ON EFFECTS DUE TO INTERACTION OF PLASMONS

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Explicit expressions are derived for the probabilities of four-plasmon decays of longitudinal waves with a random phase in the plasma. Some particular cases are considered. Qualitative analysis of the effects of four-plasmon decays is carried out for longitudinal waves and it is shown that they result in a broadening of the noise spectrum.

1. INTRODUCTION

It can be assumed by now that the system of fundamental equations of a weakly-turbulent plasma has been completely derived for the simplest case of a plasma without a magnetic field (see, for example, [1,2]). In most cases it is possible to confine oneself in the derivation of these equations to the first nonlinear terms, which are quadratic in the number of plasmons $N(\mathbf{k})$ (or in the energy of the electric field), that is, only processes involving first and second-order scattering, and decays in which three plasmons participate, need be taken into account. In some cases, however, it may be important to make allowance for terms of higher order of smallness, and, in particular, to take into account effects of four-plasmon decays. Such a situation may arise, for example, if we consider nonlinear interactions between plasmons of any one particular type (t, s, or l plasmons), when by virtue of the dispersion relations and the energy and momentum conservation laws the probabilities of the three-plasmon decay processes are identically equal to zero [1]). Generally speaking, it is necessary to take into account here the next higher-order terms of the expansion in powers of the electric field, that is, in particular, processes of four-plasmon decay.

The present paper is devoted to the calculation of the probability of four-plasmon decays in the case when all plasmons participating in the process belong to any single type α (α = t, l, s).

In the general case, in the presence of four plasmons α, α_1, α_2, and α_3, two essentially different processes can occur:

$$\alpha \rightleftarrows \alpha_1 + \alpha_2 + \alpha_3, \tag{I}$$

$$\alpha + \alpha_1 \rightleftarrows \alpha_2 + \alpha_3. \tag{II}$$

The first corresponds to the decay of one plasmon α into three plasmons α_1, α_2, and α_3 or conversely, to the coalescence of three plasmons α_1, α_2, and α_3 into a single plasmon α; the probabilities of the two cases are obviously identical. The second process corresponds to the decay (coalescence) of two plasmons α and α_1 into two plasmons α_2 and α_3.

However, in the case considered here, that of a plasma without a magnetic field, when the dispersion relations are of the form [2])

$$\Omega_t = [\omega_{0e}^2 + k^2]^{1/2}, \qquad \Omega_l = [\omega_{0e}^2 + 3k^2 v_{Te}^2]^{1/2},$$

$$\Omega_s = \left[\frac{\omega_{0i}^2 k^2 v_{Ti}^2}{\omega_{0e}^2 + k^2 v_{Te}^2} + 3k^2 v_{Ti}^2\right]^{1/2};$$

$$\omega_{0e}^2 = \frac{4\pi e_e^2 n_e}{m_e}, \quad k^2 v_{Te}^2 \ll \omega_{0e}^2, \quad v_{Te}^2 = \frac{T_e}{m_e},$$

$$\omega_{0i}^2 = \frac{4\pi e_i^2 n_i}{m_i}, \quad v_{Ti}^2 = \frac{T_i}{m_i}, \tag{1}$$

where e_e, m_e, n_e, and T_e are the charge, mass, density, and temperature of the electrons, while e_i, m_i, n_i, and T_i are the corresponding quantities for the ions, it is easy to show that the energy and momentum conservation laws

$$\Omega_\alpha = \Omega_{\alpha_1} + \Omega_{\alpha_2} + \Omega_{\alpha_3}, \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3,$$

$$\Omega_\alpha + \Omega_{\alpha_1} = \Omega_{\alpha_2} + \Omega_{\alpha_3}, \quad \mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3,$$

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1) As in [1], we define t-plasmons as transverse waves, l-plasmons as longitudinal Langmuir waves, and s-plasmons as longitudinal sonic waves in a non-isothermal plasma.

2) As in [1], we put henceforth c = 1 for the velocity of light, and $\hbar = 1$.

absolutely forbid processes in which one plasmon decays into three (or vice versa), and permit only processes of type II, that is, the decay of two plasmons α and α_1 into two others, α_2 and α_3 (and vice versa).

We denote by $V^{\alpha\alpha_1,\alpha_2\alpha_3}(\mathbf{kk}_1\mathbf{k}_2\mathbf{k}_3)$ the probability of decay of two plasmons of type α with momenta $\mathbf{k}_2$ and $\mathbf{k}_3$ and energies $\Omega_{\alpha_2} = \Omega_\alpha(\mathbf{k}_2)$ and $\Omega_{\alpha_3} = \Omega_\alpha(\mathbf{k}_3)$ respectively, into two plasmons of the same type with momenta $\mathbf{k}$ and $\mathbf{k}_1$ and energies Ω_α and Ω_{α_1}. This probability, obviously, satisfies the relations [3)]

$$V^{\alpha\alpha_1,\,\alpha_2\alpha_3}(\mathbf{kk}_1\mathbf{k}_2\mathbf{k}_3) = V^{\alpha_2\alpha_3,\,\alpha\alpha_1}(\mathbf{k}_2\mathbf{k}_3\mathbf{kk}_1)$$
$$= V^{\alpha\alpha_1,\alpha_3\alpha_2}(\mathbf{kk}_1\mathbf{k}_3\mathbf{k}_2) = V^{\alpha_1\alpha,\alpha_2\alpha_3}(\mathbf{k}_1\mathbf{kk}_2\mathbf{k}_3) \tag{2}$$

and includes as factors δ functions which take into account the plasmon energy and momentum conservation laws during the decay process:

$$V^{\alpha\alpha_1,\,\alpha_2\alpha_3}(\mathbf{kk}_1\mathbf{k}_2\mathbf{k}_3) \sim \delta(\Omega_\alpha + \Omega_{\alpha_1} - \Omega_{\alpha_2} - \Omega_{\alpha_3})$$
$$\times\, \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3). \tag{3}$$

