THE RENORMALIZATION GROUP AND THE ULTRAVIOLET ASYMPTOTIC LIMIT OF SCATTERING

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A brief survey of the foundations of the renormalization-group method is given. The general solutions of the functional equations derived by Ovsyannikov are written out. They are used as the basis for a study of the problem of finding the high-energy asymptotic behavior of the scattering amplitude f. If the mass variable drops out at high energies, then for a fixed scattering angle f is a function of one argument, and for fixed momentum transfer it is a function of two arguments. In the former case the renormalization-group method gives improved asymptotic properties as compared with ordinary perturbation theory, and in the latter case it does not. The sum of the main logarithmic terms in the symmetric charged pion theory is found. A special hypothesis is formulated which leads to asymptotic behavior of a quasi-Regge type for both the elastic and the inelastic amplitudes.

UP to the present time the ultraviolet asymptotic behavior of Green's functions is still one of the most important unsolved problems of quantum field theory. Some years ago this problem attracted attention in connection with the well known paradox of the "zero charge" or the "ghost pole" in quantum electrodynamics. Owing to the increase of energies in accelerators it is taking on more and more actual experimental interest. Indeed, in this connection a new approach to the problem has been developed in recent years, based on the study of the analytic properties of scattering amplitudes in the complex plane of the angular momentum. Still, owing to the lack of sufficiently complete results from this approach, the attention of researchers is from time to time attracted to the more traditional methods, including the renormalization-group method (RGM). A number of papers of this kind have appeared recently.[1-7]

Along with interesting new results, some of these papers contain arguments which are not clearly enough stated and lead to conclusions which are not always correct. Therefore we think it timely to publish this paper, which contains both a concise survey of the basic points of the RGM and a detailed analysis of the possibilities of the RGM in the problem of ultraviolet asymptotic behaviors.

1. FOUNDATIONS OF THE RENORMALIZATION-GROUP METHOD

The logical basis of the RGM can be expounded very simply. Let us consider, for example, a many-body system described by the Lagrangian

$$L = L_2 + hL_4, \qquad (1.1)$$

consisting of a term L_2 which is quadratic in the field functions and their derivatives, and a term hL_4 which is a form of the fourth degree, i.e.,

$$L_2(a\varphi) = a^2L_2(\varphi), \qquad L_4(a\varphi) = a^4L_4(\varphi), \qquad (1.2)$$

where a is a constant. In this general form (1.1) can describe both nonrelativistic systems and relativistic systems, including nonlocal field theories.

The usual method for studying the system (1.1) is as follows. The term L_2 is the Lagrangian of the free field, and hL_4 describes the interaction:

$$L_0 = L_2, \qquad L_{int} = hL_4. \qquad (1.3)$$

One first studies the system with the interaction "turned off" (with h = 0), and quantizes it. The commutation relations $[\varphi(1), \varphi(2)] = iG(1, 2)$ normalize the field function φ. Then one "turns on" the interaction hL_4 and uses elementary chronological pairings $\langle T\varphi(1)\varphi(2)\rangle_0 = iD(1, 2)$ to calculate by perturbation theory (PT) the fundamental physical quantities M (matrix elements and Green's functions) in the form of expansions in powers of h.

It is clear, however, that the separation (1.3) is not unique. For example, it can be replaced by the following:

$$L = L_0' + L_{int}', \qquad L_0' = z_2^{-1}L_2,$$
$$L_{int}' = (1 - z_2^{-1})L_2 + hL_4, \qquad (1.4)$$

where z_2 is a positive real number. In (1.4) we have "transferred" part of the free Lagrangian L_0 of (1.3) into the interaction. Terms of this type usually get introduced into L_{int} as renormalization counterterms. It is known that their effect is a renormalization of the fields, the pairings, and, in the final analysis, of the complete Green's functions.

