PLASMA OSCILLATIONS IN A HIGH-FREQUENCY ELECTRIC FIELD

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We investigate the irrotational oscillations of a fully ionized plasma located in a strong high-frequency electric field. It is shown that the oscillations of the particles with respect to each other induced by the external field can lead to spatial dispersion. The presence of the high-frequency field modifies familiar branches of the dispersion relations and is also responsible for the appearance of a new branch. It is also shown that a high-frequency field has a stabilizing effect on the two-stream instability (electrons moving with respect to the ions).

INTRODUCTION

IN experiments on radiation acceleration [1] one is concerned with a fully ionized plasma located in a strong high-frequency electric field. Under these conditions the field-induced velocity of the electrons can easily be much greater than the electron thermal velocity, and the distance traversed by an electron in one period of the high-frequency field can be appreciably greater than the Debye radius.

The theoretical investigation of a plasma subject to these conditions was initiated in a paper by one of the authors;[2,3] this work was concerned with phenomena connected with the effect of a high-frequency field on collisions between charged plasma particles. However, even in [3] attention was directed to the fact that there was no theory available to describe oscillations of a plasma located in a strong high-frequency field. In the present work we have developed a theory which describes irrotational oscillations of a fully ionized plasma for the case in which the period of the external high-frequency electric field is much smaller than all of the characteristic times associated with the plasma motion. Under these conditions one can use relatively slowly varying collective variables, which are governed by the modified equations for the self-consistent field; in the limit of zero external electric field these obviously become the usual equations.

The characteristic oscillation spectra of an unmagnetized plasma are first considered within the framework of the two-fluid hydrodynamic approximation (Sec. 1) and then by means of the kinetic equation (Sec. 2). Section 3 treats the spectrum of irrotational oscillations of a plasma in a fixed magnetic field. It is shown that the field-induced oscillation of the plasma particles with respect to each other provides a mechanism by which the oscillation frequency can depend on the wave vector. In other words, a varying external field (like thermal motion) can give rise to spatial dispersion. It is obvious that the situation is similar to that of a fixed electric field.[4] However, in contrast with the dc field, a high-frequency field does not excite low-frequency oscillations. Indeed, we show below that an external high-frequency field can have a stabilizing effect on the two-stream instability in a plasma. The external electric field modifies familiar branches of the usual dispersion relation and also leads to the appearance of a new branch. In the long wavelength region the new branch is very similar to the ion-acoustic wave, the only difference being that the role of the Debye radius is played by the magnitude of the electron displacement in an oscillation period of the external field. We note that the ion-acoustic oscillations still appear in a nonisothermal plasma, although the characteristic spectrum is modified by the high-frequency field.

1. OSCILLATIONS OF A COLD UNMAGNETIZED PLASMA

We start our analysis with the case of a cold plasma in which the thermal motion of the particles can be neglected, using the two-fluid hydrodynamic equations. It will be assumed that the particle distribution is uniform in space in the equilibrium state. It will be also assumed that the electrons and ions move with fixed, but unequal, velocities, and that they oscillate under the effect of the high-frequency field $\mathbf{E}(t) = \mathbf{E}_0 \sin \omega_0 t$.

Thus, in the equilibrium state the electron and ion velocities are given by

$$\mathbf{u}_\alpha = \mathbf{u}_\alpha^{(0)} + \frac{e_\alpha}{m_\alpha \omega_0} \mathbf{E}_0 \cos \omega_0 t. \quad (1.1)$$

The subscript α denotes the electron or ion respectively.

This directed motion leads to the appearance of an electromagnetic field. The effects of this field will be neglected below, as is customary in the theory of the two-stream instability. The oscillations associated with the irrotational electric field ($\delta \mathbf{E} = -\nabla \delta \Phi$) can then be described by the following linearized two-fluid hydrodynamic equations:

$$\frac{\partial \delta \mathbf{v}_\alpha}{\partial t} + i(\mathbf{k}\mathbf{v}_\alpha)\delta \mathbf{v}_\alpha = -i\frac{e_\alpha}{m_\alpha}\mathbf{k}\delta\Phi, \quad (1.2)$$

$$\frac{\partial \delta n_\alpha}{\partial t} + i(\mathbf{k}\mathbf{u}_\alpha)\delta n_\alpha = -i n_\alpha^{(0)}\mathbf{k}\delta \mathbf{v}_\alpha, \quad (1.3)$$

where $n_\alpha^{(0)}$ is the density of particles of type α in the equilibrium state and δn_α and $\delta \mathbf{v}_\alpha$ are the nonequilibrium corrections to the density and velocity.

