THREE-PARTICLE UNITARITY CONDITION FOR COMPLEX ANGULAR MOMENTA AND THE MANDELSTAM BRANCHING POINTS

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The contribution of three-particle states to the unitarity condition for the elastic partial-wave amplitude is studied. The unitarity condition is extended to complex values of the angular momentum j in such a way, that no amplitude singularities arise for large values of Re j. It is found that the three-particle contribution should contain a sum of not only integer but also complex values of the angular momentum projection (m). This results in the appearance of Mandelstam branching points in the j-plane. In conclusion, the possibility of writing down the unitarity condition in the form of a contour integral with respect to m is discussed.

THE question of the analytic continuation of many-particle unitarity conditions to complex values of the angular momentum j is presently one of the central problems in the entire theory of complex angular momenta. The structure of the unitarity conditions for complex j determines the position and the character of the moving singularities of the partial waves $f_j(s)$ in the j plane. This is connected with the fact that the singularities of $f_j(s)$ with respect to energy, the positions of which depend on j, go over to the physical sheet only through the right-side cut,[1] the discontinuity on which is determined by the unitarity condition. Only two poles appear in this case from the unphysical sheets which are connected with the two-particle intermediate states.

The results of Mandelstam, who investigated the asymptotic behavior of some diagrams containing three-particle intermediate states, point to the existence of moving branch points in the j-plane.[2] In this connection, it is interesting to attempt to establish the possibility of the appearance of Mandelstam branch points directly from the structure of the three-particle unitarity conditions for complex j.

A hypothesis was previously advanced [3] concerning the structure of many-particle unitarity conditions near the singular points. This has made it possible to establish the character of the Mandelstam branchings and to obtain the reggeon unitarity conditions. In [3] use was made of the amplitudes f_{jlm} for the production of three particles with specified total angular momentum j, particle-pair momentum l, and particle helicity m, continued into complex j, l, and m. The use of these amplitudes is subject to objections connected with the poor convergence of the initial series with respect to l. Actually, to establish the mechanism of occurrence of Mandelstam branch points, the continuation to complex values of l and m is not essential. In the present paper we shall write down the three-particle unitarity condition in terms of the amplitudes for the production of three particles, continued into complex j, without using complex l and m. This will enable us to trace the formation of the Mandelstam branch points and to find the coefficients at the singularities. These coefficients can be expressed in terms of the amplitudes for the production of particles in states with definite complex l and m, which agrees with the previously obtained results.[3] An important role in the determination of the singularities is played by the unitarity condition for the three-particle amplitudes with respect to the energy of the pair of produced particles for complex j (Sec. 2). In the third section we discuss the connection between the form for writing down the unitarity condition, proposed in the present paper, and the form employed in [3]. In our next article,[4] using as an example very simple Feynman diagrams, we shall show that the three-particle amplitudes introduced by us can actually be continued to complex j and have the necessary properties.

1. THREE-PARTICLE UNITARITY CONDITION

The three-particle contribution to the unitarity condition for the partial amplitude of scattering of two particles $f_j(s)$ can be written in terms of the partial amplitudes for the transformation of two particles into three, F_{jm}. The amplitudes F_{jm} depend on the total angular momentum j, its projection m on the Z axis of the coordinate system, which is rigidly connected with the three particles, and the squares of the paired energies of the three particles s_{12}, s_{13}, and s_{23}:

$$s_{12}+s_{13}+s_{23}=s+m_1^2+m_2^2+m_3^2,$$

where $s^{1/2}$ is the total energy and m_1, m_2, and m_3 are the masses of the particles. For integer values of j and m, the values of F_{jm} are determined as follows[1]:

$$F_{jm}(s_{12},s_{13},s_{23})=\int\frac{d\Omega}{4\pi}A(t_1,t_2;\ s_{12},s_{13},s_{23})\,Y_{jm}^*(\vartheta,\varphi),\quad (1)$$

where the invariant amplitude A (Fig. 1) depends on the two momentum transfers t_1 and t_2 and on the paired energies of the particles. The integration is over the angles (ϑ, φ) of the momentum of the initial particles **p** in the coordinate frame defined by the momenta of the final particles (for example, as in Fig. 2).

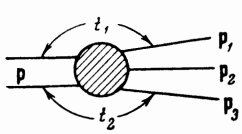

FIG. 1.

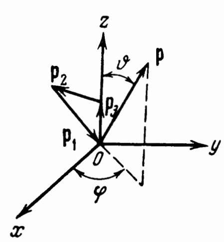

FIG. 2.

