RUNAWAY ELECTRONS IN A NONEQUILIBRIUM PLASMA

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The kinetic equation for electrons in a plasma in a stationary electric field is solved. The interaction between the electrons and nonequilibrium oscillations of the plasma is taken into account. The flux of runaway electrons and the magnitude of the critical field are determined.

1. We consider a fully ionized plasma situated in a uniform electric field. We assume that non-equilibrium longitudinal oscillations are excited in the plasma, and that these oscillations are isotropically distributed. Then the kinetic equation for the electron distribution function $f(t, v, \theta)$ in the region of epithermal velocities is written in the form[1-4]

$$\frac{\partial f}{\partial t}+\frac{eE}{m}\left(\cos\theta\frac{\partial f}{\partial v}-\frac{\sin\theta}{v}\frac{\partial f}{\partial\theta}\right)$$
$$-\frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2\left[\left(\nu_e(v)\frac{T_{\text{ef}}}{m}+D(v)\right)\frac{\partial f}{\partial v}\right.\right.$$
$$\left.\left.+\left(\nu_e(v)v+\frac{F(v)}{m}\right)f\right]\right\}$$
$$-\left[\frac{1}{2}\nu(v)+B(v)\right]\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)=0. \quad (1)$$

Here e, m, and v are the charge, mass, and velocity of the electrons, $T_{\text{ef}} = (m\overline{v^2}/3)$ is the effective temperature of the electrons, E the intensity of the constant electric field, θ the angle between the directions of **E** and **v**, and $\nu_e(v)$ the frequency of collisions between an electron having a velocity v and other plasma electrons:

$$\nu_e(v) = 4\pi e^4 N_e \ln(mv^2 R_D e^{-2}) / m^2 v^3 = \nu_0 / v^3, \quad (2)$$

where R_D is the Debye radius and N_e the electron concentration. Further,

$$\nu(v) = \nu_e(v)\left\{1-\frac{T_{e\,\text{ef}}}{mv^2}+\sum_Z Z^2\frac{N_Z}{N_e}\right\}, \quad (3)$$

where N_Z is the concentration of ions with charge Z ($N_e = \Sigma_Z Z N_Z$). Finally, D(v), F(v), and B(v) are coefficients characterizing the interaction of the electron with non-equilibrium plasma oscillations.

[1]For convenience, we divide here the diffusion coefficients into parts due to pair collisions and to interaction with waves. We can, of course, combine these parts and consider unified diffusion coefficients, a contribution to which is made both by the long-wave ($k < 1/R_D$) and the short-wave ($k > 1/R_D$) parts of the spectrum.

The coefficient F(v) does not depend in this case on the oscillation intensity. It is equal to

$$F(v) = \frac{4\pi e^4 N_e}{mv^2}\ln\frac{v}{v_T}, \quad (4)$$

where v_T is the average electron velocity.

If the velocities of the fast electrons considered here are essentially larger than the phase velocities of the waves, then the coefficients D(v) and B(v) can be represented in the form

$$D(v) = D_0 / v^3, \quad B(v) = B_0 / v^3 - D_0 / 2v^5. \quad (5)$$

The constants D_0 and B_0 are determined here by the following integrals over the spectrum of the long-wave oscillations:

$$D_0 = 2e^2\hbar\int\frac{N(k)\,\omega^2(k)}{k\,[\partial\varepsilon/\partial\omega]_{\omega(k)}}dk, \quad (6)$$

$$B_0 = e^2\hbar\int\frac{kN(k)}{[\partial\varepsilon/\partial\omega]_{\omega(k)}}dk. \quad (7)$$

Here N(k) is the spectral density of the oscillations and $\omega(k)$ is the frequency of the waves, i.e., the root of the dispersion equation $\epsilon(\omega, k) = 0$. The integration in (6) and (7) is carried out over the entire long-wave spectrum of the oscillations $k \lesssim 1/R_D$, where R_D is the Debye radius. If both plasma and ion-sound waves are excited, then it is necessary to sum in (6) and (7) over both types of oscillations. We shall henceforth neglect the variation of the logarithmic term in (2) and (4), setting under the logarithm sign $v = v_c$, where v_c is the velocity characteristic of the electrons in question, $v_c \sim v_T\gamma^{-1/2}$ (see below).