Then the change in the number of quanta $N_\alpha(\mathbf{k})$ of type α, connected with the four-plasmon decay processes, will in the classical limit have the form [4)]

$$\left(\frac{dN_\alpha}{dt}\right)_4 = \int d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3\, V^{\alpha\alpha_1,\,\alpha_2\alpha_3}(\mathbf{kk}_1\mathbf{k}_2\mathbf{k}_3)[N_\alpha(\mathbf{k}_1)N_\alpha(\mathbf{k}_2)$$
$$\times N_\alpha(\mathbf{k}_3) + N_\alpha(\mathbf{k})N_\alpha(\mathbf{k}_2)N_\alpha(\mathbf{k}_3) - N_\alpha(\mathbf{k})N_\alpha(\mathbf{k}_1)N_\alpha(\mathbf{k}_2)$$
$$- N_\alpha(\mathbf{k})N_\alpha(\mathbf{k}_1)N_\alpha(\mathbf{k}_3)]. \tag{4}$$

2. CALCULATION OF THE PROBABILITIES OF FOUR-PLASMON DECAYS

In [1,2] we developed in detail a semi-quantum method of calculating the probabilities of first and second-order scattering processes and three-plasmon decay processes. We shall therefore not dwell in detail here on the procedure for calculating the probabilities of four-plasmon decays, which coincides exactly with that used earlier, [1,2] and will describe briefly only the general method of calculation and present the final results.

The change in the number of quanta of type α, $(dN_\alpha/dt)_4^{sp}$, connected only with the spontaneous processes of four-plasmon decays, is obviously equal to

$$\left(\frac{dN_\alpha}{dt}\right)_4^{sp} = \int V^{\alpha\alpha_1,\,\alpha_2\alpha_3}(\mathbf{kk}_1\mathbf{k}_2\mathbf{k}_3)N_\alpha(\mathbf{k}_1)N_\alpha(\mathbf{k}_2)$$
$$\times\, N_\alpha(\mathbf{k}_3)\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3. \tag{5}$$

Consequently, the change in the average density is

$$\left(\frac{dU_\alpha}{dt}\right)_4^{sp} = \int \frac{\Omega_\alpha}{(2\pi)^3}\left(\frac{dN_\alpha}{dt}\right)_4^{sp} d\mathbf{k}$$
$$= \int \frac{\Omega_\alpha}{(2\pi)^3}\, V^{\alpha\alpha_1,\alpha_2\alpha_3}N(\mathbf{k}_1)N(\mathbf{k}_2)N(\mathbf{k}_3)\,d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3. \tag{6}$$

On the other hand, however, this change in energy density can be obtained directly from the nonlinear system of Maxwell's equations for the field vectors, and from the kinetic equation for the distribution function, by expanding the solution in powers of the amplitude of the electric field. Proceeding in a manner similar to that in [1], we easily obtain

$$\left(\frac{dU_t}{dt}\right)_4^{sp} = 4\pi^2 \lim_{\substack{T\to\infty\\ V\to\infty}} \frac{1}{T}\frac{1}{V}$$
$$\times \int d\mathbf{k}\, d\omega\, \Omega_t \delta(\omega^2 - \Omega_t^2)\left\langle\left|\frac{[\mathbf{k}\delta\mathbf{j}]}{k}\right|^2\right\rangle,$$

$$\left(\frac{dU_{l,s}}{dt}\right)_4^{sp} = 8\pi^2 \lim_{\substack{T\to\infty\\ V\to\infty}} \frac{1}{T}\frac{1}{V}\int \frac{d\mathbf{k}\,d\omega}{|\partial\varepsilon^l/\partial\omega|}\,\delta(\omega^2 - \Omega_{l,s}^2)$$
$$\times\left\langle\left|\frac{(\mathbf{k}\delta\mathbf{j})}{k}\right|^2\right\rangle, \tag{7*}$$

where V is the volume and T is the time interval, and

$$\delta\mathbf{j} = -\sum_q \int d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3\,d\omega_1\,d\omega_2\,d\omega_3\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$$
$$\times\,\delta(\omega+\omega_1-\omega_2-\omega_3)\,\mathbf{I}_q(\mathbf{kk}_1\mathbf{k}_2\mathbf{k}_3),$$

$$\mathbf{I}_q(\mathbf{kk}_1\mathbf{k}_2\mathbf{k}_3) = \frac{ie_q^4}{(2\pi)^4}\int \frac{\mathbf{v}\,d\mathbf{p}}{x}\left\{\mathbf{F}_1\frac{\partial}{\partial\mathbf{p}}\left[\frac{\mathbf{F}_2}{x_2+x_3}\frac{\partial}{\partial\mathbf{p}}\left(\frac{\mathbf{F}_3}{x_3}\frac{\partial f_q^{(0)}}{\partial\mathbf{p}}\right)\right]\right\},$$
$$\mathbf{F}_n = \mathbf{F}(\mathbf{k}_n\omega_n),\quad \mathbf{F} = \mathbf{E}\left(1 - \frac{\mathbf{kv}}{\omega}\right) + (\mathbf{vE})\frac{\mathbf{k}}{\omega},$$
$$x_n = \omega_n - \mathbf{k}_n\mathbf{v},\quad n = 1, 2, 3; \tag{8}$$