In fact, we must now carry out the quantization of the free field L_0'. From (1.2) we see that now it is not the φ that gets normalized, but $\varphi' = z_2^{-1/2}\varphi$; that is, $[\varphi'(1), \varphi'(2)] = iG(1, 2)$. When we change from φ to φ' in L_{int} and introduce a new coupling constant h′, we get

$$L_{int}' = (z_2 - 1) L_2(\varphi') + h' z_4 L_4(\varphi'), \qquad (1.5)$$

$$h' = h z_2^2 z_4^{-1}. \qquad (1.6)$$

We note that when we replace (1.3) by (1.5) the single-particle and four-vertex Green's functions

$$D = \frac{i}{S_0} \langle T\varphi\varphi S\rangle_0, \qquad \Gamma = \frac{1}{ihS_0}\left\langle \frac{\delta^4 S}{\delta\varphi\delta\varphi\delta\varphi\delta\varphi} \right\rangle_0 \qquad (1.7)$$

are changed in the following way (cf.[8]):

$$D \to \widetilde{D} = z_2^{-1}(\ldots, z_4 z_2^{-2} h'), \quad \Gamma \to \widetilde{\Gamma} = z_4 \Gamma(\ldots, z_4 z_2^{-2} h').$$

With the condition (1.6) the functions $\widetilde{D}$ and $\widetilde{\Gamma}$ again describe a theory with the coupling constant $h = h' z_4 z_2^{-2}$. It is clear that the simultaneous transformation of the coupling constant and the Green's functions

$$\begin{aligned} &D(\ldots, z_2 = 1, h) \to D(\ldots, z_2, z_4 h') \\ &\quad = z_2^{-1} D(\ldots, z_2 = 1, z_2^{-2} z_4 h'), \\ &\Gamma(\ldots, z_2 = 1, h) \to \Gamma(\ldots, z_2, z_4 h') \\ &\quad = z_4 \Gamma(\ldots, z_2 = 1, z_2^{-2} z_4 h'), \\ &h \to h' = z_4^{-1} z_2^2 h, \end{aligned} \qquad (1.8)$$

leads to a situation which is physically equivalent to the original one. This is natural, since observable quantities are defined by the physical properties of the system as a whole [which are fixed by the complete Lagrangian (1.1) and cannot depend on the way L is broken up into L_0 and L_{int}]. Therefore the Lagrangians L_{int} of (1.3) and (1.5) with the condition (1.6) indeed lead to physically equivalent results. This property is called renormalization invariance, and the corresponding group of continuous transformations of the Green's functions is called the renormalization group.

The formulas (1.8) reflect a peculiar self-similar property of the Green's functions.[9] They are consequences of dimensional relations and do not carry any dynamical information. The formulas (1.8) are valid, for example, for the problem of an electron gas with the Coulomb interaction,[10] for the Thirring two-dimensional fermion model,[11] for a pion field with interaction by fours,[8] and so on. Problems with three-vertex interactions, for example the quantum electrodynamics of electrons or mesons, lead to an obvious modification of (1.8). For definiteness we shall deal here with the problem of relativistic interacting bosons, i.e., the problem of pion-pion interaction.

The pion-pion interaction is a representative of the class of logarithmic theories, i.e., theories in which the basic ultraviolet asymptotic forms of the PT are logarithmic, and so also are the corresponding divergences. A well known property of such theories is the multiplicative arbitrariness in the Green's functions, of the type (1.8), which in the momentum representation is connected with the subtraction procedure, i.e., with the possibility of normalization. Thus for the propagator of a meson with mass m

$$D(k) = \frac{d(k^2)}{m^2 - k^2} = (2\pi)^{-4} \int e^{ikx} D(x)\, dx$$

we can set

$$d = 1 \text{ for } k^2 = \lambda^2 < 4m^2.$$

We can now write the function d as a function of dimensionless arguments, $d(k^2/\lambda^2, m^2/\lambda^2, h)$. The normalization condition will be

$$d(1, y; h) = 1. \qquad (1.9)$$

It can be seen that we have brought the arbitrary multiplicative factor z_2 into the argument of the function d through the normalization momentum λ. The values of λ^2 are limited by the requirement that d be real, which is due to the fact that the constant z_2 is real.

Making the analogous argument with the four-vertex function Γ, taken in the momentum representation, we put it in the form $\Gamma(x_1, \ldots, x_6; y; h)$, where, for example,

$$x_i = p_i^2/\lambda^2 \quad (1 \leqslant i \leqslant 4), \quad x_5 = \sigma = s/\lambda^2,$$

$$x_6 = \tau = t/\lambda^2; \quad s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2,$$

$$\Sigma p_i = 0,$$

and we can take

$$\Gamma(1, 1, 1, 1, 1, 1, y; h) = 1. \qquad (1.10)$$

By means of known arguments[8] the formulas (1.8)–(1.10) are converted into functional equations

$$\rho_x = \rho(x, y; h) = \rho(x/\xi, y/\xi; \rho_\xi); \qquad (1.11)$$

$$d(x, y; h) = d(x/\xi, y/\xi; \rho_\xi) d(\xi, y; h); \qquad (1.12)$$

$$\begin{aligned} &\Gamma(x_1, \ldots, x_6, y; h) = \gamma(\xi, y; h)\Gamma(x_1/\xi, \ldots, x_6/\xi, y/\xi; \rho_\xi), \\ &\gamma(x, y; h) = \Gamma(x, \ldots, x, y; h). \end{aligned} \qquad (1.13)$$

