SURFACE WAVES IN A PLASMA WITH A CURRENT

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Surface oscillations are considered in a bounded cold plasma in which charged particles flow along a constant magnetic field. The boundary of the plasma is assumed to be sharp, so that the wavelength of the oscillations is much longer than the thickness of the transition layer. General boundary conditions are derived for the joining of the solutions on this layer, with the aid of which dispersion equations are obtained for the oscillations in various particular cases. It is shown that "oblique" ($k_x \gg k_z$) surface waves are unstable in the presence of particle currents, and this leads to a smearing of the sharp plasma boundary.

1. INTRODUCTION

It is known that currents of charged particles excite various types of oscillations in a plasma.[1] These effects are as a rule investigated in the approximation of a uniform plasma which is either unbounded (in the latter case the boundary of the plasma is assumed to be sufficiently sharp, so that the thickness of the boundary transition layer is much smaller than the wavelength of the oscillations). In order to be able to carry out a consistent analysis of all the types of instabilities of a bounded plasma, it is necessary to specify boundary conditions for the perturbed electric and magnetic fields of the oscillations in an infinitesimally thin transition layer. Such boundary conditions, however, have never been formulated before for a moving plasma (except for the particular case of axially-symmetrical perturbations of a cylindrical plasma, considered in the paper of Gorbatenko[2]).

In the present paper we derive, for a system of bounded cold plasma currents moving along the magnetic field, general boundary conditions which make it possible to consider not only three-dimensional but also surface disturbances of the plasma. It is shown that in a bounded plasma with current, situated in a strong magnetic field, instability of "oblique" ($k_x > k_z$) surface waves takes place (for a plasma of cylindrical geometry this corresponds to axially-asymmetrical surface perturbations; Tsintsadze and Lominadze[3] have left out these effects, owing to an incorrect formulation of the boundary conditions). This instability leads to the convection of the plasma transversely to the magnetic field and to a smearing of the sharp boundary.

The instability of surface waves in a bounded plasma with current is equivalent to collisionless current-convective instability of an inhomogeneous plasma, which was considered by one of the authors.[4] In this sense the present paper shows how effects which are typical of an inhomogeneous plasma can be investigated within the framework of the theory of a homogeneous plasma with sharp boundary.

2. DIELECTRIC CONSTANT OF A COLD INHOMOGENEOUS PLASMA WITH PARTICLE CURRENTS

Let a plasma consist of several groups (currents) of charged particles moving relative to one another along the constant magnetic field $\mathbf{H}_0 \parallel C$ (the self-fields of the currents are neglected). We assume that the equilibrium velocities of the particles $\mathbf{U}_\alpha$ do not depend on the coordinates, and their densities $n_{0\alpha}(y)$ vary in the Y direction (α is the number of the species of the group). The temperature of each group of particles will be assumed for simplicity equal to zero. Then the behavior of particles of species α in a field $\mathbf{E}$ and $\mathbf{H}$ of a small perturbation is described by the following linearized system of hydrodynamic equations:

$$\frac{\partial \mathbf{v}_\alpha}{\partial t} + (\mathbf{U}_\alpha \nabla)\mathbf{v}_\alpha = \frac{e_\alpha}{m_\alpha}\mathbf{E} + [\mathbf{v}_\alpha \boldsymbol{\omega}_{H\alpha}] + \frac{e_\alpha}{m_\alpha c}[\mathbf{U}_\alpha \mathbf{H}];$$

$$\frac{\partial n_\alpha}{\partial t} + \operatorname{div}(n_{0\alpha}\mathbf{v}_\alpha) + \operatorname{div}(n_\alpha \mathbf{U}_\alpha) = 0. \qquad (2.1)^*$$

Here n_α and $\mathbf{v}_\alpha$ are the perturbed density and velocity, e_α and m_α are the charge and mass of the particles, $\omega_{H\alpha} = e_\alpha H_0/m_\alpha c$ is the cyclotron frequency, and c is the velocity of light.

*$[\mathbf{U}_\alpha \mathbf{H}] = \mathbf{U}_\alpha \times \mathbf{H}$.