3)Analogous relations, which are perfectly obvious from the physical point of view, are satisfied, as was shown in [1], also by the probabilities of the three-plasmon decay processes and the probabilities of scattering. This, in turn, makes it possible to write down immediately several first integrals for the obtained system of nonlinear equations, expressing the laws of energy and momentum conservation in a particle-plasmon system, and, in several cases (for example in the presence of only one type of plasmons), the total number of plasmons. The trivial manner in which these first integrals are obtained, which is a direct consequence of the semiquantum methods employed by us, offers evidence that it has definite advantages over other methods of obtaining the fundamental equations of a turbulent plasma (see, for example, [3]), in which these first integrals can be written only after a very tedious analysis of the properties of the kernels which appear in the system of final equations.

4)The normalization of N_α is chosen such that the energy density U of the quanta of type α is equal to

$$U_\alpha = \int \frac{\Omega_\alpha N_\alpha d\mathbf{k}}{(2\pi)^3}.$$

*$[\mathbf{k}\delta\mathbf{j}] = \mathbf{k}\times\delta\mathbf{j}$.

the index q = e, i denotes the species of the particles, $\mathbf{E}(\mathbf{k})$ is the Fourier component of the electric field vector, and $f_q^{(0)}(\mathbf{p})$ is the zeroth-approximation distribution function of particles of species q which we assume close to Maxwellian, while the angle brackets denote the operation of averaging over the ensemble.

Taking into account the connection between the spectral density of the field energy and the number of the corresponding quanta, and reducing (7) to the form (6), we can easily obtain the sought probabilities. Without stopping to obtain the probability of the four-plasmon decays for transverse waves, the expression for which is rather cumbersome, we confine ourselves here only to the case of longitudinal waves. In this case, recognizing that

$$\langle E_i^{l,s}(\mathbf{k}\omega)E_j^{l,s}(\mathbf{k}'\omega')\rangle = \frac{k_ik_j}{k^2}\overline{E_{l,s}^2}(\mathbf{k}\omega)\delta(\omega^2-\Omega_{l,s}^2) \times \left|\frac{\partial\varepsilon^l}{\partial\omega^2}\right|^{-1}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'),$$

$$N_{l,s}(\mathbf{k}) = \overline{E_{l,s}^2}(\mathbf{k}, \Omega_{l,s})/8\pi^2,$$

where $\epsilon^l(\omega\mathbf{k})$ is the longitudinal dielectric constant, we obtain

$$V^{\alpha\alpha_1,\alpha_2\alpha_3}(\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3) = \frac{4}{\pi}\left|\frac{\partial\varepsilon^l}{\partial\Omega_{\alpha_1}}\frac{\partial\varepsilon^l}{\partial\Omega_{\alpha}}\frac{\partial\varepsilon^l}{\partial\Omega_{\alpha_2}}\frac{\partial\varepsilon^l}{\partial\Omega_{\alpha_3}}\right|^{-1} \times\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\Omega_\alpha+\Omega_{\alpha_1}-\Omega_{\alpha_2}-\Omega_{\alpha_3})\,|Q|^2, \tag{9}$$

$$Q = \sum Q_q = \sum_q[\Phi_q(k, -k_1, k_2, k_3)+\Phi_q(k, -k_1, k_3, k_2) + \Phi_q(k, k_2, -k_1, k_3)+\Phi_q(k, k_2, k_3, -k_1) + \Phi_q(k, k_3, -k_1, k_2)+\Phi_q(k, k_3, k_2, -k_1)],$$

$$\Phi_q(k, k_1, k_2, k_3) = e_q^4\int\frac{d\mathbf{p}}{x}\left\{\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\left[\frac{\mathbf{k}_2}{x_2+x_3}\frac{\partial}{\partial\mathbf{p}}\left(\frac{\mathbf{k}_3}{x_3}\frac{\partial f_q^{(0)}}{\partial\mathbf{p}}\right)\right]\right\},$$

$$k_i = \{\mathbf{k}_i, \Omega_\alpha(k_i)\}. \tag{10}$$

In the case of longitudinal Langmuir waves, recognizing that

$$\Omega_l \gg kv_{Te}, \quad \Omega_l - \Omega_{l_1} \ll |\mathbf{k}-\mathbf{k}_1|v_{Te}$$

and that the contribution of the ions can be completely neglected, the integral (10) can be readily calculated, and accurate to small quantities $\sim k_2v_{Te}^2/\omega_{0c}^2$ we shall have

$$Q = Q_e = -\frac{e_e^2}{4\pi m_e^2 v_{Te}^2\omega_{0e}^2}[\cos\widehat{\mathbf{k}\mathbf{k}_2}\cos\widehat{\mathbf{k}_1\mathbf{k}_3} + \cos\widehat{\mathbf{k}\mathbf{k}_3}\cos\widehat{\mathbf{k}_1\mathbf{k}_2}]. \tag{11}$$

