{}^* v_k \cdot u_q v_q{}^*}{\varepsilon_k + \varepsilon_q}. \qquad (12)$$

Using the relations

$$2u_k v_k{}^* = \frac{|\Delta_k|}{\varepsilon_k} e^{i\varphi_1}, \quad 2u_q v_q{}^* = \frac{|\Delta_q|}{\varepsilon_q} e^{i\varphi_2} \qquad (13)$$

and regarding $|\Delta_k|$, $|\Delta_q|$ and T_{kq}^n, T_{kq}^s as weakly energy dependent in the interval $\Delta\epsilon \sim \Delta$ in which the integral (12) diverges, we obtain

$$\Delta \tilde{E} = -N_1(0) N_2(0) \{\langle |T_n|^2\rangle - \langle |T_s|^2\rangle\}$$

$$\times \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \frac{\Delta_1 \Delta_2}{\varepsilon_1 \varepsilon_2 (\varepsilon_1 + \varepsilon_2)} \cos(\varphi_1 - \varphi_2), \qquad (14)$$

where $N_1(0)$ and $N_2(0)$ are the densities of electron states in the left and right metals, and $\langle |T|^2\rangle$ is the average of $|T_{kq}|^2$ with respect to the angles between the vectors **k** and **q** on the Fermi surface.

The term in the energy depending on the relative phase difference $\varphi_1 - \varphi_2$ corresponds to a current given by[2]

$$J = \frac{2e}{\hbar} N_1(0) N_2(0) \{\langle |T_n|^2\rangle - \langle |T_s|^2\rangle\}$$

$$\times \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \frac{\Delta_1 \Delta_2}{\varepsilon_1 \varepsilon_2 (\varepsilon_1 + \varepsilon_2)} \sin(\varphi_1 - \varphi_2). \qquad (15)$$

We note that exactly the same result is obtained in the technique of Ambegaokar and Baratoff,[3] in which the tunnel current (but not ΔE) is calculated directly.

In order to obtain a formula analogous to (1), it is necessary to compare (15) with the current J_{NN} in the normal state. In the present case the magnitude of this current is given by the expression

$$J_{NN} = \frac{4\pi e}{\hbar} N_1(0) N_2(0) \{\langle |T_n|^2\rangle + \langle |T_s|^2\rangle\}\, V, \qquad (16)$$

where V is the applied bias. The factor associated with V in this formula represents the reciprocal of the junction's resistance R_{NN} in the normal state.

Comparing (15) with (16), we obtain the following expression[1)] for the superconducting tunnel current

$$J = \frac{\pi}{2} \Delta R_{NN}{}^{-1} \frac{\langle |T_n|^2\rangle - \langle |T_s|^2\rangle}{\langle |T_n|^2\rangle + \langle |T_s|^2\rangle} \sin(\varphi_1 - \varphi_2). \qquad (17)$$

The maximal current occurs in the case when $|\sin(\varphi_1 - \varphi_2)| = 1$. Comparing expression (17) with formula (1), we obtain

$$\frac{J_{max}}{J_S} = \left| \frac{\langle |T_n|^2\rangle - \langle |T_s|^2\rangle}{\langle |T_n|^2\rangle + \langle |T_s|^2\rangle} \right| \leqslant 1. \qquad (18)$$

The formula obtained proves the assertion stated at the beginning of this article, according to which the maximal value J_{max} of the critical Josephson current is always smaller than the value of J_S given by formula (1).

In conclusion, we note that relation (1) may also be broken in connection with the use of a Hamiltonian of the form (2) in which, however, no assumption is made that $T_{kq}^* = T_{-k,\,-q}$. In this case, as one can easily show, the magnitude of the critical Josephson current also turns out to always be less than J_S. Such a Hamiltonian may possibly be applied to the description of tunnel transitions in the absence of reflection symmetry of the barrier or in the presence in it of trapping-type inclusions, so

1)For simplicity, we confine our attention to the case of identical superconductors: $\Delta_1 = \Delta_2 = \Delta$; however, all of the results in fact remain unchanged for $\Delta_1 \neq \Delta_2$. In particular, the final formula (18) is valid in the general case.

that the acts of absorption and emission of electrons by the tunnel barrier are uncorrelated, and consequently there is no symmetry with respect to the replacement of t by −t (if, in general, similar transitions may be described by the perturbation Hamiltonian H_T).

[1] B. D. Josephson, Phys. Letters **1**, 251 (1962).
[2] P. W. Anderson, Weak Superconductivity: Josephson Tunneling Effect (1963 Preprint).
[3] V. Ambegaokar and A. Baratoff, Phys. Rev. Letters **10**, 486 (1963).
[4] P. W. Anderson and J. M. Rowell, Phys. Rev. Letters **10**, 230 (1963).
[5] J. M. Rowell, Phys. Rev. Letters **11**, 200 (1963).
[6] Yanson, Svistunov, and Dmitrenko, JETP **47**, 2091 (1964), Soviet Phys. JETP **20**, 1404 (1965).
[7] M. D. Fiske, Revs. Modern Phys. **36**, 221 (1964).
[8] R. A. Ferrell and R. E. Prange, Phys. Rev. Letters **10**, 479 (1963).
[9] A. Messiah, Quantum Mechanics, Volume II, North-Holland Publishing Co., Amsterdam, 1961.

Translated by H. H. Nickle
157