In a precisely similar way we can write functional equations for any Green's function $G(\{x_\mu\}, y; h)$ (where $x_\mu = p_i p_j/\lambda^2$, $y = m^2/\lambda^2$):

$$G(\{x_\mu\}, y; h) = \frac{G(\xi, \{x_i\}, y; h)}{G(1, \{x_i/\xi\}, y/\xi; \rho_\xi)} G\left(\left\{\frac{x_\mu}{\xi}\right\}, \frac{y}{\xi}; \rho_\xi\right) \tag{1.14}$$

(where $\mu = 1, 2, 3, \ldots, i = 2, 3, \ldots$), or, what is the same thing,

$$G(\{x_\mu\}, y; h) = d^\alpha(\xi, y; h)\gamma^\beta(\xi, y; h) G(\{x_\mu/\xi\}, y/\xi; \rho_\xi). \tag{1.15}$$

Here $d(\xi, y; h)$ and $\gamma(\xi, y; h)$ are nothing other than the constants z_2 and z_4 determined from (1.8), (1.12), and (1.13), and ρ_x is the invariant charge, given by

$$\rho_x = \rho(x, y; h) = hd^2(x, y; h)\gamma(x, y; h) \rightarrow hz_2^2 z_4^{-1}. \tag{1.16}$$

Equations (1.11)—(1.15) are the functional equations of the RG. We can also get from them differential equations which are useful in a number of cases,

$$\frac{\partial \rho(x, y; h)}{\partial \ln x} = \frac{\partial}{\partial \xi}\rho\left(\xi, \frac{y}{x}; \rho_x\right)\Big|_{\xi=1} \equiv \tilde{\rho}\left(\frac{y}{x}; \rho_x\right), \tag{1.17}$$

$$\frac{\partial \ln G(\{x_\mu\}, y; h)}{\partial \ln x_1} = \frac{\partial}{\partial \xi}\ln G\left(\xi, \left\{\frac{x_i}{x_1}\right\}, \frac{y}{x_1}; \rho_{x_1}\right)$$
$$= \tilde{G}\left(\left\{\frac{x_i}{x_1}\right\}, \frac{y}{x_1}; \rho_{k_1}\right). \tag{1.18}$$

Integrating Eq. (1.18), we get

$$\ln\frac{G(\{x_\mu\}, y; h)}{G(x_1 = x_0, \{x_i\}, y; h)} = \int_{x_0}^{x_1} \frac{du}{u}\tilde{G}\left(\left\{\frac{x_i}{u}\right\}, \frac{y}{u}; \rho_u\right). \tag{1.19}$$

The one-argument function $\gamma(x, y; h)$ is defined by fixing relations between the arguments $x_1, \ldots, x_6$ in Γ. These connections must be fixed so that the function $\Gamma(x, y; h)$ will be real. In the case in which all of the momenta are on the mass shell, this corresponds to points that lie inside a small triangle in the Mandelstam plane. In particular, the choices that correspond to the usual charge normalizations in meson theory are

$$x_1 = \ldots = x_4 = x, \quad x_5 = 4y + {}^4/_3(x - y),$$
$$x_6 = {}^4/_3(x - y)$$

or

$$x_1 = \ldots = x_4 = x, \quad x_5 = x_6 = {}^4/_3 x.$$

For $|x| \gg y$ the functions defined in this way coincide with each other in the main logarithmic approximation of PT. This sort of choice of the connections between the variables is conditioned by the fact that by its definition the invariant charge is a real quantity.

It follows from the arguments in Sec. 26 of[8] that for arbitrary $x < y$ and in each order of PT the invariant charge so defined is a real quantity. This fact is in obvious agreement with the analyticity properties of the invariant charge, which are considered in the Appendix.