Expressing with the aid of Maxwell's equations the magnetic field of the perturbation $\mathbf{H}$ in terms of the electric field $\mathbf{E}$, and writing the dependence of the field on the coordinates and on the time in the form

$$\mathbf{E}=\mathbf{E}(y)\exp\{-i\omega t+ik_x x+ik_z z\}, \quad (2.2)$$

we obtain from (2.1)

$$v_{\alpha x}=\frac{e_\alpha}{m_\alpha\Delta_\alpha}\Big\{i\frac{\Omega_\alpha^2}{\omega}E_x+\omega_{H\alpha}\frac{\Omega_\alpha}{\omega}E_y-ik_xU_\alpha\frac{\Omega_\alpha}{\omega}E_z-i\frac{\omega_{H\alpha}}{\omega}U_\alpha\frac{\partial E_z}{\partial y}\Big\};$$

$$v_{\alpha y}=\frac{e_\alpha}{m_\alpha\Delta_\alpha}\Big\{-\omega_{H\alpha}\frac{\Omega_\alpha}{\omega}E_x-i\frac{\Omega_\alpha^2}{\omega}E_y-\frac{\omega_{H\alpha}}{\omega}k_xU_\alpha E_z-\frac{\Omega_\alpha}{\omega}U_\alpha\frac{\partial E_z}{\partial y}\Big\};$$

$$v_{\alpha z}=i\frac{e_\alpha E_z}{m_\alpha\Omega_\alpha},$$

$$n_\alpha=\frac{n_{0\alpha}}{\Omega_\alpha}\Big\{k_xv_{\alpha x}+\frac{i}{n_{0\alpha}}\frac{\partial}{\partial y}(n_{0\alpha}v_{\alpha y})+k_zv_{\alpha z}\Big\},$$

$$\Omega_\alpha=\omega-k_zU_\alpha,\qquad \Delta_\alpha=\omega_{H\alpha}^2-\Omega_\alpha^2. \quad (2.3)$$

The perturbed electric current density can be written in the form

$$j_{x,y}=\sum_\alpha e_\alpha n_{0\alpha}v_{\alpha x,y},\qquad j_z=\sum_\alpha e_\alpha(n_{0\alpha}v_{z\alpha}+n_\alpha U_\alpha). \quad (2.4)$$

The electric induction vector, as is well known, is equal to

$$D_i=\delta_{ik}E_k+4\pi ij_i/\omega=\hat{\varepsilon}_{ik}E_k \qquad (i,k=x,y,z), \quad (2.5)$$

where the components of the tensor-operator of the dielectric constant $\hat{\epsilon}_{ik}$, according to (2.3) and (2.4), are

$$\hat{\varepsilon}_{xx}=\hat{\varepsilon}_{yy}=1+\sum_\alpha\frac{\omega_{0\alpha}^2\Omega_\alpha^2}{\omega^2\Delta_\alpha},$$

$$\hat{\varepsilon}_{xy}=-\hat{\varepsilon}_{yx}=i\sum_\alpha\frac{\omega_{0\alpha}^2\omega_{H\alpha}\Omega_\alpha}{\omega^2\Delta_\alpha},$$

$$\hat{\varepsilon}_{xz}E_z=\sum_\alpha\frac{\omega_{0\alpha}^2}{\omega^2\Delta_\alpha}\Big[\Omega_\alpha k_xU_\alpha E_z+\omega_{H\alpha}U_\alpha\frac{\partial E_z}{\partial y}\Big],$$

$$\hat{\varepsilon}_{yz}E_z=-i\sum_\alpha\frac{\omega_{0\alpha}^2}{\omega^2\Delta_\alpha}\Big[\omega_{H\alpha}k_xU_\alpha E_z+\Omega_\alpha U_\alpha\frac{\partial E_z}{\partial y}\Big],$$

$$\hat{\varepsilon}_{zx}E_x=\sum_\alpha\frac{\omega_{0\alpha}^2}{\omega^2\Delta_\alpha}\Big[\Omega_\alpha k_xU_\alpha E_x-\frac{\omega_{H\alpha}U_\alpha}{n_{0\alpha}}\frac{\partial}{\partial y}(n_{0\alpha}E_x)\Big],$$

$$\hat{\varepsilon}_{zy}E_y=i\sum_\alpha\frac{\omega_{0\alpha}^2}{\omega^2\Delta_\alpha}\Big[\omega_{H\alpha}k_xU_\alpha E_y-\frac{\Omega_\alpha U_\alpha}{n_{0\alpha}}\frac{\partial}{\partial y}(n_{0\alpha}E_y)\Big],$$

$$\hat{\varepsilon}_{zz}E_z=\Big(1-\sum_\alpha\frac{\omega_{0\alpha}^2}{\Omega_\alpha^2}\Big)E_z+\sum_\alpha\frac{\omega_{0\alpha}^2}{\omega^2\Delta_\alpha}\Big[(k_xU_\alpha)^2E_z-\frac{U_\alpha^2}{n_{0\alpha}}\frac{\partial}{\partial y}\Big(n_{0\alpha}\frac{\partial E_z}{\partial y}\Big)-\frac{\omega_{H\alpha}k_xU_\alpha^2}{\Omega_\alpha}\frac{\partial(\ln n_{0\alpha})}{\partial y}E_z\Big], \quad (2.6)$$

where $\omega_{0\alpha}=(4\pi e_\alpha^2 n_{0\alpha}/m_\alpha)^{1/2}$ is the Langmuir frequency of particles of species α.

