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POSSIBILITY OF OSCILLATING DISTRIBUTIONS OF TURBULENT SOUND WAVES IN A PLASMA

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We investigate the turbulent acoustic-wave distributions that can be established in a plasma consisting of cold ions and hot electrons, which move with respect to the ions with supersonic velocity. In addition to a stationary distribution of turbulent fluctuations it is found that there can be another kind of distribution: this is an "oscillatory" distribution, in which the amplitude of the random waves is a weak function of time and in which the angular distribution varies periodically.

In this work we consider the possible distributions of turbulent acoustic waves that can be established in a plasma consisting of cold ions and hot electrons which move with respect to the ions with a velocity u, which exceeds the acoustic velocity s. It is usually assumed that the "runaway" of electrons in such a plasma leads, after an appropriate time interval, to a time-independent distribution of turbulent fluctuations. We show in this note that another kind of stationary fluctuation distribution is possible: this is an oscillatory distribution in which the amplitude of the random waves is a weak function of time and in which the angular distribution varies periodically.

We start with the equation

$$(ks)^{-1}\frac{\partial I}{\partial t}+\sqrt{\frac{\pi m}{2M}}\,\varepsilon\left(1-\frac{u}{s}\cos\theta\right)I = \frac{8\pi^2e^2T_i}{T_e^3}Ik\frac{\partial}{\partial k}\int\limits_{s/u}^{1}(1-\cos^2\theta\cos^2\theta')k^3I'd\cos\theta', \tag{1}$$

where $I \equiv I(k, \theta; t)$ is the correlation function for the scalar potential φ, $I' \equiv I(k, \theta'; t)$,

$$\langle\varphi(\mathbf{k},\omega)\varphi(\mathbf{k}',\omega')\rangle = \delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\{I(k,\theta;t)\times\delta(\omega-ks)+I(k,\pi-\theta;t)\delta(\omega+ks)\}, \tag{2}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{u}$, $\cos\theta > s/u$; m, M and $T_{e,i}$ are the electron and ion masses and temperatures respectively; ϵ is a small parameter that characterizes the slope of the plateau on the electron distribution function,[3] $\epsilon \sim (e^2T_i \sim \Lambda/aT_e^2)^{1/2}$, a is the Debye radius, Λ is the Coulomb logarithm (we assume for simplicity that $1 - s/u \ll 1$).

The relation in (1) can be obtained quite simply using a method developed by Kadomtsev and Petviashvili.[1] Equation (1) takes account of the nonlinear interaction between waves as well as the feedback effect of the waves on the particle distribution function, which leads to the formation of a plateau on the electron distribution function. This equation applies when k is sufficiently large ($k \gg (s\tau_i)^{-1}(1 - s/u)^{-1}(M/m)^{1/2}$, where τ_i^{-1} is the ion collision frequency, in which case the effect of collisions on the growth rate of the acoustic waves can be neglected.)

We note that (1) is invariant under the transformation

$$I \to \alpha^{-3}I,\quad k \to \alpha k,\quad t \to \alpha^{-1}t \tag{3}$$

(α is an arbitrary parameter) so that it is convenient to introduce the variables Ik^3 and $x = (\pi m/2M)^{1/2}\epsilon ukt$, which are invariant under this transformation; this substitution allows a reduction in the number of variables (cf. [4]).

Greatest interest attaches to the self-similar solutions of (1) [in the sense of the transformation in (3)], that is to say, solutions for which Ik^3 depends only on x (and not explicitly on t); it is precisely these solutions which correspond to the initial fluctuation distributions that obtain in a plasma. When t is not too large, so that the linear theory still applies, the correlation function depends on time in the form $\exp\{2\gamma t\}$, the growth rate γ being proportional to k. In other words, when x is not too large the function Ik^3 depends on x and is not an explicit function of t (if the possible dependence of the factor that multiplies the exponential is neglected). Because (1) is symmetric with respect to the transformation (3) the function Ik^3, which does not depend explicitly on t for small x, will not depend explicitly on t for any x; this function is thus a self-similar solution of (1).