We note that in a recent paper by Huang and Low[7] the authors use instead of Eq. (1.11) an equation [Eq. (9)—(10) in[7]] in which the last argument is not an invariant charge but the scattering amplitude itself—i.e., a complex function of many arguments. Therefore the arguments of Huang and Low[7] are equivalent to the introduction of complex renormalization constants, that is, of non-hermitian Lagrangians, which is in contradiction with the unitarity of the theory.

From the arguments we have given it is clear that by this method the pion-nucleon interaction can be given a treatment which is no worse than that of the $\pi\pi$ interaction. This is also in contradiction with[7].

2. SOLUTION OF THE EQUATIONS

In what follows we shall need general solutions of the equations of the RG. Such general solutions have been found by Ovsyannikov.[12] In particular, the general solution of the equations (1.11) for ρ_x is implicitly defined by the equation

$$f(y/x; \rho_x) = f(y; h). \tag{2.1}$$

Here $f(y; h)$ is an arbitrary function of two arguments which is invertible with respect to each of them. When we set $\xi = u$, $x_1 = y$ in (1.14), we find that the general solution of (1.14) must necessarily be of the form

$$G(\{x_\mu\}, y; h) = \frac{R(\{x_i\}, y; h)}{R(\{x_i/x_1\}, y/x_1; \rho_{x_1})} Q\left(\left\{\frac{x_i}{x_1}\right\}, \frac{y}{x_1}, \rho_{x_1}\right),$$
$$R(\{x_i\}, y; h) = G(x_1 = y; \{x_i\}, y; h),$$
$$Q(\{x_i\}, y; h) = Q(1, \{x_i\}, y; h). \tag{2.2}$$

Similarly, setting $\xi = x_1$ in (1.15), we get

$$G(\{x_\mu\}, y; h) = g(x_1, y; h) Q(\{x_i/x_1\}, y/x_1; \rho_{x_1}),$$
$$g(x, y; h) = d^\alpha(x, y; h)\gamma^\beta(x, y; h). \tag{2.3}$$

It can easily be shown by direct verification that the equations (1.14) and (1.15) are satisfied for arbitrary functions R and Q. In other words, the solution of Eqs. (1.14) and (1.15) is expressed in terms of two arbitrary functions of a large number of arguments.

Analogous solutions can also be obtained in the many-charge theory (cf.[12]). For reference we give the general solutions obtained in exactly the same way for the RG equations in the case in which the dependence on the argument y drops out (in the zero-mass theory):

$$\varphi(\rho_x) = x\varphi(t); \tag{2.1a}$$

$$G(\{x_\mu\};\ h) = \frac{R(\{x_i\}; h)}{R(\{x_i/x_1\};\ \rho_{x_i})} Q\left(\left\{\frac{x_i}{x_1}\right\};\ \rho_{x_1}\right)$$

$$= g(x_1, y; h) Q\left(\left\{\frac{x_i}{x_1}\right\};\ \rho_{x_1}\right). \tag{2.2a}$$

The solutions (2.1)—(2.3) contain all of the information that is contained in the equations of the RG. The arbitrariness which is included in these solutions is very great, and to get concrete results one must have additional information of a "dynamical" nature. This is indeed natural, since in the derivation of the RG equations essentially only dimensional considerations have been used, and not data on the dynamics of the system.

We see that renormalization invariance diminishes the number of independent arguments by unity. For example, the scattering amplitude $f(s/\lambda^2, t/\lambda^2, m^2/\lambda^2; h)$, which is a function of four arguments, is represented in the form of an arbitrary function of three arguments:

$$f\left(\frac{s}{\lambda^2}, \frac{t}{\lambda^2}\ \frac{m^2}{\lambda^2};\ h\right) = \gamma\left(\frac{s}{\lambda^2}, \frac{m^2}{\lambda^2}, h\right)$$

$$\times Q\left(\frac{t}{s}, \frac{m^2}{s};\ \bar{\rho}\left(\frac{s}{\lambda^2}, \frac{m^2}{\lambda^2};\ h\right)\right) \tag{2.4}$$

(some of the arguments come in only through the invariant charge). Furthermore, it must be noted that for large s the dependence on the mass in PT often drops out. The number of arguments of Q is then diminished by another unit, and γ becomes a function of ρ only. If we fix the scattering angle $[\theta = \theta(t/s) = \text{const}]$, we get a function of a single argument—the invariant charge.

Contrary to the conclusion of Huang and Low [Eq. (21) of[7]], this function can be obtained from PT (its determination does not require information about complex values of the coupling constant).