Substituting this expression in (9), we obtain finally the sought probability

$$V^{ll_1,l_2l_3}(\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3) = \frac{e_e^4}{m_e^4v_{Te}^4}[\cos\widehat{\mathbf{k}\mathbf{k}_2}\cos\widehat{\mathbf{k}_1\mathbf{k}_3} + \cos\widehat{\mathbf{k}\mathbf{k}_3}\cos\widehat{\mathbf{k}_1\mathbf{k}_2}]^2 \times\frac{\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\Omega_l+\Omega_{l_1}-\Omega_{l_2}-\Omega_{l_3})}{(4\pi)^3}. \tag{12}$$

For sound oscillations the situation is somewhat more complicated. In the region of short waves, $kv_{Te} \gg \omega_{0e}$, where the principal role is played by ions, the contribution of the electrons in the integral (10) can be neglected; we then obtain for the probability an expression which is perfectly analogous to (12), namely:

$$V^{ss_1,s_2s_3}(\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3) = \frac{e_i^4}{m_i^4v_{Ti}^4}[\cos\widehat{\mathbf{k}\mathbf{k}_2}\cos\widehat{\mathbf{k}_1\mathbf{k}_3} + \cos\widehat{\mathbf{k}\mathbf{k}_3}\cos\widehat{\mathbf{k}_1\mathbf{k}_2}]^2 \times\frac{\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\Omega_s+\Omega_{s_1}-\Omega_{s_2}-\Omega_{s_3})}{(4\pi)^3}. \tag{12'}$$

In the region of long waves, $kv_{Te} \ll \omega_{0e}$ the expression for the probability V^{ss_1,s_2s_3} has a very complicated form, and will not be presented here. We note only that to determine it we must take into account not only the ions, but also the electrons, and that the term connected with the ions (that is, Q_i) has rather sharp maxima in regions where $\Omega_{s_{2,3}} - \Omega_{s_1} < |\mathbf{k}_{2,3} - \mathbf{k}_1|v_{Ti}$, where inside these regions the ionic term Q_i is larger by a factor $(\Omega_s/kv_{Ti})^2$ than the corresponding electronic term Q_e, while outside these regions the contribution from the ions has the same order of magnitude as the contribution from the electrons, that is, $Q_e \approx Q_i$. Taking into account that, as can be readily verified,

$$Q_e = \frac{e_e^4n_e}{T_e^3}\frac{1}{kk_1k_2k_3}, \tag{13}$$

we can obtain an estimate for the probability V^{ss_1,s_2s_3} in the region of long waves:

$$V^{ss_1,s_2s_3}(\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3) \approx \frac{e_i^4}{m_i^4v_{Ti}^4}\frac{\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\Omega_s+\Omega_{s_1}-\Omega_{s_2}-\Omega_{s_3})}{(4\pi)^3} \times kk_1k_2k_3\left(\frac{v_{Te}}{\omega_{0e}}\right)^4\left(\frac{T_i}{T_e}\right)^2. \tag{14}$$

Inasmuch as $kv_{Te}/\omega_{0e} \ll 1$ and $T_i/T_e \ll 1$, it follows that the contribution from the four-plasmon decay processes, connected with small k, gives a summary effect which is much smaller than the corresponding contribution from the region of large k. And since furthermore, unlike in the case of Langmuir waves, the linear decrement of the attenuation of the sound waves is finite as $k \to 0$, in a large number of cases the role of the four-plasmon decay effects can be allowed for sufficiently well by confining oneself only to the long-wave region $k > \omega_{0e}/v_{Te}$, assuming that outside this region the influence of the four-plasmon decays on the dynamics of the waves is negligibly small and that when $k \ll \omega_{0e}/v_{Te}$ the probability V^{ss_1,s_2s_3} is identically equal to zero.

3. ANALYSIS OF THE EFFECTS OF FOUR-PLASMON DECAYS; SOME PARTICULAR CASES

We now attempt, using several particular cases, to ascertain the new effects which can result from four-plasmon decay processes. Bearing in mind that the structure for the expression for V^{ll_1,l_2l_3} and V^{ss_1,s_2s_3} is perfectly analogous, we confine ourselves only to the analysis of the case of Langmuir waves.

1. We note first several general properties of four-plasmon decay processes. Multiplying the right and left sides of (4) in turn by Ω_α and $\mathbf{k}$, integrating over all of $\mathbf{k}$-space, and taking into account relations (2) and (3), we can easily verify that

$$\int \Omega_\alpha\left(\frac{dN_\alpha}{dt}\right)_4 d\mathbf{k} \equiv 0, \qquad \int \mathbf{k}\left(\frac{dN_\alpha}{dt}\right)_4 d\mathbf{k} \equiv 0. \tag{15}$$

These relations are obviously the consequence of the laws of conservation of plasmon energy and momentum during scattering.

On the other hand, inasmuch as the conservation laws allow only the process in which two plasmons decay into two other plasmons, the total number of plasmons in such decays must also be conserved. Indeed, integrating (4) over all of $\mathbf{k}$-space and taking into account relations (2), we can easily verify that

$$\int d\mathbf{k}\left(\frac{dN_\alpha}{dt}\right)_4 \equiv 0, \tag{16}$$

i.e., four-plasmon decays do not change the total number of plasmons.

2. If we assume that the noise is strictly one-dimensional, that is, $N_\alpha(\mathbf{k}) = N_\alpha(k_\parallel)\,\delta(\mathbf{k}_\perp)$, and integrate (4) with respect to $\mathbf{k}_\perp$ with (12) taken into account, we get

$$\int d\mathbf{k}_\perp\left(\frac{dN_\alpha}{dt}\right)_4 \equiv 0, \tag{17}$$

i.e., in the case of the purely one-dimensional spectrum four-plasmon decays do not change the density of the number of plasmons. This can be readily understood physically. Indeed, in the one-dimensional case the conservation laws (3) cause the probability of the decays to differ from zero only if $k_2 = k$ and $k_3 = k_1$ or $k_2 = k_1$ and $k_3 = k$, that is, if the two quanta k and k_1 decay into precisely the same two quanta.

