SOVIET PHYSICS JETP VOLUME 22, NUMBER 3 MARCH, 1966

A CHARACTERISTIC OF THE GENERALIZED FREE FIELD

A. N. VASIL'EV

Leningrad State University

Submitted to JETP editor January 6, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) 49, 781-783 (September, 1965)

A transformation of a local field is considered which consists in the transition from a field $\varphi(p)$ to a field $\varphi_a(p) \equiv a(p^2)\varphi$, where $a(p^2)$ is an infinitely differentiable function of power growth, and $\varphi(p)$ is a field operator in the momentum representation. If a is not a polynomial, then the field φ_a is in general nonlocal. However, if $\varphi(p)$ decreases sufficiently rapidly for $|p^2| \equiv |p_0^2 - \mathbf{p}^2| \to \infty$ [in particular, like $\exp(-\mu|p^2|)$ for some $\mu > 0$], then the field φ_a is local for any a and the initial field can only be a generalized free field.

1. INTRODUCTION

THE distinction of the class of generalized free fields is of fundamental interest, since it allows one to separate out the theories which manifestly cannot describe real processes. It is known that a local field which satisfies one of the following three conditions: a) $\varphi(p)$ has an infinite zero (i.e., φ as well as all its derivatives are zero) for $p_0^2 - \mathbf{p}^2 \equiv p^2 = -d^2 < 0$; b) $\varphi(p) = 0$ for $0 \leq p^2 < m^2$ and has an infinite zero at $p = 0$; and c) $\varphi(p) = 0$ for $p^2 > M^2$, is a generalized free field.[1] This result was obtained with the assumption that the theory described by the field operator $\varphi(x)$ satisfies the following requirements: 1) the matrix elements $\langle\alpha|\varphi(x_1)\ldots\varphi(x_n)|\beta\rangle$ of products of field operators must be continuous functionals in the space S_{4n} of infinitely differentiable and rapidly decreasing functions of 4n variables; 2) there exist unitary representations $U(a, \Lambda)$ of the inhomogeneous Lorentz group and $\varphi(\Lambda x + a) = U(a, \Lambda)\varphi(x)U^+(a, \Lambda)$; 3) the spectrum of the energy-momentum operator lies in the upper light cone; 4) the theory has a unique vacuum, i.e., such a state that $U(a, \Lambda)\psi_0 = \psi_0$; and 5) the field $\varphi(x)$ is local, i.e., $[\varphi(x), \varphi(y)] = 0$ for $(x - y)^2 < 0$.

In the present paper we shall show that a field which satisfies all the requirements enumerated above is a generalized free field if $\varphi(p)$ [the Fourier transform of $\varphi(x)$] decreases sufficiently rapidly for $|p^2| \to \infty$. The main instrument for the proof of this assertion is the a transformation of the given theory, which is defined as the transition from the theory described by the field operator $\varphi(p)$ to a theory with field operators $\varphi_a(p) \equiv a(p^2)\varphi(p)$, where $a(p^2)$ is a real function. We shall choose a from the space O_M of infinitely differentiable functions with power growth.

We assume that the field φ_a also satisfies all requirements 1) to 5) for arbitrary $a \in O_M$. Choosing a function $a(p^2)$ which has an infinite zero at $p^2 = -d^2$, we find that φ_a must be a generalized free field. Since a can be chosen arbitrarily, φ itself is a generalized free field. Starting from this, we can formulate the following criterion: a necessary and sufficient condition for a field φ satisfying requirements 1) to 5) to be a generalized free field is that φ_a satisfy 1) to 5) for arbitrary $a \in O_M$.

It will be shown that, if $\varphi(x)$ satisfies conditions 1) to 5) and $\varphi(p)$ decreases sufficiently rapidly as $|p^2| \to \infty$, then φ_a always satisfies conditions 1) to 5).

2. PROOF OF LOCALITY OF THE FIELD φ_a

Let us thus consider the theory which is obtained from the given one by the transformation $\varphi(p) \to a(p^2)\varphi(p)$. We assume that the original theory satisfies 1) to 5). In terms of the Wightman functional W,

$$W_n(x_1 \ldots x_n) \equiv \langle\psi_0|\varphi(x_1)\ldots\varphi(x_n)|\psi_0\rangle$$

it is easy to see that the theory with φ_a always satisfies conditions 1) to 4). If, in addition, the theory with φ allows the construction of asymptotic states and an S matrix,[2] then the theory with φ_a also allows the construction of such quantities and has, up to a factor, the same asymptotic states and the same S matrix.