The dynamical information drawn from perturbation theory is introduced in the right members of the Lie differential equations (1.17)—(1.19), and the solution of these equations leads to renormalization-invariant expressions. The expansions of these expressions correspond to the orders of PT that are used.

The question arises: in what sort of cases will the resulting renormalization-invariant expression have better approximative properties than the original PT?

It is well known that in the ultraviolet asymptotic forms of logarithmic theories the mass variable drops out from the one-particle Green's functions. There then remains only the dependence on the invariant charge ρ. It is clear that in the region where $\rho_s < h \ln s$ we get an improvement of the approximative properties. As we have just now shown, there is an analogous situation for the scattering amplitude for a fixed scattering angle. In this case, using the PT expansion

$$f(\sigma, \tau; h) = \sum_n h^n f_n\left(\ln\sigma,\ \ln\frac{\tau}{\sigma}\right), \tag{2.5}$$

we get

$$Q\left(\frac{\tau}{\sigma},\ \rho_\sigma\right) = f\left(1,\ \frac{\tau}{\sigma},\ \rho_\sigma\right) = \sum_n \rho_\sigma{}^n f_n\left(0,\ \ln\frac{\tau}{\sigma}\right). \tag{2.6}$$

This formula, together with (2.4), indeed gives an effective improvement of the PT. If, on the other hand, t is fixed (diffraction scattering), then PT gives a double series for Q:

$$Q = \sum_{n=0} \rho_\sigma{}^n \sum_{k=0} A_k\left(\frac{m^2}{t}\right)\left(\ln\frac{t}{s}\right)^k,$$

and owing to this there is no improvement of the approximative properties.[4,7]

We note that in summing the main logarithmic asymptotic forms we can make use of the fact that they break up into three one-argument terms:

$$f(\sigma, \tau, y; h) \approx \sum_n h^n [a_n \ln^n \sigma + b_n \ln^n \tau$$

$$+ c_n \ln^n (4y - \tau - \sigma)].$$

Owing to this the problem actually reduces to the finding of a one-argument function from its expansion in a PT series. It is practically convenient (though not obligatory) to calculate the weight function of the corresponding one-dimensional spectral integral of the Mandelstam representation.

In particular, for the symmetrical charged pion model

$$L = {}^1/_4\pi^2 h:\ (\varphi_\sigma\varphi_\sigma)^2: \tag{2.7}$$

we get

$$A(s, t, u) = h + 7I(s) + 2I(t) + 2I(u),$$

$$u = 4m^2 - t - s; \tag{2.8}$$

$$I(x) = \int_4^\infty \frac{\rho(x')\,dx'}{x' - x - i\varepsilon},$$

$$\rho(x) = \frac{h^2}{(1 + 11h\ln x)^2}\left(\frac{x - y}{x}\right)^{1/2}. \tag{2.9}$$

Here A(s, t, u) is the well known structural component of the amplitude for $\pi\pi$ scattering. The other two, B and C, are obtained from A by means of crossing symmetry.

3. THE QUASI-REGGE BEHAVIOR OF THE AMPLITUDES

Recently many papers have appeared in which the asymptotic behaviors of elastic cross sections and amplitudes for $|s| \gg |t|$, m^2 were studied. There has been active discussion of a hypothesis according to which the amplitude in this region is of the Regge form:

$$f(s, t) \sim \beta(t) s^{\alpha(t)}. \tag{3.1}$$

From the considerations developed above it is clear that the equations of the RG by themselves are not sufficient for getting this kind of information about the amplitudes, and other considerations must be introduced. In particular, by summing a definite class of diagrams (the ladder diagrams) in the $h\varphi^3$ theory Arbuzov, Logunov, Tavkhelidze, and Faustov[5] succeeded in deriving the asymptotic form (3.1) by means of the Lie equations (1.18). The legitimacy of applying the method of the RG to find the asymptotic form of the sum of such ladder diagrams is due to certain properties of these diagrams, which have been studied in papers by these and other authors.