We now change to new variables:

$$u = v\left(\frac{T_{\text{ef}}}{m}+\frac{D_0}{\nu_0}\right)^{-1/2}(1+\Delta)^{1/2},$$

$$\mu = \cos\theta, \tau = t\nu_0(1+\Delta)^{5/2}\left(\frac{T_{\text{ef}}}{m}+\frac{D_0}{\nu_0}\right)^{-3/2},$$

$$\Delta = \ln(v_c/v_T)/\ln(mv_c^2R_D/e^2) \ll 1. \quad (8)$$

Equation (1) is then rewritten in the form

$$\frac{\partial f}{\partial \tau}+\gamma\left(\mu\frac{\partial f}{\partial u}+\frac{1-\mu^2}{u}\frac{\partial f}{\partial \mu}\right)-\frac{1}{u^2}\frac{\partial}{\partial u}\left(\frac{1}{u}\frac{\partial f}{\partial u}+f\right)$$
$$-\frac{\alpha}{u^3}\left(1-\frac{\beta}{u^2}\right)\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial f}{\partial \mu}\right]=0. \quad (9)$$

Here γ, α, and β are dimensionless parameters

$$\gamma = eE(T_{\mathrm{ef}}+mD_0/\nu_0)/m\nu_0(1+\Delta)^2, \quad (10)$$

$$\alpha=\left(\frac{1}{2}+\frac{1}{2}\sum_Z Z^2\frac{N_Z}{N_e}+\frac{B_0}{\nu_0}\right)(1+\Delta)^{-1}, \quad \beta={}^1/_2\alpha. \quad (11)$$

The parameter γ characterizes the value of the electric field intensity. The parameter α characterizes the relative role of the angular scattering of the electrons. In particular, in an equilibrium singly-ionized plasma we have $D_0/\nu_0 = T_e\Delta/m$, $\alpha = 1$, and $\gamma = E/E_{c0}$, where

$$E_{c0} = m\nu_0/eT_e = 4\pi e^3 N_e \ln\Lambda / T_e, \quad (12)$$

E_{c0} is the critical field ($\ln\Lambda$ is the Coulomb logarithm). Equation (9) coincides in this case with that considered earlier.[1,11] If the plasma contains an appreciable number of multiply charged ions, then

$$\alpha=\frac{1}{2}+\frac{1}{2N_e}\sum_z Z^2N_Z.$$

In a non-equilibrium plasma with $T_e \gg T_i$, even the scattering of the electrons by the ion-sound oscillations of the plasma is already significant. In this case we have as before $\gamma = E/E_{c0}$, and

$$\alpha=\frac{1}{2}+\frac{1}{2N_e}\sum_Z Z^2N_Z+\frac{ZI(T_e/T_i,Z)}{4\ln\Lambda},$$

where I is the function considered by Silin and Gorbunov,[5] with approximate value

$$I\approx\frac{ZT_e}{2T_i}\frac{1}{\ln(Z^2T_e^3M/T_i^3m)}.$$

The parameter α is always larger than unity (or equal to it). The parameter β, to the contrary, is smaller than $1/2$. Inasmuch as we are considering below only the region of high velocities $u^2 \gg 1$, the term β/u^2 can always be neglected.

It was shown earlier[6,7] that to determine the distribution function of the electrons in a plasma which is situated in a constant electric field it is necessary to find the stationary solution of Eq. (9) with a constant total particle flux in velocity space (the flux of the runaway electrons)

$$S=-2\pi\int_{-1}^{1}\left(f+\frac{1}{u}\frac{\partial f}{\partial u}-\gamma\mu u^2 f\right)d\mu=-S_0. \quad (13)$$

The flux of the runaway electrons S_0 is then determined by the boundary condition

$$f_{u\to\infty}\to 0. \quad (14)$$

We shall find below the stationary solution of Eq. (9) with boundary conditions (13) and (14) for arbitrary values of the parameter $\alpha \geq 1$. We note that solutions of this type are meaningful only when the flux of the runaway electrons is small, $S_0 \ll N_e$. For $\alpha \geq 1$ this condition, as will be shown below, is satisfied if $\gamma \ll \alpha^{1/2}$.

2. Proceeding to solution of (9), we consider first the case

$$\alpha\gamma \gg 1. \quad (15)$$

In this case the angular scattering of the electrons is large, $\alpha \gg 1$, so that the direction of the electron velocity changes more vigorously as a result of collisions than the magnitude of the velocity. We can therefore expect the distribution function to be close to spherically symmetrical in an extensive region of velocities. The solution of (9) can then be naturally sought in the form of a series in Legendre polynomials $P_i(\mu)$:

$$f(u,\mu,\tau)=\sum_{i=0}^{\infty}f_i(u,\tau)P_i(\mu). \quad (16)$$

Substituting the expansion (16) in (9) and using the orthogonality and other properties of Legendre polynomials, we obtain for the functions $f_i(u, \tau)$ in place of (9) the following chain of coupled equations:

$$\frac{\partial f_i}{\partial \tau}+\gamma\left[\frac{i}{2i-1}u^{i-1}\frac{\partial}{\partial u}(u^{1-i}f_{i-1})+\frac{i+1}{2i+3}u^{-i-2}\frac{\partial}{\partial u}(u^{i+2}f_{i+1})\right]$$
$$-\frac{1}{u^2}\frac{\partial}{\partial u}\left[\frac{1}{u}\frac{\partial f_i}{\partial u}+f_i\right]+\frac{\alpha}{u^3}i(i+1)f_i=0. \quad (17)$$