It can be shown that the asymptote for the self-similar solutions of (1) is of the form

$$I(k,\theta;t)=A(ak)^{-3}\rho(x)\{(1-\cos\theta)+(1-s/u)\lambda(x)\}^{-1}\quad(x\gg 1),$$
$$A=a^3T_e^3u\varepsilon(4\pi e)^{-2}e^{-2}(T_is)^{-1}(\pi m/2M)^{1/2},\tag{4}$$

where ρ and λ are given by the equations

$$x\rho d\lambda/dx=\Psi_0(\lambda),\quad xd\rho/dx=\Psi_1(\lambda),$$
$$\Psi_0=D^{-1}\left\{(1-\lambda)\ln\left(1+\frac{1}{\lambda}\right)+1\right\},$$
$$\Psi_1=D^{-1}\left\{\ln\left(1+\frac{1}{\lambda}\right)+\frac{1-\lambda}{\lambda(1+\lambda)}\right\},\tag{5}$$
$$D=\ln^2\left(1+\frac{1}{\lambda}\right)-\frac{1}{\lambda(1+\lambda)}.$$

The functions ρ and λ are periodic functions of the variable (x/x_0) with period

$$P=\rho_0|c_++c_-|,\tag{6}$$

where ρ_0 and x_0 are parameters determined by the initial distribution of fluctuations:

$$c_+=\int_0^\infty\Psi_0^{-1}\exp\{f^+(\lambda)\}\,d\lambda,\quad c_-=\int_{-1}^{-\infty}\Psi_0^{-1}\exp\{f^-(\lambda)\}\,d\lambda,$$
$$f^\pm=\int_{\lambda_\pm}^{\lambda}\Psi_1\Psi_0^{-1}\,d\lambda,\qquad\lambda_+=1,\qquad\lambda_-=-2.$$

From (4) we see that the angular distribution of the fluctuations varies periodically in time. When $\ln(x/x_0)=nP$ $(n=0,\pm1,\ldots)$ almost all of the turbulent waves are propagated along the electron stream or along the surface of the Cerenkov cone

$$I(k,\theta;t)=A(ak)^{-3}\rho_0\begin{cases}\delta(1-\cos\theta) & (\ln(x/x_0)\to nP-0)\\ \delta(\cos\theta-s/u) & (\ln(x/x_0)\to nP+0).\end{cases}\tag{7}$$

If $\ln(x/x_0)=P(n+\nu)$, where $\nu=c_+(c_++c_-)^{-1}$, all of the turbulent waves (regardless of θ) have the same amplitude

$$I(k,\theta;t)=A(ak)^{-3}\rho_0(1-s/u)^{-1}\quad(\cos\theta>s/u).\tag{8}$$

The correlation function averaged over θ increases as k is reduced, going as k^{-3}, and is a weak function of the time

$$\bar I(k;t)\equiv{}^1/_2\int I(k,\theta;t)\,d\cos\theta={}^1/_2A(ak)^{-3}\rho_0\beta(x),\tag{9}$$

where β is an oscillatory factor approximately equal to unity.

The distinguishing feature of the oscillating distribution of fluctuations is the fact that the oscillation period is expressed in terms of the amplitude of the random acoustic waves

$$P=2(ak)^3\bar I(k)A^{-1}|c_++c_-|.\tag{10}$$

For large t the asymptote for the general solutions of (1) is analogous to the asymptote for the self-similar solutions. Noting that the asymptotic equation (1) is invariant under the transformation

$$I\to\alpha^{-3}(t)I,\qquad k\to\alpha(t)k\tag{11}$$

(α is an arbitrary function of t which is bounded by a polynomial as $t\to+\infty$) it is easy to show that the asymptote of the general solutions of Eq. (1) is given by the relations (4)—(9) as before, where $x=\alpha(t)k$. For the self-similar solutions $\alpha(t)$ is a linear function of t; for the stationary solutions $\alpha(t)=\text{const}$.

We note that (10) also applies in the general case; however, the quantity P, which characterizes the period of the oscillations in the distribution of random waves as ln k varies, will not generally give the oscillation period as ln t varies.

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[3] Vedenov, Velikhov, and Sagdeev, Nuclear Fusion **1**, 82 (1961).

[4] G. Birkhoff, Hydrodynamics, (Princeton University Press, 1960).

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131