3. BOUNDARY CONDITIONS ON THE DISCONTINUITY

Let us consider a cold plasma with a dielectric constant (2.6), in which the particle flux densities $n_{0\alpha}(y)$ vary within a certain layer $-\delta/2<y<\delta/2$ near $y=0$, the plasma being homogeneous in the regions $y<-\delta/2$ (region 1) and $y>\delta/2$ (region 2), with $n_{0\alpha}=\text{const}$ (we assume, for example, that $n_{0\alpha}^{(2)}>n_{0\alpha}^{(1)}$). We assume that the thickness of the transition layer δ is much smaller than the wavelength of the oscillations λ ($\delta/\lambda\ll 1$), i.e., we consider the problem of a homogeneous plasma with sharp boundary (discontinuity).[2,3]

We have already noted that the choice of the boundary conditions at such a plasma in the form of a continuity condition for the tangential components of the oscillation fields in an infinitesimally thin transition layer (discontinuity) results in failure to take into account oscillations of the surface-wave type. Indeed, any surface perturbation should lead to the occurrence of nonzero surface currents and charges, so that besides the normal component of the electric field, the tangential components of the magnetic field should also become discontinuous. To take this into account, we write the boundary condition on the discontinuity in the form

$$\int_1^2 \operatorname{div}\mathbf{D}(y)\,dy=0, \quad (3.1)$$

where the integration is carried out along an infinitesimally thin transition layer ($\delta\ll\lambda$). Specifying D in the form (2.5), we represent condition (3.1) in the following form, which is convenient for further analysis:

$$ik_x\int_1^2 D_x(y)\,dy+D_y\Big|_1^2+ik_z\int_1^2 D_z(y)\,dy=0. \quad (3.1a)$$

We consider this boundary condition for two different cases: a) a rarefied plasma, when the thickness of the skin layer is $\delta_s=c/\omega_{0e}>\delta$ (ω_{0e} is the electron Langmuir frequency), and b) a dense plasma, when $\delta_s<\delta$.

a) Rarefied plasma, $\delta_s>\delta$. Substituting in (3.1a) the corresponding expressions from (2.6) and integrating over an infinitesimally thin layer ($\delta\to 0$) with account of the fact that the tangential components of the electric field E_x and E_z are continuous, while the derivatives with respect to y as well as the normal components E_y and the density $n_{0\alpha}(y)$ experience a finite jump on the discontinuity, we obtain, accurate to terms $\sim \delta/\lambda$,

$$D_y^{(2)} - D_y^{(1)} = i \sum_\alpha \frac{4\pi e_\alpha^2 k_z U_\alpha}{m_\alpha \omega^2 [\omega_{H\alpha}^2 - (\omega - k_z U_\alpha)^2]}$$

$$\times \Big\{ \omega_{H\alpha} E_x [n_{0\alpha}^{(2)} - n_{0\alpha}^{(1)}]$$

$$+ (\omega - k_z U_\alpha)[n_{0\alpha}^{(2)} E_y^{(2)} - n_{0\alpha}^{(1)} E_y^{(1)}]$$

$$+ U_\alpha \left[n_{0\alpha}^{(2)} \frac{\partial E_z^{(2)}}{\partial y} - n_{0\alpha}^{(1)} \frac{\partial E_z^{(1)}}{\partial y} \right]$$

$$+ \frac{\omega_{H\alpha} k_z U_\alpha}{\omega - k_z U_\alpha} E_z [n_{0\alpha}^{(2)} - n_{0\alpha}^{(1)}] \Big\}. \quad (3.2)$$

We see that the electric induction vector experiences a finite jump on the plasma boundary (discontinuity). Inasmuch as with the aid of Maxwell's equation*

$$\operatorname{rot} \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}$$

the normal component of $\mathbf{D}$ can be expressed in terms of the tangential components of the magnetic field, condition (3.2) also determines a jump in the latter on the plasma boundary. We note that in the absence of currents, $U_\alpha = 0$, we obtain the usual condition for the continuity of the normal component $D_y^{(2)} = D_y^{(1)}$.