The unitarity condition for the partial scattering amplitude $f_j(s)$ for integer j can be written in the form

$$\frac{1}{2i}[f_j-f_j^*]=\frac{p}{8\pi\sqrt{s}}f_jf_j^*+\int d\Gamma\sum_{m=-j}^{j}F_{jm}F_{jm}^*,$$

$$\int d\Gamma=\frac{1}{8(2\pi)^3}\int\limits_{(m_1+m_2)^2}^{(\sqrt{s}-m_3)^2}\frac{q_{12}}{\sqrt{s_{12}}}ds_{12}\frac{p_3}{\sqrt{s}}\int\limits_{-1}^{+1}\frac{dx}{2}.\qquad (2)$$

Here $\mathbf{q}_{12}$ is the momentum of the relative motion of particles 1 and 2 in their c.m.s. in the intermediate state, $\mathbf{p}_3$ is the momentum of the third particle in the common c.m.s., and x is the cosine of the angle between them.

The definition (1) is not convenient for continuation into complex j, inasmuch as Y_{jm} contains only branch points in j and m. It is more convenient to use in their place the associated Legendre functions, which are entire functions of j and m:

$$P_{jm}(\cos\vartheta)\,e^{im\varphi}=\left[\frac{\Gamma(j+m+1)}{\Gamma(j-m+1)}\right]^{1/2}Y_{jm}(\vartheta,\varphi).\qquad (3)$$

Introducing

$$f_{jm}=\left[\frac{\Gamma(j+m+1)}{\Gamma(j-m+1)}\right]^{1/2}F_{jm},\quad f_{jm}=\int\frac{d\Omega}{4\pi}P_{jm}\,e^{-im\varphi}A,\qquad (4)$$

we can rewrite the unitarity condition (2) in the form

$$\frac{1}{2i}(f_j-f_j^*)=\frac{p}{8\pi\sqrt{s}}f_jf_j^*+2\int d\Gamma\sum_{m=0}^{j}\frac{\Gamma(j-m+1)}{\Gamma(j+m+1)}f_{jm}f_{jm}^*.\qquad (5)$$

In formula (5) we have excluded summation over negative values of m with the aid of the relation that follows from the properties of P_{jm}:

$$f_{j-m}=(-1)^m\frac{\Gamma(j-m+1)}{\Gamma(j+m+1)}f_{jm}.\qquad (6)$$

It is understood here that the term with m = 0 in the sum (5) does not contain the factor 2 in front of the integral.

The correct continuation of the right side of the unitarity condition into complex values of j is a continuation which has no singularities at sufficiently large Re j and which decreases as Re j → ∞, inasmuch as the left side has these properties.[1] Let us imagine that we have succeeded in continuing the quantities f_{jm} with respect to j for fixed integer m. We shall show below, for several very simple diagrams, that such a continuation is actually possible.[4] It can then be thought that in order to generalize the right side of (5) to include complex j it is sufficient to extend the summation with respect to m in (5) to infinity. In fact, this can always be done for integer j, since for such values of j the quantities f_{jm} vanish when m > j [see (4)]. To be sure, $\Gamma(j-m+1)$ has at the same points m poles each, but these poles are offset by the second-order zeros of the product $f_{jm}f_{jm}^*$. Therefore, whereas for complex j the summation with respect to m goes to infinity, for integer j the sum automatically terminates at m = j, as is required by the condition (5).

Such reasoning, however, is incorrect if account is taken of the signature of the quantities f_{jm}, i.e., the fact that the amplitudes f_{jm} are continued into complex j separately from even and odd j. (This, of course, always takes place

[1] We use spherical functions normalized such that

$$\int|Y_{jm}|^2d\Omega=4\pi/(2j+1).$$

in relativistic theory.) If we consider, for example, only the positive signature (as will be done henceforth for concreteness), then only even values of j are physical. This means that for odd j the quantities f_{jm} (obtained by continuation from even j) are not described by formula (4) and, generally speaking, do not vanish when $m > j$. Then replacement of the sum over m from zero to j by the sum to infinity is in its literal form incorrect, for owing to the presence of $\Gamma(j-m+1)$ there occur poles in j for all integer odd j. This contradicts the condition for the continuation of the amplitudes $f_j(s)$, according to which there should be no singularities at sufficiently large values of Re j.