Let us now consider the problem of the locality of the theory with φ_a. According to the results of Borchers,[3] it suffices to prove the mutual local commutativity of the fields φ and φ_a. Let us consider the expression

$$J(g, a) \equiv \int \ldots \int dx_1 \ldots dx_n g(x_1 \ldots x_n) \times \langle \psi_0 | \varphi(x_1) \ldots \varphi(x_{i-1}) \varphi(x_i) \varphi_a(x_{i+1}) \varphi(x_{i+2}) \ldots | \psi_0 \rangle,$$

where $g \in S_{4n}$ is such that $g(x_1 \ldots x_n) = 0$ for $(x_i - x_{i+1})^2 < 0$ and

$$g(x_1 \ldots x_{i-1} x_i x_{i+1} x_{i+2} \ldots x_n) = -g(x_1 \ldots x_{i-1} x_{i+1} x_i x_{i+2} \ldots x_n).$$

The vanishing of $J(g, a)$ for all g, n, and i is a necessary and sufficient condition for the mutual local commutativity of the fields φ and φ_a.

In the momentum representation

$$J(g, a) = \int \ldots \int dp_1 \ldots dp_n g(p_1 \ldots p_n) a(p_{i+1}^2) \times \langle \psi_0 | \varphi(p_1) \ldots \varphi(p_{i-1}) \varphi(p_i) \varphi(p_{i+1}) \ldots | \psi_0 \rangle.$$

Integrating over $p_1, \ldots, p_i, p_{i+2}, \ldots, p_n, \mathbf{p}_{i+1}$ and summing over the signs of $p_{i+1\,0}$, we can write $J(g, a)$ in the form

$$J(g, a) = \int dp^2 a(p^2) f(p^2).$$

Let us now note the following: $J(g, a_0) = 0$ for any polynomial a_0, since $\varphi(x)$ always commutes locally with the field $\varphi_{a_0}(x) \equiv P(\Box)\varphi(x)$ (P is an arbitrary polynomial, and $\Box$ the d'Alembertian operator). We note further that f must be a continuous linear functional in O_M and hence, must be expressible in the form $f(\xi) = DF(\xi)$, where $\xi \equiv p^2$, D is some polynomial of differential operators, and $F(\xi)$ is a continuous function which decreases more rapidly than any power of ξ for $|\xi| \to \infty$.

Applying the operator D to the other side, we have

$$\int d\xi\, a_0(\xi) F(\xi) = 0$$

for any polynomial $a_0(\xi)$. These equations imply for the Fourier transform $\tilde{F}(\alpha)$ of the function $F(\xi)$ that $\tilde{F}(\alpha)$ has an infinite zero at $\alpha = 0$.

The manner of decrease of $F(\xi)$ for $|\xi| \to \infty$ determines the analytic properties of $\tilde{F}(\alpha)$. It may turn out for a sufficiently fast fall-off of $F(\xi)$ that $\tilde{F}(\alpha)$ can be continued analytically into some region including the entire real axis. For such theories the condition of mutual local commutativity is satisfied for any function $a(p^2)$, as we set out to prove.

It should be noted that such a rapid decrease of $F(\xi)$ must be a consequence of the rapid fall-off of the Wightman function $W_n(p_1, \ldots, p_n)$ and not of the function $g(p_1, \ldots, p_n)$. Otherwise, the function $g(x_1, \ldots, x_n)$ could be continued analytically into a region including all real values of the coordinates, which is impossible since $g(x_1, \ldots, x_n) = 0$ for $(x_1 - x_{i+1})^2 < 0$. In particular, a theory of precisely this type is one for which $\varphi(p)$ decreases not more slowly than $\exp(-\mu|p^2|)$ for some $\mu > 0$. Since $\varphi(p)$ is an operator distribution, the condition of exponential decrease must be formulated in the following way: there exists a $\mu > 0$ such that $\exp(\mu|p^2|)\varphi(p)$ is also an operator distribution (the matrix elements of products of these field operators must be continuous functionals in S_{4n}). For such theories the function $F(\xi)$ decreases no more slowly than $e^{-\mu|\xi|}$ for $|\xi| \to \infty$, and hence $\tilde{F}(\alpha)$ can be continued analytically into the strip $|\mathrm{Im}\,\alpha| < \mu$ including the entire real axis.

3. CONCLUSIONS

Thus, the field operators of a locally interacting field can not decrease very rapidly for $|p^2| \to \infty$. The estimate obtained for the rate of decrease can evidently be sharpened, since the exponential decrease guarantees the analytic continuation into a strip, and the general result remains true for a wider class of theories in which $\tilde{F}(\alpha)$ can be continued analytically into an arbitrary region including the real axis (for example, with boundaries coming arbitrarily close to the real axis for $|\mathrm{Re}\,\alpha| \to \infty$).

In conclusion I use this opportunity to thank L. B. Prokhorov for discussions and his constant interest in this work.

[1] O. W. Greenberg, J. Math. Phys. 3, 859 (1962).
[2] D. Ruelle, Helv. Phys. Acta 35, 147 (1962).
[3] H. J. Borchers, Nuovo Cimento 15, 784 (1960).

Translated by R. Lipperheide
104