We must recognize, however, that the noise distribution is, strictly speaking, never purely one dimensional and there is always a certain number of plasmons with $\mathbf{k}_\perp \neq 0$. Allowance for this circumstance causes the four-plasmon decays to reduce the number of plasmons with $\mathbf{k}_\perp = 0$, while the number of plasmons with $\mathbf{k}_\perp \neq 0$ increases, so that the one-dimensional spectrum "broadens" in the direction of $\mathbf{k}_\perp$, becoming practically three-dimensional.

3. Using the expression obtained above for the probability V^{ll_1,l_2l_3} and the expressions given in [1] for the probabilities of the scattering of l-plasmons by subthermal particles (or the corresponding cross sections), we can now write the final equation for the plasmon-number density $N_l(\mathbf{k})$, in which account is taken of the nonlinear corrections which are both quadratic and cubic in the number of plasmons. Going over for convenience to dimensionless variables

$$\mathbf{x} = \mathbf{k}r_D, \quad \tau = \omega_{0e}t, \quad \psi(\mathbf{x}) = N_l\omega_{0e}r_D^{-3}/(2\pi)^3,$$

where $r_D^{-1} = \omega_{0e}/v_{Te}$, we can write it in the form

$$\frac{d\psi}{d\tau} = \psi\left\{\gamma + \int K(\mathbf{x}\mathbf{x}_1)\psi(\mathbf{x}_1)d\mathbf{x}_1\right\} + \int P(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3)$$
$$\times[\psi(\mathbf{x}_1)\psi(\mathbf{x}_2)\psi(\mathbf{x}_3) + \psi(\mathbf{x})\psi(\mathbf{x}_2)\psi(\mathbf{x}_3)$$
$$-\psi(\mathbf{x})\psi(\mathbf{x}_1)\psi(\mathbf{x}_2) - \psi(\mathbf{x})\psi(\mathbf{x}_1)\psi(\mathbf{x}_3)]\,d\mathbf{x}_1\,d\mathbf{x}_2\,d\mathbf{x}_3, \tag{18}$$

where the first term on the right side corresponds to an account of only the linear terms (γ—the Landau damping decrement), the second corresponds to an account of second-order scattering effects, and the third, finally, to an account of four-plasmon decay effects. According to (12),

$$P(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3) = {}^1/_{24}\,\pi\delta(\mathbf{x}+\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}_3)\,\delta(x^2+x_1^2-x_2^2-x_3^2)$$
$$\times[\cos\widehat{\mathbf{x}\mathbf{x}_2}\cos\widehat{\mathbf{x}_1\mathbf{x}_3} + \cos\widehat{\mathbf{x}_1\mathbf{x}_2}\cos\widehat{\mathbf{x}\mathbf{x}}]^2. \tag{19}$$

The concrete form of the kernel $K(\mathbf{x}\mathbf{x}_1)$ depends essentially on the region of wave numbers x of interest to us. In particular if we confine ourselves here to the case of sufficiently long-wave oscillations with $k < (\omega_{0e}/v_{Te})(m_eT_i/m_iT_e)^{1/2}$, when the decisive role is played by scattering by the ions, then, according to [1,2], the expression for $K(\mathbf{x}\mathbf{x}_1)$ can be represented in the form[5])

[5])See also[4].

$$K(\mathbf{x}\mathbf{x}_1) = -\varepsilon \frac{x^2 - x_1^2}{|\mathbf{x} - \mathbf{x}_1|} \cos^2 \widehat{\mathbf{x}\mathbf{x}_1},$$

$$\varepsilon = \frac{3\sqrt{\pi}}{4\sqrt{2}} \left(\frac{e_i^2 m_i T_i}{e_e^2 m_e T_e} \right)^{1/2} \frac{T_e^2}{(T_i + T_e)^2}. \qquad (20)$$

We now turn to the case when the initial distribution of the noise is isotropic and consequently $\psi(\mathbf{x}) = \psi(x)$. Going over in (18) to a new unknown function $\varphi(x) = 4\pi x^2 \psi(x)$ and integrating (18) over the angles in $\mathbf{x}$-space, we obtain

$$\frac{d\varphi}{d\tau} = \gamma\varphi + \varphi \int \bar{K}(xx_1)\varphi(x_1)\,dx_1 + \int \bar{P}(xx_1x_2x_3)[\varphi(x_1)\varphi(x_2)\varphi(x_3) + \varphi(x)\varphi(x_2)\varphi(x_3) - \varphi(x)\varphi(x_1)\varphi(x_2) - \varphi(x)\varphi(x_1)\varphi(x_3)]\,dx_1dx_2dx_3, \qquad (21)$$

where

$$\bar{K}(xx_1) = \varepsilon \frac{x_1 - x}{3}\left(1 + \frac{x_1}{x}\right) \times \begin{cases} \dfrac{5x^2 + 2x_1^2}{5x^2} & \text{for } x > x_1 \\ x\dfrac{5x_1^2 + 2x^2}{5x_1^2} & \text{for } x < x_1 \end{cases},$$