We can, however, attempt to ascribe to the infinite sum with respect to m an addition which is lacking in the case of physical (even) j and which cancels out the poles in the case of odd values of j. Then the unitarity condition for complex j takes the form

$$\frac{1}{2i}(f_j - f_{j^*}{}^*) = \Delta_2 f_j + \Delta_3 f_j, \qquad \Delta_2 f_j = \frac{p}{8\pi\sqrt{s}} f_j f_{j^*}{}^*,$$

$$\Delta_3 f_j = 2\int d\Gamma \left\{ \sum_{m=0}^{\infty} \frac{\Gamma(j-m+1)}{\Gamma(j+m+1)} f_{jm} f_{j^*m}^* + \mathrm{tg}\,\frac{\pi j}{2}\Lambda(j) \right\}. \qquad (7)^*$$

Here $f_{j^*}^*$ and $f_{j^*m}^*$ are the analytic continuation of the quantities f_j^* and f_{jm}^* to complex values of j. For physical (even) j, Eq. (7) takes on the form (5). The function $\Lambda(j)$ is defined for odd integer j by the condition that the poles be cancelled out in the sum over m. Therefore for odd j

$$\Lambda(j) = \sum_{m=j+1}^{\infty} (-1)^m \frac{\pi}{2} \times \frac{1}{\Gamma(-j+m)\Gamma(m+j+1)} f_{jm} f_{jm}{}^*. \qquad (8)$$

To continue $\Lambda(j)$ to complex values of j, we make, for integer odd j, a shift in the summation over m in (8):

$$\Lambda(j) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\pi}{2} \frac{f_{j\,j+n} f_{j\,j+n}^*}{\Gamma(n)\Gamma(2j+n+1)} \qquad (9)$$

In (9) we replaced $(-1)^{j+n}$ by $(-1)^{n+1}$ in view of the fact that j is odd.

Now the continuation of the sum (9), like the continuation of the sum over m in (7), possibly reduces to the continuation of the quantities $f_{j\,j+n}$ and $f_{j\,j+n}^*$ from odd j at a fixed value of n. This continuation does not coincide, generally speaking, with the continuation of f_{jm} in m for fixed unphysical j to the point $m = j + n$. To emphasize this circumstance, we shall denote the continued functions $f_{j\,j+n}$ and $f_{j\,j+n}^*$ by $\varphi_{j\,j+n}$ and $\varphi_{j^*\,j^*+n}^*$. The fact that the functions $\varphi_{j\,j+n}$ and $f_{j\,j+n}$ do not coincide for arbitrary j is illustrated by Fig. 3. To obtain the continuation of $\varphi_{j\,j+n}$ we first continued f_{jm} to complex values of j for fixed m, using the even values of $j \geq m$ (continuation along the horizontal lines in Fig. 3, starting with the points marked by the circles). Then, using the values of the obtained f_{jm} for odd integer $j < m$ (crosses on Fig. 3), we carry out the continuation along the inclined lines joining the crosses ($n = -j + m$ is fixed). On the other hand, the continuation of f_{jm} in m with fixed j would mean continuation along the vertical straight lines. We see that the quantities $f_{j\,j+n}$ and $\varphi_{j\,j+n}$ coincide only for odd integer j; moreover, the functions $f_{j\,j+n}$ vanish for even j [for in this case they are determined by formula (4)], while $\varphi_{j\,j+n}$ generally speaking do not vanish.

Thus, our hypothesis consists essentially in the fact that the three-particle contribution to the unitarity condition has for complex j the form

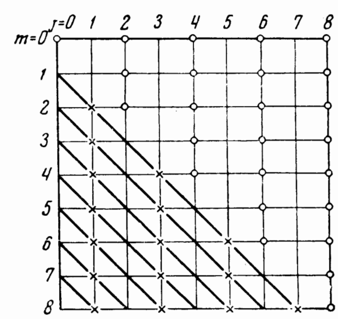

FIG. 3.

$$\Delta_3 f_j = 2\int d\Gamma \left\{ \sum_{m=0}^{\infty} \frac{\Gamma(j-m+1)}{\Gamma(j+m+1)} f_{jm} f_{j^*m}^* + \mathrm{tg}\,\frac{\pi j}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\pi}{2} \frac{\varphi_{j\,j+n}\varphi_{j^*\,j^*+n}^*}{\Gamma(n)\Gamma(2j+n+1)} \right\}. \qquad (10)$$

More accurately speaking, we think that the singularities of $\Delta_3 f_j$ are correctly defined by expression (10), although the series in m and in n can diverge, and the integration domain $d\Gamma$ can be changed when j is complex.[4,5]

2. BRANCH POINTS IN THE j PLANE

We shall show how the unitarity condition (10) leads to the appearance of branch points in the j plane. The Mandelstam branch points occur when

*tg = tan.

account is taken of pair interactions of the produced particles. This means that in order to establish the mechanism of their occurrence, it is necessary to take such interactions into account in the production amplitudes f_{jm} and $\varphi_{j\,j+n}$. To this end we consider the unitarity condition with respect to energy of a pair of produced particles. It is shown symbolically in Fig. 4. By Im is meant, of course, not the total imaginary part, but the discontinuity at the two-particle singularity with respect to the paired energy.

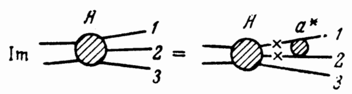

FIG. 4.