It was already indicated above that in the case considered here we can expect the distribution function to be close to spherically symmetrical. This means that in the chain of Eqs. (17) we can confine ourselves to the fundamental harmonics f_0 and f_1. Neglecting therefore terms of order f_2 in comparison with f_0 in the equation for f_1, and neglecting also small terms of order $1/\alpha$, we get

$$\frac{\partial f_0}{\partial \tau}+\frac{\gamma}{3u^2}\frac{\partial}{\partial u}(u^2f_1)-\frac{1}{u^2}\frac{\partial}{\partial u}\left\{\frac{1}{u}\frac{\partial f_0}{\partial u}+f_0\right\}=0, \quad (18)$$

$$\frac{\partial f_1}{\partial \tau}+\gamma\frac{\partial f_0}{\partial u}+\frac{2\alpha}{u^3}f_1=0. \quad (19)$$

The stationary solution of (18) and (19) is of the form

$$f_1=-\frac{\gamma}{2\alpha}u^3\frac{\partial f_0}{\partial u},$$

$$f_0=C_0\exp\left\{-\int_0^u\frac{u\,du}{1+u^6\gamma^2/6\alpha}\right\}-C_1$$

$$=C_0\exp\left\{-\frac{\alpha^{1/3}}{(6\gamma)^{2/3}}\left[\frac{1}{2}\ln\frac{(w+1)^2}{w^2-w+1}\right.\right.$$

$$+\sqrt{3}\operatorname{arc\,tg}\frac{\sqrt{3}w}{2-w}\Big]\Big\}-C_1, \qquad (20)^*$$

$$w=u^2(\gamma^2/6\alpha)^{1/3}.$$

The constant C_0 is determined here by the normalization conditions

$$C_0=N_e/(2\pi)^{3/2},$$

where N_e is the electron density in the main velocity region, i.e., at $u^2 \lesssim (6\alpha/\gamma^2)^{1/3}$. The constant C_1 is proportional to the flux of the runaway electrons: $C_1 = 4\pi S_0$.

Using condition (14), we find

$$S_0=\sqrt{\frac{2}{\pi}}N_e\exp\Big\{-\int_0^\infty\frac{u\,du}{1+\gamma^2u^6/6\alpha}\Big\}$$
$$=\sqrt{\frac{2}{\pi}}N_e\exp\Big\{-\frac{2^{1/3}\pi}{3^{7/6}}\Big(\frac{\alpha}{\gamma^2}\Big)^{1/3}\Big\}. \qquad (21)$$

In the case in question, consequently, the flux of runaway electrons is proportional to

$$\exp(-\mathrm{const}\,\alpha^{1/3}/E^{2/3}).$$

We have neglected above all the harmonics except f_0 and f_1. Substituting the obtained solution (20) in (17), we verify that with condition (15) the solution (20) is valid in the region of velocities $u^2 \ll \alpha/\gamma$. The flux of runaway electrons (21) is formed in the velocity region $u^2 \sim (\alpha/\gamma^2)^{1/3}$. It is seen therefore that the distribution function remains spherically symmetrical in that region of velocities where the flux of the runaway electrons is formed, provided condition (15) is satisfied. In the opposite case, the distribution in this velocity region has a directional character and the flux is no longer determined by (21); this case will be considered in the next section.

In (18) and (19) we took into account only the principal terms. Using now the complete system of equations (17) for the calculations, we can find the correction to the expression (20) for the distribution function. It takes the form of a factor

$$\exp\Big\{-\frac{11}{30(6\alpha)^{1/3}\gamma^{4/3}}\int_0^{\gamma^2u^6/6\alpha}\frac{w^{5/3}\,dw}{(1+w)^4}\Big\}. \qquad (22)$$

The expression for the flux of runaway particles, with account of the correction factor (22), is written in the form

$$S_0=\sqrt{\frac{2}{\pi}}N_e\exp\Big\{-\frac{2^{1/3}\pi}{3^{7/6}}\Big(\frac{\alpha}{\gamma^2}\Big)^{1/3}\Big[1+\frac{11}{81(6\alpha\gamma)^{2/3}}\Big]\Big\}. \qquad (23)$$

Under the conditions (15) the correction term is always much smaller than the principal term.