$$\bar{P}(xx_1x_2x_3) = \frac{\pi\sqrt{2}}{12} \frac{\delta(x^2 + x_1^2 - x_2^2 - x_3^2)}{xx_1x_2x_3} F(xx_1x_2x_3),$$

$$F(xx_1x_2x_3) = c \begin{cases} \left.\begin{matrix} x \text{ for } x < x_1 \\ x_1 \text{ for } x_{\_} > x_1 \end{matrix}\right\}, \text{ when } x_2x_3 > xx_1 \\ \left.\begin{matrix} x_2 \text{ for } x_2 < x_3 \\ x_3 \text{ for } x_2 > x_3 \end{matrix}\right\}, \text{ when } x_2x_3 < xx_1 \end{cases} \qquad (22)$$

and c is some slow function of x, x_1, x_2, and x_3 of the order of unity.

A. As the first example we consider the interaction between narrow wave packets. Thus, we assume that at the initial instant of time $\varphi(x) = A(0)\delta(x - a) + B(0)\delta(x - b)$ where, say, $a \leq b$, and see to what effects an account of the four-plasmon decays will lead[6]. It is easy to verify that the four-plasmon interaction leads to the decay of any two given packets with $x = a_1$ and $x = a_2 > a_1$, generally speaking, into four packets, two of which have the same $x = a_1, a_2$, while the two others (satellites) are located at the point[7] $a_1' = (2a_1^2 - a_2^2)^{1/2}$ and $a_2' = (2a_2^2 - a_1^2)^{1/2}$.

If we confine ourselves to sufficiently short times, so long as the amplitudes of the satellites are still small, so that we can neglect the interaction between the satellites themselves and between the satellites and the main packets, then we obtain the following expressions for the amplitudes $A'(t)$, $A(t)$, $B(t)$, and $B'(t)$ where B' is the amplitude of the "violet" satellite with $x = (2b^2 - a^2)^{1/2}$ and A' is the amplitude of the "red" satellite with $x = (2a_2 - b^2)^{1/2}$:

$$\frac{dA'}{d\tau} = \gamma(a')A' + \hat{k}_1 A^2 B, \quad \frac{dB'}{d\tau} = \gamma(b')B' + \hat{k}_2 AB^2,$$

$$\frac{dA}{d\tau} = \gamma(a)A + AB[q + \hat{k}_2 B - 2\hat{k}_1 A],$$

$$\frac{dB}{d\tau} = \gamma(b)B + AB[-q + \hat{k}_1 A - 2\hat{k}_2 B], \qquad (23)$$

where

$$a' = (2a^2 - b^2)^{1/2}, \quad b' = (2b^2 - a^2)^{1/2},$$

$$\hat{k}_1 = \frac{\pi}{12\sqrt{2}} c \frac{(2a^2 - b^2)^{1/2}}{a^2 b} \leqslant \hat{k}_2 = \frac{\pi}{12\sqrt{2}} \frac{c}{b^2},$$

$$q = \varepsilon \frac{b^2 - a^2}{3} \frac{5b^2 + 2a^2}{5b^3}.$$

It follows hence that the most important role is played by the nonlinear effects only in the region of small x, where the linear damping decrement is small or is in general equal to zero, and that the four-plasmon interaction need be taken into account only for sufficiently close-lying wave packets, when

$$b - a < \frac{A}{b^2} \lesssim \frac{1}{b^2}. \qquad (24)$$

In accordance with this, we analyze the solutions of the system (23) in the case when the linear damping and the scattering processes can be neglected, and when the evolution of the packets is determined only by the four-plasmon decays with characteristic time of the process $\tau_4 \approx \omega_{0e}^{-1} \times (kr_D n_e T_e/U)^2$, where U is the total energy of the plasmons in the packet.

Neglecting in (23) the terms proportional to γ and q, and recognizing that in this case the total energy and the total number of plasmons are first integrals, we obtain the relations

$$A(\infty) = B(\infty) = 0, \quad A'(\infty) = {}^1\!/_3(2A(0) + B(0)), \quad B'(\infty) = {}^1\!/_3(2B(0) + A(0))$$

for $2a^2 > b^2$, and

$$A'(\infty) = B(\infty) = 0, \quad A(\infty) = {}^1\!/_2(B(0) + 2A(0)),$$

$$B'(\infty) = {}^1\!/_2 B(0)$$

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[6]An account of only scattering leads, as can be readily seen, only to a complete transfer of energy from the packet with large k to a packet with small k, without exciting any waves with other values of k.

[7]If $2a_1^2 - a_2^2 < 0$, then only a "violet" satellite is produced with $x = (2a_2^2 - a_1^2)^{1/2}$.

for $2a^2 < b^2$, characterizing the redistribution of the energy over the spectrum [8].

We see thus that two wave packets decay into four (or three) wave packets with wave numbers lying outside the interval [ab]. If we now take into account the interaction between the two produced satellites, and the interactions between these satellites and the main packets, then we find that this leads to the occurrence, generally speaking, of three more red satellites with

$$x = (5a^2 - 4b^2)^{1/2}, \quad (4a^2 - 3b^2)^{1/2}, \quad (3a^2 - 2b^2)^{1/2}$$

and three "violet" ones with

$$x = (3b^2 - 2a^2)^{1/2}, \quad (4b^2 - 3a^2)^{1/2}, \quad (5b^2 - 4a^2)^{1/2}.$$

The latter, in turn, lead to the occurrence of nine more red and nine more violet satellites, etc. As a result, the entire region outside the interval [ab], from $x = x_{min} = [M(a^2 - b^2) + a^2]^{1/2}$, where M is the maximum integer at which the radicand is still positive ($a < b$!), to $x \approx 1$, turns out to be filled with wave packets centered at the points

$$x = (l(a^2 - b^2) + a^2)^{1/2},$$
$$(l(b^2 - a^2) + b^2)^{1/2} \quad (l = 0, 1, 2, \ldots)$$

and the initial spectrum, containing initially only two lines $x = a, b$, gradually spreads out and becomes a line spectrum.