For the partial waves f_{jm} of the amplitude A, the unitarity condition can be written in the form

$$\frac{1}{2i}[f_{jm}(s_{12}+i\varepsilon,x)-f_{jm}(s_{12}-i\varepsilon,x)]$$

$$=\frac{q_{12}}{8\pi\sqrt{s_{12}}}\int_{-1}^{+1}\frac{dx'}{2}f_m(s_{12}-i\varepsilon,x,x')f_{jm}(s_{12}+i\varepsilon,x'). \quad (11)$$

Here m is the projection of the total angular momentum on the momentum of particle 3; x is the cosine of the angle between the momentum of relative motion of particles 1 and 2 ($\mathbf{q}_{12}$) and the momentum of the third particle $\mathbf{p}_3$ in the final state; x′ is the same for particles in the intermediate state. The quantity f_m is connected in simple fashion with the amplitude of elastic scattering of particles 1 and 2:

$$f_m(s_{12}-i\varepsilon,x,x')=\int_0^{2\pi}\frac{d\varphi}{2\pi}e^{-im\varphi}a(s_{12}-i\varepsilon,z). \quad (12)$$

The cosine of the scattering angle in the c.m.s. of particles 1 and 2 is expressed in terms of x, x′ and φ:

$$z=xx'+[(1-x^2)(1-x'^2)]^{1/2}\cos\varphi. \quad (13)$$

Inasmuch as we assume that f_{jm} can be continued to complex j for integer m, and the quantity f_m does not depend on j, the unitarity condition (11) retains the same form for arbitrary j. Henceforth, as in the three-particle unitarity condition (10), we shall pay no attention to the possibility of change in the contours of integration with respect to x′.[4,5]

The unitarity condition (11) for the amplitudes f_{jm} with complex j and integer m in itself yields nothing of interest, inasmuch as in this case the continuation in j does not affect the quantity f_m, which is directly connected with the amplitude of the pair interaction. In this sense, the situation is entirely different for the quantities $\varphi_{j\,j+n}$. The unitarity condition for $\varphi_{j\,j+n}$ can be readily obtained from (11) and is of the form

$$\frac{1}{2i}[\varphi_{j\,j+n}(s_{12}+i\varepsilon,x)-\varphi_{j\,j+n}(s_{12}-i\varepsilon,x)]$$

$$=\frac{q_{12}}{8\pi\sqrt{s_{12}}}\int_{-1}^{+1}\frac{dx'}{2}f_{j+n}(s_{12}-i\varepsilon,x,x')\varphi_{j\,j+n}(s_{12}+i\varepsilon,x'). \quad (14)$$

The function f_{j+n} is an analytic continuation of the quantity f_m, defined by formula (12). Inasmuch as f_m does not depend on j, it is obvious that f_{j+n} is obtained by simple continuation in m.

The quantities $\varphi_{j\,j+n}(s_{12}\pm i\epsilon, x)$ have been continued in j in such a way that they have no singularities at large values of Re j. With decreasing Re j, the singularities appear on one of the edges of the cut in s_{12}, i.e., in one of the functions $\varphi_{j\,j+n}(s_{12}+i\epsilon, x)$ or $\varphi_{j\,j+n}(s_{12}-i\epsilon, x)$. From (14) it is readily seen that the poles of $\varphi_{j\,j+n}(s_{12}-i\epsilon, x)$ appear simultaneously with the poles of $f_{j+n}(s_{12}-i\epsilon, x, x')$. If we rewrite the right side of (14) in such a way that the argument of the function f_{j+n} is $s_{12}+i\epsilon$, while the argument of $\varphi_{j\,j+n}$ is of the form $s_{12}-i\epsilon$, then it is clear that the poles of $f_{j\,j+n}(s_{12}+i\epsilon, x)$ appear together with the poles of $f_{j+n}(s_{12}+i\epsilon, x, x')$. Thus, to determine the poles $\varphi_{j\,j+n}(s_{12}, x)$ it is sufficient to investigate the poles of $f_{j+n}(s_{12}, x, x')$.

Formula (12) for f_{j+n} can be rewritten (for integer j + n) in the form

$$f_{j+n}(s_{12},x)=\frac{1}{2\pi}\int_C\frac{dy}{\sqrt{1-y^2}}(y+i\sqrt{1-y^2})^{-(j+n)}a(s_{12},z), \quad (15)$$

where $y = \cos\varphi$. The contour of integration encompasses the cut $(1-y^2)^{1/2}$ between +1 and −1 (Fig. 5), and $(1-y^2)^{1/2} > 0$ on the upper edge of the cut. In order to obtain a continuation of f_{j+n} to complex j such that it decreases at large j, it is necessary also to swing the contour of integration with respect to y around the singularities of $a(s_{12}, z)$. Of course, $a(s_{12}, z)$ has as a function of z two cuts going to the right from the

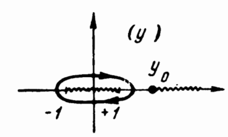

FIG. 5.

point $z_0^{(1)} > 1$ and to the left from $z_0^{(2)} < -1$. It is easy to see from (13) that when $-1 \le x$ and $x' \le 1$ this leads to analogous singularities with respect to y with $y_0^{(1)} \ge z_0^{(1)} > 1$ and $y_0^{(2)} \le z_0^{(2)} < -1$.