The boundary condition (3.2) can also be written in terms of the components of the electric field

$$\Big\{ E_y + \sum_\alpha \frac{\omega_{0\alpha}^2}{\omega_{H\alpha}^2 - (\omega - k_z U_\alpha)^2} \Big[E_y - \frac{i\omega_{H\alpha}}{\omega - k_z U_\alpha} E_x$$

$$- \frac{i\omega_{H\alpha} U_\alpha}{\omega(\omega - k_z U_\alpha)} (k_x E_z - k_z E_x) - \frac{iU_\alpha}{\omega} \Big(\frac{\partial E_z}{\partial y}$$

$$- ik_z E_y \Big) \Big] \Big\} \Big|_1^2 = 0. \quad (3.3)$$

b) Dense plasma, $\delta_S < \delta$. In oder to obtain the boundary condition on the discontinuity in this case, we consider the projection of Maxwell's equations on the Z axis, i.e., on the direction of the magnetic field:

$$(\operatorname{rot}\operatorname{rot}\mathbf{E})_z = \frac{\omega^2}{c^2} (\hat{\varepsilon}_{zx} E_x + \hat{\varepsilon}_{zy} E_y + \hat{\varepsilon}_{zz} E_z) \equiv \frac{\omega^2}{c^2} D_z. \quad (3.4)$$

We note here that the transition to the limit $\delta_S = c/\omega_{0e} \to 0$ for $\omega \ll \omega_{Hi}$ corresponds to a transition to ideal magnetohydrodynamics, when the inertia of the electrons can be neglected, $m_e \to 0$. Inasmuch as in the case of low-frequency oscillations only the term $\hat{\epsilon}_{zz} E_z$ contains terms $\sim 1/m_e$, it follows from (3.4) that $E_z \to 0$ (this corresponds to infinite conductivity of the plasma along the force lines, $\sigma_\parallel \to \infty$). Therefore, for $\delta_S \ll \delta$ we can neglect in (3.4) all the terms proportional to E_z (except $\hat{\epsilon}_{zz} E_z$), i.e., terms proportional to δ_S/λ and δ_S/δ, by virtue of which we obtain from (3.4)

$$D_z \cong \frac{c^2}{\omega^2} ik_z \Big(ik_x E_x + \frac{\partial E_y}{\partial y} \Big). \quad (3.5)$$

Substituting (3.5) in (3.1a) and integrating over the infinitesimally thin layer (with account of the continuity of E_x), we obtain, accurate to terms proportional to δ/λ, the following boundary condition:

$$D_y^{(2)} - D_y^{(1)} = \frac{k_z^2 c^2}{\omega^2} (E_y^{(2)} - E_y^{(1)}). \quad (3.6)$$

This condition can also be expressed in terms of jumps of the tangential components of the magnetic field.

4. POTENTIAL SURFACE WAVES

We consider a rarefied plasma with a thick skin layer, $\delta_S \gg \delta$, and assume that the oscillations are almost potential, $\mathbf{E} \approx -\nabla\psi$. Then the boundary condition on a discontinuity (3.3) takes the form

$$\Big\{ \frac{\partial\psi}{\partial y} + \sum \frac{\omega_{0\alpha}^2}{\omega_{H\alpha}^2 - (\omega - k_z U_\alpha)^2} \Big(\frac{\partial\psi}{\partial y} + \frac{\omega_{H\alpha} k_x}{\omega - k_z U_\alpha} \psi \Big) \Big\} \Big|_1^2 = 0. \quad (4.1)$$

On the other hand, from the Poisson equation, or, what is the same, from the equation div $\mathbf{D} = 0$, we obtain in the homogeneous regions 1 and 2 on both sides of the boundary (discontinuity) the following equation for potential oscillations:

$$A_{1,2} \frac{\partial^2 \psi_{1,2}}{\partial y^2} - (k_x^2 \varepsilon_\perp^{(1,2)} + k_z^2 \varepsilon_\parallel^{(1,2)}) \psi_{1,2}(y) = 0; \quad (4.2)$$

$$\varepsilon_\perp = 1 + \sum_\alpha \frac{\omega_{0\alpha}^2 (\omega - k_z U_\alpha)^2}{\omega^2 [\omega_{H\alpha}^2 - (\omega - k_z U_\alpha)^2]},$$

$$\varepsilon_\parallel = 1 - \sum_\alpha \frac{\omega_{0\alpha}^2}{(\omega - k_z U_\alpha)^2} + \sum_\alpha \frac{\omega_{0\alpha}^2 (k_x U_\alpha)^2}{\omega^2 [\omega_{H\alpha}^2 - (\omega - k_z U_\alpha)^2]},$$

$$A = \varepsilon_\perp + \sum_\alpha \frac{\omega_{0\alpha}^2 (k_z U_\alpha)^2}{\omega^2 [\omega_{H\alpha}^2 - (\omega - k_z U_\alpha)^2]}. \quad (4.3)$$