B. We now turn to an investigation of the evolution in time of one sufficiently narrow wave packet [9]. We first neglect the four-plasmon interaction, and take into account only scattering, after which we analyze the effects to which this interaction leads.

Thus, we assume that the spectral distribution of the noise is in the form of a narrow wave packet with maximum at the point $x = x_n$ and with characteristic width $\Delta x = \Delta x_n$. In this case, as follows from (22), accurate to small quantities of order $\Delta x_n/x_n$, the kernel $K(x, x_1)$ can be approximated by a simpler expression and we can put [10]

$$\overline{K}(x, x_1) = {}^{14}/_{15}\varepsilon(x - x_1). \quad (25)$$

[8] In actual fact, these relations are, of course, not exact, inasmuch as we have assumed in their derivation that A′, B′ ≪ A, B; these equations characterize only the tendency of the process.

[9] An analogous problem under different assumptions was considered by Galeev et al.[5]

[10] As will be shown below, an account of the nonlinear interaction causes the initial distribution of the noise to shift towards smaller wave numbers k, and the characteristic width of the spectral distribution decreases rapidly with time, and the spectrum assumes the form of a narrow wave packet. Therefore the case considered here has in practice a sufficiently broad range of application.

Then (21) takes the form

$$\frac{d\varphi}{d\tau'} = \varphi\Big[\gamma'(x) - x\int\varphi(x_1)\,dx_1 + \int x_1\varphi(x_1)\,dx_1\Big],$$
$$\tau' = {}^{14}/_{15}\varepsilon\tau, \qquad \gamma' = {}^{15}/_{14}\frac{\gamma}{\varepsilon}. \quad (26)$$

Hence

$$\varphi(x, \tau') = \varphi_0(x)\exp\Big[\gamma'\tau' - x\int_0^{\tau'}\alpha\,d\tau' + \int_0^{\tau'}\beta\,d\tau'\Big], \quad (27)$$

where $\varphi_0(x)$ is the initial noise distribution, and the quantities α and β are only functions of the time τ' and are determined from the equations

$$\alpha(\tau') = \int\varphi(x, \tau')\,dx = \exp\Big(\int_0^{\tau'}\beta\,d\tau'\Big)\int dx\,\varphi_0(x) \times \exp\Big(\gamma'\tau' - x\int_0^{\tau'}\alpha\,d\tau'\Big),$$

$$\beta(\tau') = \int x\varphi(x, \tau')\,dx = \exp\Big(\int_0^{\tau'}\beta\,d\tau'\Big) \times \int dx\,x\varphi_0(x)\exp\Big(\gamma'\tau' - x\int_0^{\tau'}\alpha\,d\tau'\Big). \quad (28)$$

Taking into account (28), we can eliminate completely the function $\beta(\tau)$ and represent the solution in the form

$$\varphi(x, \tau') = \varphi_0(x)\,\alpha(\tau')\exp\Big(\gamma'\tau' - x\int_0^{\tau'}\alpha\,d\tau'\Big) \times\Big[\int dx\,\varphi_0(x)\exp\Big(\gamma'\tau' - x\int_0^{\tau'}\alpha\,d\tau'\Big)\Big]^{-1} \quad (29)$$

where the unknown function $\alpha(\tau)$ is determined from the equation

$$\frac{d\alpha}{d\tau} = \int\varphi(x, \tau)\,\gamma'(x)\,dx. \quad (30)$$

Thus, the problem has been reduced to a solution of Eq. (30) for the function $\alpha(\tau)$.

By way of illustration, let us consider the case when the damping decrement does not depend on x, that is, $\gamma' = -\gamma_0' = \text{const}$, and the initial noise distribution is of the form

$$\varphi_0(x) = \frac{A_0 x^n}{n!\,x_0^{n+1}}\exp\Big(-\frac{x}{x_0}\Big), \quad A_0 = \int\frac{\omega_{0e}N_0(k)\,dk}{8\pi^3 n_e T_e} = \text{const}. \quad (31)$$

Substituting expression (31) in (29), we readily obtain

$$\varphi(x, \tau') = A_0\frac{x^n}{n!}\Big[\frac{\mu(\tau')}{x_0}\Big]^{n+1}\exp\Big(-\frac{x}{x_0}\mu(\tau') - \gamma_0'\tau'\Big),$$
$$\mu(\tau') = 1 + \frac{A_0 x_0}{\gamma_0'}[1 - \exp(-\gamma_0'\tau')]. \quad (32)$$

The maximum value of the spectral density $\varphi_m(\tau')$ is equal to

$$\varphi_m(\tau') \approx A_0 \frac{\mu(\tau')}{x_0} \frac{n^n e^{-n}}{n!} \exp(-\gamma_0'\tau')$$

and is reached at the point $x = x_n = nx_0/\mu(\tau')$, while the width of the wave packet is $\Delta x_n = x_0/\mu(\tau')$.