Let us assume first for simplicity that $a(s_{12}, z)$ has only a right-side cut. Then formula (15) can be written in the form

$$f_{j+n}(s_{12}, x, x') = \frac{1}{\pi}\int_{y_0}^{\infty} \frac{dy}{\sqrt{y^2-1}}\,(y+\sqrt{y^2-1})^{-(j+n)}\,a_1(s_{12}, z), \tag{16}$$

where a_1 is the absorption part of the function a. Expression (16) can be directly continued to complex j, since it shows that $f_{j+n} \to 0$ when Re j increases. The singularities of f_{j+n} with respect to j are now determined by the asymptotic behavior of $a_1(s_{12}, z)$ with respect to y (or z, which is linearly connected with y). If the paired amplitude $a(s_{12}, z)$ has a Regge pole with trajectory $\alpha(s_{12})$, then it makes to the asymptotic value of a_1 a contribution $\sim z^{\alpha(s_{12})} \sim y^{\alpha(s_{12})}$. We see therefore that f_{j+n} has a pole with respect to j when $j + n = \alpha(s_{12})$. According to the foregoing, $\varphi_{j\,j+n}$ has a similar singularity.

It is now easy to see that in the unitarity condition (10) the second term under the integral sign has poles in j when $j = \alpha(s_{12}) - n$, $n = 1, 2, \ldots$ It is obvious that after integration of these poles with respect to s_{12}, branch points arise at $j = \alpha[(\sqrt{s} - m_3)^2] - n$. Of course, in the discontinuity $\Delta_3 f_j$, which is determined by the unitarity condition (10), there are also branch points $j = \alpha^*[(\sqrt{s} - m_3)^2] - n$, connected with the poles of the functions $\varphi^*_{j^*\,j^*+n}$. The obtained branch points are precisely the branch points obtained by Mandelstam.[2]

Let us consider in greater detail the residue of the function $\varphi_{j\,j+n}$ at the pole with respect to j, and the coefficient in the two-particle amplitude at the arising branch point. From (16) we see that near the pole $f_{j+n}(s_{12}, x, x')$ is of the form

$$f_{j+n}(s_{12}, x, x') = \frac{r(s_{12})}{\pi}(1-x^2)^{\alpha/2}(1-x'^2)^{\alpha/2} \times \frac{1}{j+n-\alpha(s_{12})}, \tag{17}$$

if the asymptotic expression of $a_1(z)$ is of the form

$$r(s_{12})\,(2z)^{\alpha(s_{12})}.$$

$r(s_{12})$ is related in simple fashion with the residue of the two-particle partial amplitude $f_l(s_{12})$ at the pole with respect to l for $l = \alpha(s_{12})$. From the unitarity condition (14) we can now easily obtain the residue of the function $\varphi_{j\,j+n}(s_{12} + i\epsilon, x)$ at the pole for $j = \alpha - n$. To this end it is convenient to reverse the signs of $i\epsilon$ in the right side of (14). We have

$$\varphi_{jj+n}(s_{12}+i\varepsilon, x) = \frac{1}{j+n-\alpha(s_{12})}\,2i\,\frac{q_{12}}{8\pi\sqrt{s_{12}}}\,\frac{r(s_{12})}{\pi}(1-x^2)^{\alpha/2} \times \int_{-1}^{+1}\frac{dx'}{2}(1-x'^2)^{\alpha/2}\varphi_{jj+n}(s_{12}-i\varepsilon, x') \tag{18}$$

(we shall henceforth take the values of $\alpha(s_{12})$ and $r(s_{12})$ on the upper edge of the cut $s_{12} \to s_{12} + i\epsilon$).