We now proceed to a solution of (9) in the region of high velocities $u^2 \sim \alpha/\gamma$. We introduce in place of u a new variable $x = u^2\gamma/\alpha$. The solution will be sought, as before, in the form of a series in Legendre polynomials (16). Replacing in (17) the variable u by x and omitting small terms we find

$$f(x,\mu)=\frac{3S_0}{4\pi\alpha}\sum_{i=0}^{\infty}f_i(x)P_i(\mu), \qquad (24)$$

where S_0 is the flux of the runaway electrons, defined above. Further,

$$f_0(x)=C_0+\frac{1}{2x^2}-\frac{2}{5}f_2(x)-\frac{3}{5}\int^x\frac{f_2}{x}dx,\quad f_1(x)=\frac{1}{x}, \qquad (25)$$

and the remaining functions $f_i(x)$ are determined by the chain of equations

$$-\frac{i(i-1)}{2i-1}xf_{i-1}+\frac{2i}{2i-1}x^2\frac{df_{i-1}}{dx}+\frac{(i+1)(i+2)}{2i+3}xf_{i+1}$$
$$+\frac{2(i+1)}{2i+3}x^2\frac{df_{i+1}}{dx}+i(i+1)f_i=0. \qquad (26)$$

We can seek the solution of the chain of equations (26) in the form of a series in powers of x:

$$f_i=\sum_{k=0}^{\infty}a_{ik}x^{i+2k-2}. \qquad (27)$$

Substituting the series (27) in (26), we arrive at the following relation between the coefficients a_{ik}:

$$\frac{i[i+4k-5]}{2i-1}a_{i-1,k}+\frac{(i+1)(3i+4k-4)}{2i+3}a_{i+1,k-1}$$
$$+i(i+1)a_{ik}=0. \qquad (28)$$

From relations (28), with account of (25), we can readily determine the coefficients a_{ik}. In fact, for $k = 0$ we get from (28)

$$a_{i0}=-\frac{i-5}{(i+1)(2i-1)}a_{i-1,0},$$

or, taking (25) into account

$$a_{20}=\frac{1}{3},\qquad a_{30}=\frac{1}{30},$$
$$a_{40}=\frac{1}{1050},\qquad a_{50}=a_{60}=\ldots=0.$$

We then determine analogously the coefficients a_{ik}, etc.

Accurate to terms of order x^4, the functions $f_i(x)$ take on the form

$$f_0=\frac{1}{2}x^{-2}+C_0-\frac{1}{5}\ln x+\frac{1}{100}x^2+\frac{11}{25200}x^4,\quad f_1(x)=x^{-1},$$

$$f_2(x)=\frac{1}{3}-\frac{1}{70}x^2-\frac{1}{1260}x^4,\qquad f_3(x)=\frac{1}{30}x+\frac{1}{900}x^3,$$

$$f_4(x)=\frac{1}{1050}x^2-\frac{1}{10500}x^4. \qquad (29)$$

The constructed solution is valid, of course, only for relatively small values of x. When $x \gg 1$ the distribution function has, as in [7,8], a sharply directional character

*arc tg = $\tan^{-1}$.

$$f=\frac{S_0}{2\pi\alpha\ln x}\exp\left\{\frac{x}{\ln x}(\mu-1)\right\}. \tag{30}$$

The dependence of the distribution function on the angle θ for different values of x is shown in Fig. 1. It is seen from the figure that at small values of x the distribution function is close to spherically symmetrical. When $x \sim 1$ it is gradually transformed into one which has sharp directivity along the electric field. The distribution function (30) and the functions (24) in (29) were joined together at $x = 3$. The constant C_0 in (29) was chosen here such that the values of the functions coincided at $\theta = 0$. The distribution (30) with $x = 3$ is shown in Fig. 1 by the solid line, while the distribution (29) is shown dashed; we see that for other values of θ they also agree quite well.

Thus, the distribution function in the region of the values of the velocity $u^2 \ll \alpha/\gamma$ is close to spherically symmetrical, and for $u^2 \gg \alpha/\gamma$ it is, to the contrary, sharply directional. The condition $u_c^2 = \alpha/\gamma$ denotes the precise equality of the force of friction of the electrons in the plasma, $F_{fr} = \alpha m\nu_e(v)v$ and the force eE exerted on the electrons by the constant electric field. In other words $u_c = \sqrt{\alpha/\gamma}$ is, in an elementary definition,[9-11] the limit of electron runaway. Near this limit there occurs, however, only a transformation of the distribution function from symmetrical to directional. The flux of the runaway electrons is formed much earlier, at $u^2 \sim \alpha^{1/3}\gamma^{-2/3}$.