The time variation of the total energy U of the wave packet is obviously determined by the expression

$$U(\tau') = \int \omega_{0e}\left[1 + \frac{3}{2}\frac{k^2 v_{Te}^2}{\omega_{0e}^2}\right]\frac{N(k\tau)}{8\pi^3}dk$$
$$= n_e T_e A_0\left[1 + \frac{3}{2}(n+1)(n+2)\frac{x_0^2}{\mu(\tau')}\right]\exp(-\gamma_0'\tau'),$$

where the second turn takes into account the so-called nonlinear damping.

Thus, when $\tau < \gamma_0^{-1}$ the nonlinear effects can be quite appreciable and can lead to a narrowing of the wave packet and to its displacement towards smaller wave numbers k. Subsequently, however, that is, when $\tau > \gamma_0^{-1}$, the deformation of the packet practically ceases, and x_n and Δx_n reach their stationary values,

$$x_n(\infty) = \frac{nx_0\gamma_0}{\gamma_0 + x_0 A_0}, \qquad \Delta x_n = \frac{x_n(\infty)}{n},$$

and the packet attenuates without changing its shape, with a damping decrement given by the linear theory. In other words, when $\tau \gg \gamma_0^{-1}$ the nonlinear effects turn out to be insignificant, and the evolution of the packet can be described within the framework of the linear theory.

In the general case, when the damping decrement γ depends on x, the form of the function $\alpha(\tau)$ depends essentially on the form of functions $\gamma(x)$ and $\varphi_0(x)$, and Eq. (30) must be solved separately in each concrete case.

In the above study of the evolution of a narrow wave packet we have neglected completely the effects of four-plasmon decays. We now consider the conditions under which this can be done, and analyze qualitatively the phenomena which occur when these conditions are violated. An analysis of Eq. (21) shows that the presence of four-plasmon decays leads to a broadening of the initial spectrum, with the characteristic time of this broadening being, in the case of spherical symmetry, practically independent of the width of the spectrum. In particular, for a narrow wave packet of width Δx_n and with center at the point x_n, the characteristic spreading time τ_4 has, as can be readily seen[11], an order of magnitude

$$\tau_4 = \frac{1}{\omega_{0e}}\left(\frac{k_n r_D n_e T_e}{U}\right)^2, \tag{33}$$

that is, it is determined, apart from the packet energy, only by the coordinate k_n of its center.

On the other hand, as shown by the just performed investigation, the scattering processes lead to a narrowing of the wave packet and to a simultaneous shift of this packet to the region of small wave numbers k. The characteristic times of both these processes have the same order of magnitude and are equal to

$$\tau_{\text{scat}} = \frac{1}{\omega_{0e}}\frac{n_e T_e}{\Delta k_n r_D U}\varepsilon^{-1}. \tag{34}$$

It follows from (33) and (34) that decay processes can be neglected only when the coordinates of the center of the wave packet and its width satisfy the condition

$$k_n^2 \Delta k_n \gg U / n_e T_e \varepsilon r_D^3. \tag{35}$$

Thus, if at the initial instant of time the wave packet satisfies condition (35), then its evolution is at first determined only by scattering processes (and in the case of strong damping—in general by the linear theory), and the packet narrows down gradually and moves into the region of small wave numbers k until condition (35) is violated and four-plasmon decays come into play and prevent its further narrowing.

The ultimately obtained width of the packet Δk_n^{stat} can be estimated with the aid of (33), if it is recognized that, as we have just seen, the quantity $\Delta k_n/k_n$ is practically independent of the time. Denoting by Δk^0 the initial width of the packet and by k^0 the initial coordinate of its center, we get

$$\Delta k_n^{\text{stat}} = \left(\frac{\Delta k^0}{k^0}\right)^{2/3} r_D \left(\frac{U}{n_e T_e \varepsilon}\right)^{1/3}, \quad \frac{\Delta k_n^{\text{stat}}}{k_n^{\text{stat}}} \approx \frac{\Delta k^0}{k^0}. \tag{36}$$

C. In conclusion let us dwell briefly on the case of three-dimensional spectra. Since our aim here is not a more or less detailed analysis of (18) for the general case, we shall merely point out that, in analogy with the case of spherical symmetry, the scattering effects lead to a narrowing of three-dimensional wave packets and their displacement towards smaller k with a characteristic time

$$\tau_{\text{scat}} = \frac{1}{\omega_{0e}}\frac{n_e T_e}{\Delta k_n r_D U}\varepsilon^{-1}, \tag{37}$$

where Δk_n is the width of the packet. On the other hand, four-plasmon decay processes limit this narrowing, leading to a spreading of the packet, a result of which is the establishment of a certain stationary width.

The characteristic time of the four-plasmon in-

[11]Here $U = \int \frac{\Omega_e N_e d\mathbf{k}}{(2\pi)^3}$ is the total energy of the wave packet.

teraction for a three-dimensional packet can be readily estimated from (18); its order of magnitude is

$$\tau_4 = \frac{1}{\omega_{0e}}\left(\frac{\Delta k_n r_D n_e T_e}{U}\right)^2, \qquad (38)$$

that is, unlike the spherically symmetrical case, the time is directly proportional to the square of the width of the packet. It follows therefore that for narrow wave packets with $\Delta k_n \ll (U/\epsilon n_e T_e)^{1/3}$ an account of the processes of four-plasmon interaction is essential and, in particular, makes it possible to estimate the stationary width of the packet.

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