Substituting (18) in the three-particle unitarity condition (10), we obtain for the term containing the singularity in j, for $j = \alpha - 1$:

$$\Delta_3 f_j = -\frac{1}{2i}\int \frac{p_3\,ds_{12}}{8\pi^2\sqrt{s}}\,\frac{\operatorname{ctg}(\pi\alpha/2)}{\Gamma(2\alpha)}\,\frac{r(s_{12})}{j+1-\alpha(s_{12})} \times N_\alpha(s_{12}, s-i\varepsilon)N_\alpha(s_{12}, s+i\varepsilon), \tag{19*}$$

$$N_\alpha(s_{12}, s\pm i\varepsilon) = 2i\,\frac{q_{12}}{8\pi\sqrt{s_{12}}}\int_{-1}^{+1}\frac{dx}{2}(1-x^2)^{\alpha/2}\varphi_{\alpha-1,\alpha} \times (s_{12}-i\varepsilon, x, s\pm i\varepsilon). \tag{20}$$

The quantity N_α is connected with the amplitude for the production of three particles in the state with total angular momentum $j = \alpha - 1$, its projection on the momentum of the particle 3, equal to α, and the orbital angular momentum of particles 1 and 2, equal to α.[3] This can be seen from an expansion of the amplitudes $f_{jm}(x)$ in terms of states with definite pair momentum l:

$$f_{jm}(x) = \sum_{k=0}^{\infty}(2l+1)P_l^{-m}(x)f_{jlm} \qquad (l = m+k), \tag{21}$$

which is valid also for non-integer m. (The functions $P_{m+k}^{-m}(x)$ form an orthogonal system when Re m > −1.) From (21) we have

$$f_{jlm} = \frac{\Gamma(l+m+1)}{\Gamma(l-m+1)}\int_{-1}^{+1}\frac{dx}{2}P_l^{-m}(x)f_{jm}(x) \quad (l = m+k). \tag{22}$$

If we now go over in the right side of (22) to the quantity $\varphi_{j\,j+n}$ in accordance with our general recipe, and then put $j = \alpha - 1$, $l = m = \alpha$, then the integral arising in (22) will coincide with the integral in (20) $[P_\alpha^{-\alpha}(x) \sim (1-x^2)^{\alpha/2}]$.

We shall show in our next paper[4] that formula (20) actually does hold for s_{12} close to $(m_1 + m_2)^2$, and that for larger s_{12} the residue is determined by the analytic continuation in s_{12}, at which the region of integration with respect to x changes.

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*ctg = cot.

In the presence of two cuts for the amplitude $a(s_{12}, z)$ it is necessary to introduce a signature with respect to $j + n$, i.e., to swing over the left cut with respect to z to the right, replacing $(-1)^{j+n}$ by ± 1, depending on the parity of $j + n$ (inasmuch as the continuation of $\varphi_{j\,j+n}$ is carried out from odd j, the sign is determined by the parity of n). As a result, the poles of the quantities $\varphi_{j\,j+n}$ with even and odd n turn out to be shifted relative to one another. From a formula of the type (16) with two cuts it is easy to verify that the poles of f_{j+n} (and consequently also of $\varphi_{j\,j+n}$) are located in the case of odd n at the points $j + n = \alpha, \alpha - 2, \ldots$ and $j + n = \beta - 1, \beta - 3, \ldots$, if α and β are the poles with even and odd signature, respectively. The poles of $\varphi_{j\,j+n}$ for even n are located at the points $j + n = \alpha - 1, \alpha - 3, \ldots$ and $j + n = \beta, \beta - 2, \ldots$ Consequently the branch points of the amplitudes $f_j(s)$ are located at $j = \alpha[(\sqrt{s} - m_3)^2] - 1, \alpha - 3, \ldots$, and $j = \beta[(\sqrt{s} - m_3)^2] - 2, \beta - 4$, etc. This result is valid, of course, for the amplitudes $f_j(s)$ with positive signature with respect to j, which were the only ones considered so far. For amplitudes with negative signature, the branch points are located at $j = \alpha - 2, \alpha - 4, \ldots$, and $j = \beta - 1, \beta - 3, \ldots$.

Let us now discuss briefly which precisely are the Feynman diagrams which lead to the appearance of branch points. From the method of constructing the quantities $\varphi_{j\,j+n}$ it is clear that for the existence of Mandelstam branch points it is necessary that the continuation of the amplitudes f_{jm} require the introduction of a signature with respect to j. (We recall that in the opposite case the functions f_{jm} with odd $j < m$, which serve as a basis for the construction of the amplitudes $\varphi_{j\,j+n}$, are equal to zero.) Let us consider the diagrams shown in Fig. 6. The amplitudes $f_{jm}^{(1)}$ and $f_{jm}^{(2)}$, which enter into the diagrams of Fig. 6, are shown in Fig. 7. We shall consider in our next paper [4] the diagrams of Fig. 7 with the aid of the unitarity condition with respect to the energy of the pair of particles 1 and 2. It will turn out then that the continuation of this unitarity condition to complex j for the diagram of Fig. 7a does not require the introduction of the signature, while for the diagram of Fig. 7b the signature is essential. Therefore the continuation of the amplitudes themselves, which is determined by these unitarity conditions, requires the introduction of a signature in the case of Fig. 7b and requires none for Fig. 7a. Consequently, $\varphi_{j\,j+n}^{(1)} = 0$, and $\varphi_{j\,j+n}^{(2)} \neq 0$.