The nonequilibrium potential of the field is determined from the equation

$$k^2 \delta\Phi = 4\pi \sum_\alpha e_\alpha n_\alpha. \quad (1.4)$$

This equation is now employed to eliminate the nonequilibrium potential; using (1.2) and (1.3) we find that the function

$$\nu_\alpha = e_\alpha \delta n_\alpha \exp\left(-i\int dt\, \mathbf{k}\mathbf{u}_\alpha\right)$$

is described by the following two equations:

$$\nu_e'' + \omega_{Le}^2(\nu_e + \nu_i \exp\{i\mathbf{k}\mathbf{u}t + ia\sin\omega_0 t\}) = 0,$$

$$\nu_i'' + \omega_{Li}^2(\nu_i + \nu_e \exp\{-i\mathbf{k}\mathbf{u}t - ia\sin\omega_0 t\}) = 0, \quad (1.5)$$

where

$$\mathbf{u} = \mathbf{u}_e^{(0)} - \mathbf{u}_i^{(0)}, \quad a = \frac{\mathbf{k}\mathbf{v}_E}{\omega_0}, \quad \mathbf{v}_E = \frac{\mathbf{E}_0}{\omega_0}\left(\frac{e}{m} - \frac{e_i}{m_i}\right),$$

$$\omega_{Le}^2 = \frac{4\pi n_e^{(0)} e^2}{m}, \quad \omega_{Li}^2 = \frac{4\pi n_i^{(0)} e_i^2}{m_i}.$$

To obtain the oscillations of interest here, which are characterized by frequencies much smaller than ω_0, we now employ the familiar method of averaging.[5,6] Using the expansion

$$e^{ia\sin\omega_0 t} = \sum_{l=-\infty}^{\infty} J_l(a) e^{il\omega_0 t}, \quad (1.6)$$

as a first approximation for the quantities averaged over the period $2\pi/\omega_0$ we have[1)]

$$\langle \nu_e\rangle'' + \omega_{Le}^2(\langle\nu_e\rangle + \langle\nu_i\rangle e^{i\mathbf{k}\mathbf{u}t} J_0(a)) = 0,$$

$$\langle \nu_i\rangle'' + \omega_{Li}^2(\langle\nu_i\rangle + \langle\nu_e\rangle e^{-i\mathbf{k}\mathbf{u}t} J_0(a)) = 0. \quad (1.7)$$

We then obtain the following dispersion equation for the characteristic frequencies of irrotational oscillations of a plasma in a high-frequency electric field:

$$1 = \frac{\omega_{Le}^2}{(\omega - \mathbf{k}\mathbf{u})^2} + \frac{\omega_{Li}^2}{\omega^2} - \frac{\omega_{Le}^2\omega_{Li}^2}{\omega^2(\omega-\mathbf{k}\mathbf{u})^2}[1 - J_0^2(a)]. \quad (1.8)$$

This equation yields a number of interesting features concerning the oscillations and the stability of the cold plasma. However, before considering these features it will be useful to consider the accuracy of the method of averaging which has been used here. The most direct way to evaluate the accuracy is to examine the corrections to the spectrum in (1.8). To find these corrections we note that in the approximation used here, in which all plasma frequencies are small compared with ω_0, the following equation can be used to find $\delta\nu_\alpha$ (the rapidly varying parts of ν_α):

$$\delta\nu_e'' + \omega_{Le}^2 \sum_{l\neq 0} J_l(a) e^{-il\omega_0 t + i\mathbf{k}\mathbf{u}t}\langle\nu_i\rangle = 0,$$

$$\delta\nu_i'' + \omega_{Li}^2 \sum_{l\neq 0} J_l(a) e^{il\omega_0 t + i\mathbf{k}\mathbf{u}t}\langle\nu_e\rangle = 0. \quad (1.9)$$