Choosing the solutions of (3.2) in the form $\psi_1 \sim e^{\kappa_1 y}$ and $\psi_2 \sim e^{-\kappa_2 y}$ (region 1 for $y < 0$, region 2 for $y > 0$), we obtain the characteristic

* rot = curl

equations for the determination of κ_1 and κ_2. Substituting then these expressions into the joining condition (4.1), we obtain the dispersion equation for the potential oscillations, with account of the sharp boundary.

To illustrate this method, we consider several concrete examples.

1. "Plasma with current—vacuum" interface. Assume that in region 2 ($y > 0$) there is a plasma consisting of stationary ions and electrons moving relative to them with velocity U (plasma with current), while in region 1 ($y < 0$) there is vacuum. Assuming that the frequency of the oscillations is $\omega \lesssim \omega_{Hi} \ll \omega_{He}$, we obtain accurate to terms $\sim m_e/m_i$ and $\omega_{Hi}^2/\omega_{0i}^2 \ll 1$ for the region 2 ($y > 0$)

$$\psi_2 \sim e^{-\varkappa_2 y}, \quad \varkappa_2 = \left[k_x^2 - k_z^2 \frac{m_i}{m_e} \frac{\omega_{Hi}^2 - \omega^2}{(\omega - k_z U)^2}\right]^{1/2} \quad (k_x \gg k_z). \tag{4.4}$$

In region 1 we have $\epsilon_{\perp}^{(1)} = \epsilon_{\parallel}^{(1)} = A_1 = 1$,

$$\varkappa_1 = (k_x^2 + k_z^2)^{1/2}.$$

The boundary joining condition (4.1) has in this case the form

$$\frac{\partial \psi_2}{\partial y} - k_x \left[\frac{\omega_{Hi}^2 - \omega^2}{\omega_{Hi}(\omega - k_z U)} - \frac{\omega_{Hi}}{\omega}\right] \psi_2(y) = 0. \tag{4.5}$$

Substituting in (4.5) the solution of (4.2) with account of (4.4), we obtain the following dispersion equation for the oscillations:

$$1 - \frac{k_z^2}{k_x^2} \frac{m_i}{m_e} \frac{\omega_{Hi}^2 - \omega^2}{(\omega - k_z U)^2} - \left[\frac{\omega_{Hi}}{\omega} - \frac{\omega_{Hi}^2 - \omega^2}{\omega_{Hi}(\omega - k_z U)}\right]^2 = 0. \tag{4.6}$$

For low-frequency oscillations, $\omega \ll \omega_{Hi}$, this equation reduces to the form

$$k_x^2(\omega - k_z U)^2 - k_z^2 \omega_{Hi}^2 \frac{m_i}{m_e} - k_x^2 \omega_{Hi}^2 \frac{(k_z U)^2}{\omega^2} = 0. \tag{4.6a}$$

For small k_z, when

$$k_x^2 \gg k_z^2 \frac{m_i}{m_e} \frac{\omega_{Hi}^2}{\omega^2}, \quad \omega \gg k_z U,$$

we obtain from this

$$\omega \cong (\omega_{Hi} |k_x| U k_z / k_x)^{1/2}, \tag{4.7}$$

i.e., the oscillations are unstable if k_x has the proper sign. We note that in this case, according to (4.4), $\kappa_2^2 > 0$ and the wave attenuates exponentially on moving from the boundary inside the plasma.

If

$$k_x^2 \ll k_z^2 \frac{m_i}{m_e} \omega_{Hi}^2/\omega^2,$$

then according to (4.6a) we get aperiodic instability of the surface waves, with an increment

$$\gamma \cong |k_x| U (m_e / m_i)^{1/2}. \tag{4.8}$$

In this case κ_2 is complex, and the surface wave has an oscillating structure with respect to y, becoming attenuated on the average in space.