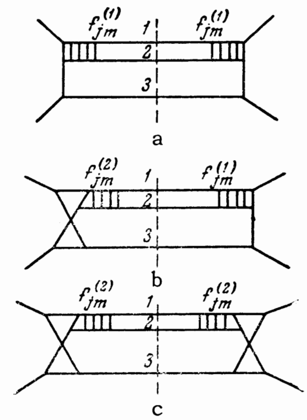

FIG. 6.

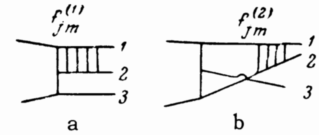

FIG. 7.

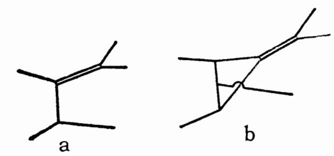

FIG. 8.

This result is natural if we regard reggeons as particles, for in this case the diagrams of Fig. 7 are transformed into the diagrams of Fig. 8. It is clear that the production amplitude of the particle shown in Fig. 8a has one cut in the momentum transfer, and therefore requires no signature; the diagram of Fig. 8b contains two cuts (by virtue of the presence of the third spectral function), and therefore it is continued into complex j with signature. Thus, the branch points should be missing from diagrams of Fig. 6a and b, and should appear in the diagram of Fig. 6c, in agreement with the results of Mandelstam [2] and Wilkin [6].

UNITARITY CONDITION IN THE FORM OF A CONTOUR INTEGRAL IN m

In the present section we wish to compare the form which we obtained for the unitarity condition continued into complex j, with its continuation in the form of a contour integral in m. This integral was written in [3] in the following fashion (for concreteness we again consider henceforth a positive signature in j):

$$\Delta_3 f_j = \int d\Gamma\, f_{j0} f^*_{j^*0} + 2\int d\Gamma \left\{ \int_C \frac{dm}{4i}\left(\operatorname{ctg}\frac{\pi m}{2} + \chi^+(j, m)\right) \times \frac{\Gamma(j - m + 1)}{\Gamma(j + m + 1)} f_{jm^+} f^{+*}_{j^*m^*} + \int_C \frac{dm}{4i}\left(-\operatorname{tg}\frac{\pi m}{2} + \chi^-(j, m)\right) \times \frac{\Gamma(j - m + 1)}{\Gamma(j + m + 1)} f_{jm^-} f^{-*}_{j^*m^*} \right\}. \tag{23}$$

Here the functions f^+_{jm} and f^-_{jm} are the analytic continuation of the amplitudes f_{jm} with even and odd values of m, respectively. We take into account here the fact that, regardless of the concrete method of continuing f_{jm}, this continuation must take into account the presence of a signature with respect to m. This can be seen, for example, from the structure of the unitarity condition (11). It includes the f_m which, as we have already seen, has different continuations from even and from odd m. The contour C encircles the real axis and the poles $\Gamma(j-m+1)$ (Fig. 9). The poles $\Gamma(j-m+1)$ must be included inside the contour C, for otherwise the integral (23) would have singularities for all odd integer j. The function

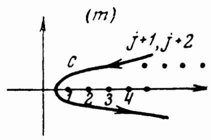

FIG. 9.

$\chi(j, m)$ should in any case be chosen such that for integer even j the unitarity condition has the usual form.

Let us consider for concreteness the integral in formula (23), corresponding to continuation from even m. The term with $\cot(\pi m/2)$ makes the following contribution to this integral:

$$2\int d\Gamma\Big\{\sum_{m\,\mathrm{even}}^{\infty}\frac{\Gamma(j-m+1)}{\Gamma(j+m+1)}f_{jm}^{+}f_{j^*m}^{+*} + \frac{\pi}{2}\tan\frac{\pi j}{2}\sum_{n\,\mathrm{odd}}^{\infty}\frac{f_{jj+n}^{+}f_{j^*j^*+n}^{+*}}{\Gamma(n)\Gamma(2j+n+1)} + \frac{\pi}{2}\cot\frac{\pi j}{2}\sum_{n\,\mathrm{even}}^{\infty}\frac{f_{jj+n}^{+}f_{j^*j^*+n}^{+*}}{\Gamma(n)\Gamma(2j+n+1)}\Big\}. \tag{24}$$

The first two terms in (24) coincide in form with the unitarity condition (10), if we consider for the continuation in j only the sum over the even m. The last term in (24) vanishes for all integer values of j: for odd j we have $\cot(\pi j/2) = 0$, and for even j and n, the function $f_{j\,j+n}^{+} = 0$, in accordance with the continuation condition, and the pole $\cot(\pi j/2)$ is cancelled out by the second-order zero of the quantity $f_{j\,j+n}^{+}f_{j^*j^*+n}^{+*}$.