3. We now consider the opposite limiting case

$$\alpha\gamma \ll 1. \tag{31}$$

Since $\alpha \geq 1$ always, the condition (31) is realized only for small values of the parameter

$$\gamma \ll 1. \tag{32}$$

This case is close to the case $\alpha = 1$ considered in [1,7,8]. Further calculations are perfectly analogous to those carried out in [7]. We therefore

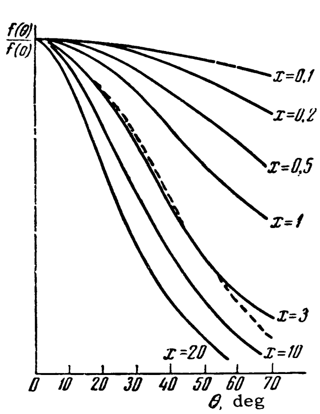

FIG. 1

present here only the results of the corresponding calculations.

At not very high values of the velocities $u < (\alpha/\gamma)^{1/4}$ the distribution function is close to symmetrical. The solution of Eq. (9) in the vicinity $u \lesssim (\alpha/\gamma)^{1/4}$ is obtained in the form of a series in Legendre polynomials (16) and (17). The functions $f_i(u)$ are determined by expanding them in series of powers of $y = \gamma u^4/\alpha$. In this region the distribution function is of the form

$$f(y,\mu)=f_0\sum_{i,k=0}^{\infty}a_{ik}y^{2k+i}P_i(\mu), \tag{33}$$

where $a_{0k} = \delta_{0k}$, and all the succeeding coefficients are determined by the relations

$$a_{i,k}=\frac{ia_{i-1,k}}{(2i-1)[i(i+1)+4(i+2k)/\alpha]}+\frac{(i+1)a_{i+1,k-1}}{(2i+3)[i(i+1)+4(i+2k)/\alpha]}-\frac{1}{3}\left[i(i+1)+\frac{4}{\alpha}(i+2k)\right]^{-1}\sum_{\rho=0}^{k-1}a_{i\rho}a_{1,k-\rho-1}. \tag{34}$$

The coefficients a_{ik} are listed in Table I (for different values of α).

We see from Table I that with increasing indices the coefficients a_{ik} decrease very rapidly, so that the series (33) converge quite well at not too large values of y. The function f_0 is of the form

$$f_0(y)=\frac{N_e}{(2\pi)^{3/2}}\exp\left\{-\frac{u^2}{2}+\frac{\alpha}{24}\sum_{k=0}^{\infty}\frac{a_{1k}}{1+k}y^{2+2k}\right\}-\frac{S_0}{4\pi}. \tag{35}$$

We see from this that when $y \ll 1$ the distribution function is symmetrical and is nearly Maxwellian. When $y \gtrsim 1$ the function f_0 deviates noticeably from Maxwellian. At the same time, the distribution function becomes oriented along the electric field.

When $u \gg (\alpha/\gamma)^{1/4}$ (i.e., $y \gg 1$) the distribution function becomes sharply directional and expansion in Legendre polynomials is ineffective. In this region we can use expansion in powers of the small parameter $\alpha\gamma$ in the exponential, i.e., seek a solution in the form

$$f=\frac{N_e}{(2\pi)^{3/2}}\exp\left\{\frac{1}{\gamma}[\varphi_0(z,\mu)+(\alpha\gamma)^{1/2}\varphi_1(z,\mu)+\alpha\gamma\varphi_2(z,\mu)+\ldots]\right\}, \tag{36}$$

where $z = \gamma u^2$. The functions φ_0, φ_1, and φ_2 are of the form $\varphi_0 = -z/2 + z^2/4$

$$\varphi_1=\frac{2^{3/2}}{\alpha}z(1-z)^{1/2}\left(\sqrt{\frac{1+\mu}{2}}-1\right)-2^{1/2}[1-(1-z)^{1/2}],$$