It follows from (4.6) that the high-frequency oscillations with $\omega \sim k_z U \gg \omega_{Hi}$ (but $\omega \ll \omega_{He}$) also are unstable, with increment

$$\gamma \cong \left(\frac{k_x}{|k_x|} \omega_{Hi} k_z U\right)^{1/2}. \tag{4.9}$$

The surface waves considered here are the limiting case of collisionless current-convective perturbations in a plasma with inhomogeneous current, which were considered earlier.[4] The instability of these waves should lead to effects of the same type as the Rayleigh-Taylor instability in a plasma contained by a magnetic field against the force of gravity,[5] i.e., to convection of the plasma transversely to the magnetic field and to smearing of the initially sharp boundary, so that after some time, when the transition layer becomes sufficiently broad ($\delta \sim \lambda$), the instability of the plasma should be considered using the methods developed in [4].

b) Plasma layer with current. Let us consider stability of a plasma layer with a current of thickness a, surrounded on both sides with vacuum. The solution of (4.2) inside the layer $-a/2 < y < a/2$ can be chosen (with account of reflection of the waves on the boundaries of the layer) in the form $\psi_2(y) \sim \cosh \kappa_2 y$, where κ_2 is defined, as in the preceding case, by (4.4). Substituting this solution into the joining condition (4.1) on the boundary $y = a/2$, we obtain the following dispersion equation for oscillations with $\omega \ll \omega_{Hi}$:

$$\varkappa_2 \operatorname{sh} \frac{\varkappa_2 a}{2} - k_x \frac{\omega_{Hi} k_z U}{\omega(\omega - k_z U)} \operatorname{ch} \frac{\varkappa_2 a}{2} = 0. \tag{4.10*}$$

In the case of large layer thickness, when

$$\varkappa_2 a \gg 1, \quad \operatorname{sh}(\varkappa_2 a/2) \approx \operatorname{ch}(\varkappa_2 a/2) \approx \exp(\varkappa_2 a/2),$$

we arrive at the earlier equation (4.6a) for a semi-bounded plasma. To the contrary, in the case of a thin layer, $\kappa_2 a \ll 1$, when

$$\operatorname{sh}(\varkappa_2 a/2) \approx \varkappa_2 a/2, \quad \operatorname{ch}(\varkappa_2 a/2) \approx 1,$$

we obtain (for $k_x^2 \gg k_z^2 (m_i/m_e) \omega_{Hi}^2/\omega^2$)

$$\omega \approx [2\omega_{Hi} k_z U / k_x a]^{1/2}, \tag{4.11}$$

i.e., when k_z and k_x have the appropriate signs, we get aperiodic instability of the thin layer, with

*sh = sinh, ch = cosh.

an increment that increases with decreasing layer thickness.

5. NONPOTENTIAL OSCILLATIONS WITH $\delta_S > \delta$

Let us consider low-frequency ($\omega \ll \omega_{Hi}$) non-potential oscillations of a homogeneous bounded rarefied plasma with current, where the thickness of the skin layer exceeds the thickness of the boundary layer, $\delta_S = c/\omega_{0e} > \delta$.

Expressing with the aid of Maxwell's equations the components of the electric field of the oscillations E_y and E_z in terms of E_x, and representing all quantities in the form

$$\sim \exp\{-i\omega t + ik_x x + ik_z z - \varkappa_y y\},$$

we reduce the boundary condition (3.3) to the form

$$\left\{\frac{\varkappa_y}{k_x}\frac{\omega_{0i}^2}{\omega_{Hi}^2} - \frac{\omega_{0e}^2}{\omega_{He}}\left[\frac{1}{\omega - k_z U} - \frac{1}{\omega} - \frac{\omega k_z U}{c_A^2 k_z^2(\omega - k_z U)}\right]\right\}\Bigg|_1^2 = 0, \tag{5.1}$$

where $c_A = H_0/[4\pi n_0 m_i]^{1/2}$ is the Alfven velocity.

The dispersion equation of the oscillations in the homogeneous region is in the general case very complicated. However, since we are considering a rarefied plasma, we can confine ourselves to slow perturbations of the convective type, for which $k_x \gg k_z$ and $\epsilon_{xx} = \epsilon_{yy} \ll c^2k_x^2/\omega^2$, and $\epsilon_{yz} \to 0$. Then the dispersion equation of the oscillations of the Alfven type takes on the form (cf. [6])

$$\varepsilon_\parallel\left(1 - \frac{\omega^2}{c^2k_z^2}\varepsilon_\perp\right) + \frac{k_x^2 - \varkappa_y^2}{k_z^2}\varepsilon_\perp = 0; \tag{5.2}$$

from which we get for κ_y the expression

$$\varkappa_y^2 = k_x^2 - \frac{\omega^2}{c^2}\varepsilon_\parallel + k_z^2\frac{\varepsilon_\parallel}{\varepsilon_\perp}; \tag{5.3}$$

where $\epsilon_\parallel$ and $\epsilon_\perp$ are defined by (4.3).