A function having such properties cannot decrease in the entire right half-plane of j.[1] On the other hand, we seek a decreasing continuation of the unitarity condition, and therefore the second term in (24) should be eliminated by a suitable choice of $\chi^{+}(j, m)$ in the integrand of (23). Naturally, the formulation of this requirement on $\chi^{+}(j, m)$ still does not define it in a unique manner. The second term is eliminated, for example, if $\chi = -\cot(\pi j/2)$. Then the entire first contour integral in (23) is equal to

$$2\int d\Gamma\Big\{\sum_{m\,\mathrm{even}}^{\infty}\frac{\Gamma(j-m+1)}{\Gamma(j+m+1)}f_{jm}^{+}f_{j^*m}^{+*} + \frac{\pi}{2}\Big[\mathrm{tg}\frac{\pi j}{2}+\mathrm{ctg}\frac{\pi j}{2}\Big] \times \sum_{n\,\mathrm{odd}}^{\infty}\frac{f_{jj+n}^{+}f_{j^*j^*+n}^{+*}}{\Gamma(n)\Gamma(2j+n+1)}. \tag{25}$$

The expression (25) now coincides with (10), if for arbitrary j

$$f_{jj+n}^{+}f_{j^*j^*+n}^{+*} = \sin^2(\pi j/2)\varphi_{jj+n}\varphi_{j^*j^*+n}^{*}, \quad n = 1, 2, \ldots, \tag{26}$$

For odd j, the functions $f_{j\,j+n}^{+}$ and $\varphi_{j\,j+n}$ should coincide in the manner of the analytic continuation of these quantities. It is obvious that (26) agrees with this condition. At the same time we see that $f_{j\,j+n}^{+}$ and $\varphi_{j\,j+n}$ do not coincide for arbitrary j, as noted above.

The proposed choice of the function χ, which leads to expressions (25) and (26), is noncontradictory only in the case when, without regard to the presence of a signature in m, the quantities f_{jm}^{+} continued from even m and j vanish for even j and for all integer $m > j$. For even values of $m > j$ this condition is obvious, since even m and even j constitute physical points for f_{jm}^{+}. For even j and odd m, this requirement follows from formula (26).

This raises the question whether such a requirement is natural. In some cases it can be verified that in spite of the presence of a signature in m, this requirement is satisfied Let, for example, the amplitude f_{jm}^{+} be described by the diagram shown in Fig. 10, with the partial wave, connected with the irreducible block B, not requiring the introduction of a signature with respect to m. Then, even if the properties of the amplitude a call for the introduction of a signature with respect to m (the existence of two cuts for a), it is easy to see that the amplitude f_{jm}^{+}, connected with the diagram of Fig. 10, vanishes for all integer $m > j$ (for even j). We do not know, however, how general this property is.

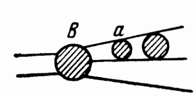

FIG. 10.

Let us now imagine that the amplitudes f_{jm}^{+} continued from even j and m, do not vanish for even j and odd m. Then the proposed choice of the function $\chi(j, m)$ is incorrect. This can be seen both from (26) (this was already mentioned above), and directly from (25). In fact, in this case the poles $\cot(\pi j/2)$, which occur for even

j, are not compensated by anything, since the quantities $f^{+}_{j\,j+n}$ do not vanish (in this case $m = j + n$ is an odd number). We can construct, however, a function $\chi(j, m)$ such which does not lead to these difficulties. We write the integral with f^{+}_{jm} in the unitarity condition (23), for example, in the form

$$2\int d\Gamma \int_{C} \frac{dm}{4i}\left[\operatorname{ctg}\frac{\pi m}{2} - \operatorname{ctg}\frac{\pi j}{2}\cos^{2}\frac{\pi(j-m)}{2}\right] \times \frac{\Gamma(j-m+1)}{\Gamma(j+m+1)} f_{jm^{+}} f^{+*}_{j^{*}m^{*}}. \tag{27}$$

Calculating directly the contribution of the poles of the integrand, we arrive at expression (10) under the condition that

$$f^{+}{}_{jj+n} = \varphi_{jj+n}, \qquad n = 1, 3, 5, \ldots . \tag{28}$$

Thus, we see that a correct choice of the function $\chi^{\pm}(j, m)$ is closely related with the character of the analytic continuation of f^{+}_{jm} in m. It is therefore difficult to propose a unique prescription for its choice, without knowing the detailed properties of the continuation of $f^{\pm}_{jm}$. However, the analysis presented allows us to hope that if formula (10) is a correct analytic continuation of the unitarity condition to complex j, then this condition can be simultaneously written in the form of some contour integral with respect to m.

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