Table I

	$\alpha=1$	$\alpha=2$	$\alpha=4$	$\alpha=10$	$\alpha\gg 1$
a_{10}	$1.66\cdot 10^{-1}$	$2,50\cdot 10^{-1}$	$3.33\cdot 10^{-1}$	$4.16\cdot 10^{-5}$	$5.00\cdot 10^{-1}$
a_{20}	$8.08\cdot 10^{-3}$	$1.67\cdot 10^{-2}$	$2.78\cdot 10^{-2}$	$4.10\cdot 10^{-2}$	$5.56\cdot 10^{-2}$
a_{30}	$1,98\cdot 10^{-4}$	$5.56\cdot 10^{-4}$	$1.11\cdot 10^{-3}$	$1.86\cdot 10^{-3}$	$2.78\cdot 10^{-3}$
a_{40}	$3.15\cdot 10^{-6}$	$1.13\cdot 10^{-5}$	$2.64\cdot 10^{-5}$	$4,93\cdot 10^{-5}$	$7,94\cdot 10^{-5}$
a_{11}	$-4,34\cdot 10^{-4}$	$-1,77\cdot 10^{-3}$	$-5,36\cdot 10^{-3}$	$-1.30\cdot 10^{-2}$	$3,05\cdot 10^{-2}$
a_{21}	$-2,97\cdot 10^{-5}$	$-1,67\cdot 10^{-4}$	$-6.19\cdot 10^{-4}$	$-1.79\cdot 10^{-3}$	$-4,74\cdot 10^{-3}$
a_{31}	$-8.56\cdot 10^{-7}$	$-6,42\cdot 10^{-6}$	$-2,83\cdot 10^{-5}$	$-9,35\cdot 10^{-5}$	$-2,73\cdot 10^{-5}$
a_{12}	$1.65\cdot 10^{-6}$	$1.90\cdot 10^{-5}$	$1.35\cdot 10^{-4}$	$7,23\cdot 10^{-4}$	$4.12\cdot 10^{-3}$
a_{22}	$1.19\cdot 10^{-7}$	$1.87\cdot 10^{-6}$	$1.64\cdot 10^{-5}$	$1.03\cdot 10^{-4}$	$6.63\cdot 10^{-4}$
a_{13}	$-6.62\cdot 10^{-9}$	$-2,15\cdot 10^{-7}$	$-3.68\cdot 10^{-6}$	$-4.49\cdot 10^{-5}$	$-7,30\cdot 10^{-4}$
C_0	0.13	0.12	0.085	0.033	—0.012

$$\varphi_2=\frac{z(5z-3)}{2\alpha^2(1-z)}(\mu-1)-\frac{1}{4\alpha}\ln\frac{1+\mu}{2}$$
$$-\frac{\alpha+4-6z}{2\alpha^2}\ln\frac{3+\mu+2^{3/2}(1+\mu)^{1/2}}{8}$$
$$+\frac{2^{1/2}z}{\alpha^2(1-z)}(\alpha+4-6z)[(1+\mu)^{1/2}-2^{1/2}]$$
$$+\frac{1}{4\alpha}\ln\frac{z^2}{1-z}+\frac{1}{4}\left[\frac{2}{1-z}+\ln\frac{z}{1-z}\right]$$
$$-\left(\frac{1}{8}+\frac{1}{4\alpha}\right)\ln(\alpha\gamma)+C_0(\alpha). \qquad (37)$$

The constant $C_0(\alpha)$ is determined by joining together the distribution functions (36) and (33). Its values are listed in Table I. The distribution function (36) deviates strongly from Maxwellian. It has a sharp directional character; however, as $z \to 0$ and $z \to 1$, the directivity of the distribution (36) becomes weaker.

In the vicinity of the point $z = 1$ the series (36) diverges. Here we can find the solution (9) by changing over to a new variable $t = (z - 1)/(2\alpha\gamma)^{1/3}$, expanding the function in the experiment in powers of the small parameter $\epsilon = (2\alpha\gamma)^{1/3}$ and in powers of $(\mu - 1)$:

$$f=\frac{N_e}{(2\pi)^{3/2}}\exp\left\{\frac{1}{\varepsilon}\left[(\psi_0(t)+\varepsilon\psi_1(t)+\ldots)\right.\right.$$
$$+(\psi_0'(t)+\varepsilon\psi_1'(t)+\ldots)(\mu-1)$$
$$\left.\left.+(\psi_0''(t)+\ldots)\frac{(\mu-1)^2}{2}+\ldots\right]\right\}-\frac{S_0}{4\pi}. \qquad (38)$$

As a result of the calculation we find that the function $\psi_0(t)$ is given by the equation

$$\psi_{0t}(\psi_{0t}-t)^2=-1,\ \psi_{0t}=d\psi_0/dt. \qquad (39)$$

The remaining functions are expressed in terms of ψ_{0t} with the aid of the following relations:

$$\frac{d\psi_1}{dt}=-\frac{3\psi_{0t}+(2/\alpha-1)t}{2(3\psi_{0t}-t)^2},$$

$$\psi_1'=-\frac{t}{\psi_{0t}-t}+\frac{2\psi_{0t}}{\alpha(3\psi_{0t}-t)}$$
$$-\frac{(2\psi_{0t}-t)(3\psi_{0t}+2t/\alpha-t)}{2(3\psi_{0t}-t)^2},$$

$$\psi_0'(t)=\psi_{0t}(\psi_{0t}-t),$$

$$\psi_0''=\frac{1}{4(\psi_{0t}-t)}-\frac{1}{2\alpha(3\psi_{0t}-t)}. \qquad (40)$$

The integration constants in (39) and (40) are chosen from the condition that the distribution functions (38) and (36) coincide in the region $-t \gg 1$, where both expansions are valid.