Substituting (5.3) into the joining condition on the interface between the plasma and the vacuum, we obtain the following dispersion equation for the oscillations:

$$\frac{\omega_{0e}^2}{k_x^2c^2}\omega^4 + \omega^2(\omega - k_z U)^2 - \omega_{Hi}^2 \times \left[(k_z U)^2\left(1 - \frac{\omega^2}{k_z^2 c_A^2}\right) + \frac{m_i}{m_e}\frac{k_z^2}{k_x^2}\omega^2\right] = 0. \tag{5.4}$$

This fourth-order equation in ω always has complex roots corresponding to the instability of the oscillations. In particular, if $\omega \gg k_z c_A$, (5.4) takes the form

$$\omega^4\left(1 + \frac{\omega_{0e}^2}{k_x^2c^2}\right) - \omega^2\omega_{Hi}^2\left(\frac{k_z^2}{k_x^2}\frac{m_i}{m_e} - \frac{U^2}{c_A^2}\right) - \omega_{Hi}^2(k_z U)^2 = 0. \tag{5.5}$$

Comparing (5.5) with the dispersion equation of the low-frequency potential oscillations (4.6a), we see that an account of the non-potential nature of the oscillations is essential at electron velocities $U \gtrsim c_A$ (but in this case $\omega \gg k_z U$) and under the condition that $\omega_{0e}^2/k_x^2c^2 \gtrsim 1$, i.e., $\lambda \gtrsim \delta_S$.

Thus, if $\omega \gg k_z U$, the two highest-order roots of (5.5) take the form

$$\omega_{1,2}^2 = \frac{\omega_{Hi}^2}{1 + \omega_{0e}^2/k_x^2c^2}\left(\frac{k_z^2}{k_x^2}\frac{m_i}{m_e} - \frac{U^2}{c_A^2}\right), \tag{5.6}$$

so that the instability of the oscillations ($\omega^2 < 0$) takes place under the condition

$$U / c_A > (k_z / k_x)(m_i / m_e)^{1/2}. \tag{5.7}$$

We note that since we have assumed that $\omega \ll \omega_{Hi}$, the solutions (5.6) are valid if $\omega_{0e}^2/k_x^2c^2 \gg 1$, i.e., $\lambda \gg \delta_S$.

To the contrary, the two other roots of (5.5)

$$\omega_{3,4}^2 = -\frac{(k_z U)^2}{k_z^2 m_i/k_x^2 m_e - U^2/c_A^2} \tag{5.8}$$

are unstable at low velocity: $U < c_A(k_z/k_x)(m_i/m_e)^{1/2}$. When condition (5.7) is satisfied, they are stable, and when $U/c_A \gg (k_z/k_x)(m_i/m_e)^{1/2}$ they go over into ordinary Alfven waves with $\omega = k_z c_A$.

6. SURFACE WAVES IN A DENSE PLASMA, $\delta > \delta_S$

As was already noted in Sec. 3, the transition to a small skin layer, $\delta_S \to 0$, is equivalent to transition to ideal magnetohydrodynamics of a plasma with infinite longitudinal conductivity, so that $E_z \to 0$. The dispersion equation of the oscillations in the homogeneous plasma then assumes the form

$$\left(\varepsilon_{xx} - \frac{c^2k_z^2}{\omega^2}\right)\left[\varepsilon_{yy} - \frac{c^2(k_x^2 + k_z^2 - \varkappa_y^2)}{\omega^2}\right] - \varepsilon_{yx}\varepsilon_{xy} = 0. \tag{6.1}$$

From this, taking into account that $\epsilon_{xx} = \epsilon_{yy}$ and $\epsilon_{xy} = -\epsilon_{yx}$, we get

$$\varkappa_y^2 = -\frac{\omega^2}{c^2}\left\{\varepsilon_{xx} - \frac{c^2(k_x^2 + k_z^2)}{\omega^2} + \frac{\varepsilon_{xy}^2}{\varepsilon_{xx} - c^2k_z^2/\omega^2}\right\}. \tag{6.2}$$

Expressing with the aid of Maxwell's equations all the components of the electric field in terms of E_x, we rewrite the joining condition (3.6) in the form

$$\left\{\frac{\varepsilon_{yx} - ik_x\varkappa_y c^2/\omega^2}{\varepsilon_{xx} - (c/\omega)^2(k_x^2 + k_z^2)}\right\}\Bigg|_1^2 = 0. \tag{6.3}$$

We note that in the absence of currents, U = 0, we get according to (2.6) $\epsilon_{yx} = 0$.