Let us examine what information about the amplitudes for elastic processes can be gotten from (3.1) for the RGM, and use them for the analysis of other processes. By using (1.11) we write the equation (1.18) for the amplitude

$$\frac{\partial \ln f(\sigma, \tau, y; h)}{\partial \ln \sigma} = \tilde{f}\left(\frac{\tau}{\sigma}, \frac{y}{\sigma}\right),$$

$$\rho\left(\frac{\sigma}{\tau}, \frac{y}{\tau}; \rho_\tau\right) \equiv \varphi\left(\frac{\tau}{\sigma}, \frac{y}{\tau}; \rho_\tau\right).$$

According to (3.1) the asymptotic limit of f can be of the Regge form only if the limit

$$\lim_{\sigma\to\infty} \tilde{f}\left(\frac{\tau}{\sigma}, \frac{y}{\sigma}; \rho_\sigma\right) = \lim_{\sigma\to\infty} \varphi\left(\frac{\tau}{\sigma}, \frac{y}{\tau}; \rho_\tau\right) = \alpha\left(\frac{y}{\tau}; \rho_\tau\right) \neq 0 \tag{3.2a}$$

exists, or, when there is a finite "bare" charge[6] $H = \rho(\sigma = \infty, y; h)$ if there exists the limit

$$\lim_{\sigma\to\infty} \tilde{f}\left(\frac{\tau}{\sigma}, \frac{y}{\sigma}; \rho_\sigma\right) = \alpha\left(\frac{y}{\tau}, H\right) \neq 0. \tag{3.2b}$$

We now write the equation (1.18) for the dimensionless amplitude for the scattering process in question (which is not necessarily elastic and is not necessarily on the mass shell, but is near it) in the form

$$\frac{\partial \ln G(\sigma, \tau, a_\alpha; h)}{\partial \ln \sigma} = \nu\left(\frac{\tau}{\sigma}, a_\alpha; \rho_\sigma\right)$$
$$\equiv \frac{\partial}{\partial \xi} \ln G\left(\xi, \frac{\tau}{\sigma}, a_\alpha; \rho_\sigma\right)\Big|_{\xi=1} \tag{3.3}$$

Here t is one of the parameters of the type $(p_i + p_j)^2$, p_i^2; s is the energy of the process; $\tau = t/\lambda^2$, $\sigma = s/\lambda^2$; a_α are parameters of the type t/m^2, $(p_\alpha + p_\beta)^2/(p_\gamma + p_\sigma)^2$ which do not involve λ and are such that for $s \to \infty$, t = const they remain constant.

We now assume that a relation of the type (3.2) is valid in field theory, independently of which amplitude it is applied to; that is,

$$\lim_{\sigma\to\infty} \nu\left(\frac{\tau}{\sigma}, a_\alpha; \rho_\sigma\right) = \begin{cases} \nu_0(a_\alpha; \rho_\tau) \\ \nu_0(a_\alpha; H) \end{cases} \neq 0. \tag{3.4}$$

(It is obviously impossible to derive this sort of result directly in PT.) Then it follows from (3.3) that we have the respective formulas

$$G(\sigma, \tau, a_\alpha; h) = r(\tau, a_\alpha; \rho_\tau)\,(s/s_0)^{\nu_0(a_\alpha;\rho_\tau)} \tag{3.5a}$$

or

$$G(\sigma, \tau, a_\alpha; h) = r(\tau, a_\alpha; \rho_\tau)\,(s/s_0)^{\nu_0(a_\alpha;H)} \tag{3.5b}$$

corresponding to the two possibilities (3.2a), (3.2b).

The relations (3.5) mean that the amplitudes for all processes behave in a quasi-Regge manner. Special cases of (3.5) are the well known results of Ter-Martirosyan[13] on the asymptotic behavior of inelastic processes and those of Dremin and Chernavskii[14] on the amplitudes as functions of the virtuality. The present approach naturally gives less information than the "dynamical" analysis of the papers cited (we cannot fix the form of the function ν_0), but it has the advantage of generality.

In conclusion we are pleased to express our gratitude to I. Todorov for writing the Appendix. The writers also take great pleasure in stating that the initiative for this work is due to A. Logunov, owing to whose decisive action this research can finally be regarded as having reached a conclusion. One of us (I.G.) also expresses his unchanging gratitude to D. Stel'makh.

APPENDIX

ANALYTIC PROPERTIES OF INVARIANT CHARGES IN MESON THEORY[1)]

We shall show that in any order of PT the function $\rho(x, y; h)$ is analytic in the x plane with a cut for $x > y$. In doing so we use the results of[15-18]

*Written by I. Todorov.

on the majorization of Feynman diagrams. We shall also show that each of the factors in the right member of (1.22) (sic) is analytic in the x plane with cut as already indicated. For the function d(x) this follows from the Källen-Lehmann representation for the boson wave function (cf. e.g.,[8]), which naturally is valid also in any order of PT. The cut for the function d begins at $s = 9m^2$, i.e., $x = 9y$. The first branch point on the real axis is a singularity of the factor $\gamma(s)$.