Solving (1.9) and using the solutions to obtain a second approximation for $\langle\nu_\alpha\rangle$ we have

$$\langle\nu_e\rangle'' + \langle\nu_e\rangle\omega_{Le}^2\left\{1 + \frac{\omega_{Li}^2}{\omega_0^2}\sum_{l\neq 0} J_l^2(a)\frac{1}{l^2}\right\}$$

$$+ \langle\nu_i\rangle\omega_{Le}^2 J_0(a) e^{i\mathbf{k}\mathbf{u}t} = 0,$$

$$\langle\nu_i\rangle'' + \langle\nu_i\rangle\omega_{Li}^2\left\{1 + \frac{\omega_{Le}^2}{\omega_0^2}\sum_{l\neq 0} J_l^2(a)\frac{1}{l^2}\right\}$$

$$+ \langle\nu_e\rangle\omega_{Li}^2 J_0(a) e^{-i\mathbf{k}\mathbf{u}t} = 0. \quad (1.10)$$

1) If the external rf field is of the form $\mathbf{E}(t) = \sum \mathbf{E}_i \sin(\omega_0 t + \delta_i)$, the argument of the Bessel function of zero order will be of the form

$$(e_\alpha/m_\alpha - e_\beta/m_\beta)\omega_0^{-2}[(\mathbf{k}\mathbf{E}^{(1)})^2 + (\mathbf{k}\mathbf{E}^{(2)})^2]^{1/2};$$

$$\mathbf{E}^{(1)} = \sum_i \mathbf{E}_i\cos\delta_i, \quad \mathbf{E}^{(2)} = \sum_i \mathbf{E}_i \sin\delta_i.$$

These formulas can be useful, for example, in the case of random field phases. For circular polarization of the rf field the argument of the Bessel function will be of the form

$$(e_\alpha/m_\alpha - e_\beta/m_\beta)\omega_0^{-2}k_\perp E_0.$$

From the system in (1.10) and the second-approximation dispersion equation we have

$$\begin{vmatrix} -(\omega-\mathbf{ku})^2+\omega_{Le}^2\left\{1+\frac{\omega_{Li}^2}{\omega_0^2}\sum_{l\neq 0}J_l^2(a)\frac{1}{l^2}\right\} & \omega_{Le}^2J_0(a) \\ \omega_{Li}^2J_0(a) & -\omega^2+\omega_{Li}^2\left\{1+\frac{\omega_{Le}^2}{\omega_0^2}\sum_{l\neq 0}J_l^2(a)\frac{1}{l^2}\right\}\end{vmatrix}=0 \quad (1.11)$$

which evidently shows that the method of averaging can be used when the following inequality is satisfied:

$$\omega_0^2 \gg \omega_{Le}^2. \quad (1.12)$$

We assume below that this inequality is satisfied.[2)]

Having verified the use of the method of averaging we now analyze the dispersion equation (1.8). We shall first analyze the frequency region in which $\omega \gg \mathbf{k}\cdot\mathbf{u}$. In other words, we shall be considering a plasma in which there is no directed motion or in which the oscillations have phase velocities exceeding the velocity of the directed motion of the electrons and ions. Then (1.8) yields the following two expressions for the characteristic plasma frequencies:

$$\omega^2=\omega_{Le}^2+\omega_{Li}^2J_0^2(a), \quad (1.13)$$

$$\omega^2=\omega_{Li}^2[1-J_0^2(a)]. \quad (1.14)$$

As far as the plasma oscillations at frequencies close to the electron Langmuir frequency ω_{Le} are concerned, we note that the effect of the rf field is simply a small correction to the dependence of oscillation frequency on wave vector. In contrast with the usual Langmuir oscillations,[7] in the present case the frequency is a maximum rather than a minimum at $k = 0$ although $\omega = (\omega_{Le}^2 + \omega_{Li}^2)^{1/2}$. As $\mathbf{k}\mathbf{v}_E/\omega_0 = a$ increases it is evident from (1.13) that the frequency oscillates, assuming a minimum value ω_{Le} at points corresponding to the zeros of $J_0^2(a)$ and asymptotically approaching the same value according to the relation

$$\omega^2=\omega_{Le}^2+\omega_{Li}^2\frac{2}{\pi|a|}\cos^2\left(\frac{\pi}{4}-|a|\right). \quad (1.15)$$

At small value of a the oscillation spectrum (1.14) assumes the form

$$\omega^2=\frac{\omega_{Li}^2}{2\omega_0^2}(\mathbf{kv}_E)^2\equiv(\mathbf{kw}_s)^2. \quad (1.16)$$

We may speak of anisotropic sound in the plasma under these conditions, understanding the velocity of sound to be $w_s = \omega_{Li}V_E/\sqrt{2}\omega_0$.