The function $\psi_0(t)$ is shown in Fig. 2 of [7]. (It does not depend on α.) It is seen from that figure that the function $\psi_0(t)$ tends to a constant value at large values $t \gg 1$. The function $\psi_1(t)$ behaves similarly. Consequently in this region $t \gg 1$ $(u^2 > 1/\gamma)$ a flux of runaway electrons is formed. From the condition (14) we find its value:

$$S_0=\sqrt{\frac{2}{\pi}}N_e\exp\left\{-\frac{1}{4\gamma}-\sqrt{\frac{2\alpha}{\gamma}}\right\}. \qquad (41)$$

The flux of runaway electrons is proportional to

$$\exp\{-\mathrm{const}/E-\mathrm{const}\cdot\sqrt{\alpha}/\sqrt{E}\}.$$

When $\alpha = 1$ it coincides with that calculated in [1].

The distribution function (38) has a directional character. However, with increasing t, its directivity weakens and when $t \to \infty$, i.e., when $z \gg 1$, the distribution should again become symmetrical. It is therefore again natural to use here the expansion (16) in the Legendre polynomials. We then get from (17)

$$f_0=\frac{S_0}{4\pi}\left(\exp\left[\frac{3\alpha}{2z^2}\right]-1\right);\qquad f_1=\frac{3S_0}{4\pi z}\exp\left[\frac{3\alpha}{2z^2}\right]$$
$$f_2=\frac{S_0}{4\pi\alpha}\left(1+\frac{2\alpha}{z^2}\right)\exp\left[\frac{3\alpha}{2z^2}\right], \qquad (42)$$

where the flux S_0 is given by the expression (41). We see therefore that in the region $z \gg 1$ the distribution function is actually symmetrical, i.e., $f_0 \gg f_1 \gg f_2 \ldots$ The ratio f_0/f_1 increases to values $z \sim \sqrt{\alpha}$ when $f_0/f_1 \sim \sqrt{\alpha}$; with increasing z it again begins to decrease. When $z \gg \sqrt{\alpha}$ distribution (42) coincides with (24) and (29) with $x = u^2\gamma/\alpha = z/\alpha \ll 1$.

When $z \sim \alpha$, i.e., $x \sim 1$, the distribution function is again transformed from symmetrical to directional. The distribution function is described here by expressions (24), (29), and (30) which were analyzed in the preceding section. Of course, with decreasing α the region where the distribution function is symmetrical, $\alpha/\gamma > u^2 > 1/\gamma$ becomes narrower; when $\alpha \sim 1$ it actually disappears completely. In this case the distribution function for $u^2 = 1 + \Delta/\gamma$, where $\Delta \sim 1$, is of an intermediate nature, being neither symmetrical nor sharply directional. For the case $\alpha = 1$ it was investigated earlier[7,8] also in this intermediate region.

Thus, under the conditions (31) considered here, the distribution function for $u^2 < (\alpha/\gamma)^{1/2}$ is symmetrical; when $1/\gamma > u^2 > (\alpha/\gamma)^{1/2}$ it has a sharply directional character. When $\alpha/\gamma > u^2 > 1/\gamma$ it is again symmetrical, and when $u^2 > \alpha/\gamma$ it is again sharply directional. The runaway limit in the elementary definition, as before, is $u_c^2 = \alpha/\gamma$. At this limit there occurs a second transformation of the distribution function from symmetrical to directional. The flux of runaway electrons is formed earlier in the region $u^2 \sim 1/\gamma$.

4. When $\alpha\gamma \sim 1$ it is natural to assume that the expansion (16) in Legendre polynomials is valid for $u^2 \lesssim \alpha/\gamma$. In this case, however, unlike the case $\alpha\gamma \gg 1$, considered in Sec. 2, it is necessary to use not two but three, four, or more polynomials. A numerical calculation of the runaway-electron flux made in this manner over a wide range of values of $\alpha\gamma$ has lead to good agreement with formula (21) for $\alpha\gamma \gg 1$, and to good agreement with formula (41) when $\alpha\gamma \ll 1$. The final expression for the flux of runaway electrons for arbitrary value of $\alpha\gamma$, accurate to a pre-exponential factor, can be represented in the form

$$-\frac{dN_e}{d\tau} = S_0 = \sqrt{\frac{2}{\pi}} N_e \exp\left\{-\frac{1}{4\gamma} F(\alpha\gamma)\right\}. \quad (43)$$