Let us consider by way of an example a plasma with current, bordering on a currentless but sufficiently dense plasma, so that the condition $\delta_S < \delta$ is satisfied at all points of the transition layer. The dispersion equation for low-frequency ($\omega \ll \omega_{Hi}$) oscillations with $k_x \gg k_z$ breaks up in this case into two equations:

$$\left(1-\frac{\omega^2}{c^2k_x^2}\varepsilon_{xx}\right)\left[\varepsilon_{xy}-2i\left(\varepsilon_{xx}-\frac{c^2k_z^2}{\omega}\right)\right]=0, \qquad (6.4)$$

the first of which ($\epsilon_{xx} = c^2k_x^2/\omega^2$) describes stable magnetic-sound oscillations with frequency $\omega = k_x c_A$, and the second can be reduced to the form

$$\omega^2 - c_A^2k_z^2 + \omega_{Hi}k_zUk_x/|k_x| = 0. \qquad (6.5)$$

We see that it describes surface waves of the Alfven type. Under the condition

$$U/c_A > k_zc_A/\omega_{Hi} \qquad (6.6)$$

we get a current-convective instability of these oscillations, which leads to the smearing of the sharp plasma boundary.

In conclusion it must be noted that the excitations of surface waves in a plasma with sharp boundary ($\lambda \gg \delta$), considered in the present paper, are due in final analysis to the current gradient and constitute, as already noted, the limiting case of a collisionless current-convective instability of an inhomogeneous plasma.[4] However, effects of this kind can take place also in the absence of current, but at a finite electron temperature. Then the surface waves are the limiting case of drift waves in a plasma with sharp boundary, which were considered in a paper by one of the authors.[6]

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Errata

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Formulas (40) and (42) should read:

$$\sigma_{12}=\frac{en_e}{H}, \quad \sigma'_{12}=\frac{e^2n_e}{2mT}\operatorname{cth}\alpha,$$

$$\alpha_{12}=\frac{1}{T}\beta'_{12}=\frac{en_e}{2mT}\left\{\frac{1}{2}\operatorname{cth}\alpha+\alpha\frac{1+\operatorname{ch}^2\alpha}{\operatorname{sh}^2\alpha}-\frac{\zeta}{T}\operatorname{cth}\alpha\right\},$$

$$\beta_{12}=\frac{en_e}{m}\left\{\operatorname{cth}\alpha+\frac{1}{4\alpha}-\frac{\zeta}{\omega_H}\right\},$$

$$\gamma_{12}=\frac{n_e}{2m}\left\{\frac{3}{4}\operatorname{cth}\alpha+\alpha\frac{1+\operatorname{ch}^2\chi}{\operatorname{sh}^2\alpha}+\alpha^2\operatorname{cth}\alpha\frac{5+\operatorname{ch}^2\alpha}{\operatorname{sh}^2\alpha}-\frac{\zeta}{T}\left[\operatorname{cth}\alpha+2\alpha\frac{1+\operatorname{ch}^2\alpha}{\operatorname{sh}^2\alpha}\right]+\left(\frac{\zeta}{T}\right)^2\operatorname{cth}\alpha\right\}, \qquad (40)^*$$

$$\tilde{\alpha}_{12}=\frac{en_e}{2mT}\left\{\operatorname{cth}\alpha+\frac{3}{2\alpha}-2\frac{\zeta}{\omega_H}\right\}, \qquad \tilde{\gamma}_{12}=\frac{n_eT}{eH}\left\{\frac{15}{4}+3\alpha\operatorname{cth}\alpha+\alpha^2\frac{1+\operatorname{ch}^2\alpha}{\operatorname{sh}^2\alpha}-2\frac{\zeta}{T}\left(\frac{3}{2}+\alpha\operatorname{cth}\alpha\right)+\left(\frac{\zeta}{T}\right)^2\right\},$$

$$\sigma_{11}=e^2J_0, \quad \alpha_{11}=\frac{1}{T}\beta_{11}=\frac{e}{T}(J_1-\zeta J_0),$$

$$\gamma_{11}=\frac{1}{T}(J_2-2\zeta J_1+\zeta^2J_0) \qquad (42)$$

*ch ≡ cosh, sh ≡ sinh, cth ≡ coth.