The contribution to the meson-meson scattering amplitude from an arbitrary Feynman diagram which contains no divergences is of the form

$$T(p_1, \ldots, p_4) = \int_0^1 \ldots \int_0^1 \delta\Big(1 - \sum_\nu \alpha_\nu\Big) P(\alpha, p) \times \prod_\nu d\sigma_\nu [Q(\alpha, p) + i\varepsilon]^{-r}, \quad (A.1)$$

where $P(\alpha, p)$ is a polynomial in the scalar products of the momenta P_i, r is a positive integer, and

$$Q(\alpha, p) = \sum_{i=1}^4 A_i(\alpha) p_i^2 + \sum_{1 \leqslant i < j \leqslant 3} B_{ij}(\alpha)(p_i + p_j)^2 - \sum_\nu \alpha_\nu m_\nu^2, \quad (A.2)$$

$$m_\nu = \begin{cases} m \text{ for meson lines} \\ M \text{ for nucleon lines of the diagram.} \end{cases}$$

The coefficients $A_i(\alpha)$ and $B_{ij}(\alpha)$ in the expansion (A.2) are not uniquely determined, since there is a linear relation between the variables p_i^2 and $(p_i + p_j)^2$:

$$2\sum_{i=1}^4 p_i^2 = \sum_{1 \leqslant i < j \leqslant 3} (p_i + p_j)^2.$$

This ambiguity can be utilized to secure that all of the coefficients $A_i(\alpha)$ and $B_{ij}(\alpha)$ be nonnegative (cf.[18,19]). In the case of a divergent diagram the partial derivatives with respect to some of the internal masses are given by converging integrals of the type (A.1) with these same properties of the coefficients A and B. Therefore the singularities of any Feynman diagram are in all cases determined by the zeroes of the function $Q(\alpha, p)$ of Eq. (A.2).

It follows from (A.1) and (A.2) that taken as a function of γ the contribution from an arbitrary diagram is of the form

$$\gamma(s) = \int_0^1 \ldots \int_0^1 P(\alpha, s)\, \delta\Big(1 - \sum_\nu \alpha_\nu\Big) \times \prod_\nu d\alpha_\nu \Big[A(\alpha)s - \sum \alpha_\nu m_\nu^2 + i\varepsilon\Big]^{-r}, \quad (A.3)$$

where

$$A(\alpha) \geqslant 0. \quad (A.4)$$

Owing to (A.4), if

$$Q(\alpha, s) \equiv A(\alpha)s - \sum_\nu \alpha_\nu m_\nu^2 < 0 \quad (A.5)$$

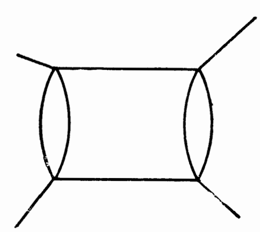
for some real value s_0 and for all α in the region of integration, then this inequality will also hold for

$$s < s_0.$$

Mestvirishvili and Todorov[17] have shown that all of the diagrams for scattering of a pseudoscalar meson by a meson are majorized by three diagrams of the type shown in the figure, with different numberings of the external momenta. Therefore for $s > 0$ (i.e., in the Euclidean region) the form Q satisfies

$$Q(\alpha, s) < 0 \text{ for } s < 4m^2, \quad (A.6)$$

so that this inequality holds for diagrams of the type shown in the figure.



According to what has been said, the inequality (A.6) remains valid for all real $s < 4m^2$. The integral (A.3) is analytic also for Im $s \neq 0$. In fact, if $A(\alpha) \neq 0$, then the denominator in the integral (A.3) cannot be zero, since it has a nonvanishing imaginary part. If, on the other hand, $A(\alpha) = 0$, then at this point $Q(\alpha, s) = Q(\alpha, 0) < 0$, and consequently the integral (A.3) again cannot have any singularities. Accordingly the analyticity of the invariant charge $\rho(x, y; h)$ in the cut x plane has been demonstrated. In particular, it follows from what we have proved that the function ρ is real for $x < 4y$.

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