The function $F(x)$ is presented in Fig. 2 and in Table II. At large values $x \gg 1$

$$F(x) = \frac{4\pi \cdot 2^{1/3}}{3^{7/6}}\left[x^{1/3} + \frac{11}{81 \cdot 6^{2/3}} x^{-1/3} + \ldots\right]. \quad (44)$$

When $x \ll 1$

$$F(x) = 1 + (32x)^{1/2}.$$

The solution of Eq. (9) constructed here is valid only for a relatively weak electric field, when the

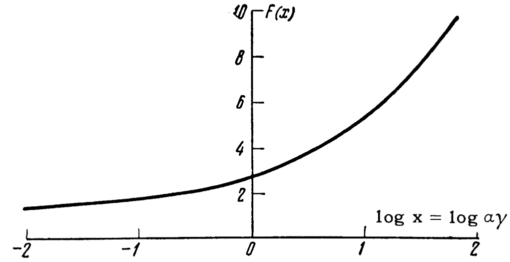

FIG. 2

flux of the runaway electrons is small $S_0 \ll N_e$. It is consequently necessary to have

$$F(\alpha\gamma) > 4\gamma. \quad (45)$$

This condition is equivalent to the condition $\gamma < \alpha^{1/2}$, which can naturally be written in the form $E < E_c$, where

$$E_c = E_{c0}\alpha^{1/2} = \frac{mv_0(1+\Delta)^2}{e(T_{\text{eff}} + mD_0/v_0)}\alpha^{1/2}; \quad (46)$$

E_c' is the critical field: when $E \gtrsim E_c$ the flux S_0 is proportional to N_e' i.e., all the electrons go over into the acceleration mode within a time of the order of $1/\nu_0$.[2)] In a non-equilibrium plasma the field E_{c0} is defined by expression (12). With increasing scattering parameter α, the field E_c increases like $\alpha^{1/2}$.

Table II

x	F	x	F	x	F	x	F
8.33	8.96	2.50	6.07	0.50	3.69	0.0833	2.28
6.67	8.33	2.00	5.65	0.25	3.025	0.0667	2.16
5.00	7.59	1.50	5.16	0.167	2.71	0.050	2.03
4.167	7.15	1.00	4.55	0.133	2.56	0.0333	1.87
3.00	6.43			0.100	2.38	0.0167	1.66

It is important to emphasize that the critical field (46) differs qualitatively from that obtained in an elementary analysis that takes no account of the concrete character of the deformation of the electron distribution function in the field. Indeed, if we assume, for example, that the electron distribution function is Maxwellian with a directional velocity v_0 then the electron frictional force increases with increasing v_0 when $v_0 < v_T$, and decreases in proportion to $1/v_0^2$ when $v_0 > v_T$. When $v_0 \sim v_T$ it has a maximum value of $F_{max} \sim \alpha m\nu(v_T)v_T$. This leads to the customary defi-

[2)]If the excitation of nonequilibrium oscillations in the plasma is the consequence of a distortion of the distribution function of the electrons in the electric field, then the parameter α itself can depend noticeably on E, thus changing E_c. If α increases for large values of E more rapidly than E^2, then, as is clear from (43) – (45), in this case the critical field cannot be attained at all.

nition[10,3,11] of the critical field: $eE_c = F_{max}$, leading in turn to the expression

$$E_c = E_{c0}\alpha, \tag{47}$$

which differs qualitatively from the results of the exact calculation (46). For example, in an equilibrium Z-fold ionized plasma, in accordance with (46), the critical field increases like $Z^{1/2}$ (for a fixed density of the electrons $N_e = ZN_Z$), while according to (47) $E_c \sim Z$. The reason for this difference is as follows: at large values of the parameter α and when $E \sim E_c$, $\gamma \sim \alpha^{1/2}$ and $\alpha\gamma \sim \alpha^{3/2} \gg 1$. In this case, which was considered in Sec. 2, the principal effect on the runaway-electron flux is exerted by the distortion of the symmetrical part of the distribution function, whereas this process is completely ignored in the elementary analysis given above.

Naturally, the estimate of the flux of the runaway electrons in a weak field $E \ll E_c$, based on analogous elementary considerations which do not take into account the deformation of the distribution function (see, for example, [9]), leads to an expression

$$S_0 \sim N_e \exp\left\{-\frac{\alpha}{2}\frac{E_{c0}}{E}\right\},$$

which has little in common with the result of the exact calculation (43).

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