According to (1.14), as a increases the oscillation frequency also increases, assuming a maximum value equal to the ion Langmuir frequency at points at which the function J_0 vanishes. The frequency asymptotically approaches the same value in accordance with the relation

$$\omega^2=\omega_{Li}^2\left\{1-\frac{2}{\pi|a|}\cos^2\left(\frac{\pi}{4}-|a|\right)\right\}. \quad (1.17)$$

The relation in (1.16) allows us to proceed by analogy with the anisotropic random motion of electrons which leads to acoustic oscillations. However, this oscillatory motion is considerably different from the thermal motion, as is evident from (1.15) and (1.17).

Let us now consider the effect of the high-frequency oscillations on the two-stream instability (with respect to the irrotational oscillations). In the limit $\omega \ll |\mathbf{k}\cdot\mathbf{u}|$ the dispersion equation (1.8) yields

$$\omega^2=\frac{\omega_{Li}^2\{(\mathbf{ku})^2-\omega_{Le}^2[1-J_0^2(a)]\}}{(\mathbf{ku})^2-\omega_{Le}^2}. \quad (1.18)$$

According to this expression, growing solutions are possible only when

$$\omega_{Le}^2>(\mathbf{ku})^2>\omega_{Le}^2[1-J_0^2(a)]. \quad (1.19)$$

The left side of this inequality arises in the usual theory of a two-stream instability (without an rf field).[7-9] The right side of the inequality in (1.19) vanishes when the field $\mathbf{E}_0$ vanishes; we note that the right-hand side of the inequality (1.19) has a strong limiting effect on the instability region. This result can be understood if we keep in mind the analogy between the effects of rapid particle oscillations in the external rf field and the effects of a thermal spread in velocity.

If a is small, we have from (1.19)

$$|\mathbf{ku}|^2>\omega_{Le}^2\omega_0^{-2}(\mathbf{kv}_E)^2.$$

Thus, for oscillation wavelengths greater than distances traversed by the electrons in a period of the external rf field the quantity $\omega_{Le}\omega_0^{-1}v_E$ is analogous to the mean thermal velocity of the particles; in the usual way, this thermal velocity must be exceeded if a two-stream instability is to occur.

In the region of high a (1.18) becomes

$$\frac{\omega^2}{\omega_{Li}^2}=1+\frac{\omega_{Le}^2}{(\mathbf{ku})^2-\omega_{Le}^2}\frac{2}{\pi|a|}\cos^2\left(\frac{\pi}{4}-|a|\right). \quad (1.20)$$

In the absence of the rf field the maximum

2)We note that the method of averaging can also be used when the quantity $\mathbf{k}\cdot\mathbf{v}_E/\omega_0$ is small, or, what is the same thing, when the strength of the high-frequency electric field is small and the oscillation wavelength is large.

growth rate arises in the region $|\mathbf{k}\cdot\mathbf{u}| = \omega_{Le}$, in which (1.18) does not apply.[8-10] Therefore let us consider this region in detail. Let $(\mathbf{k}\cdot\mathbf{u})^2 = \omega_{Le}^2(1+\Delta)$, where $|\Delta| \ll 1$. From (1.8) we now have

$$2\omega^3(\mathbf{ku}) - \omega^2\omega_{Le}^2\left[\Delta - \frac{m_e}{m_i}\right] - 2\omega(\mathbf{ku})\omega_{Li}^2 + \omega_{Le}^2\omega_{Li}^2[\Delta + J_0^2(a)] = 0. \qquad (1.21)$$

If $|\Delta| \ll J_0^2$ [i.e., far from the right-hand boundary of the instability region (1.19)], (1.21) shows that when $|\omega|/|\mathbf{k}\cdot\mathbf{u}| \ll J_0^2$ the maximum growth rate is realized when $\Delta = m_e/m_i$; this growth rate is

$$\gamma = \omega_{Le}\frac{\sqrt{3}}{2}\left(\frac{\omega_{Li}^2J_0^2}{2\omega_{Le}^2}\right)^{1/3}, \quad J_0^2 \gg \frac{(27)^{1/4}}{4}\frac{\omega_{Li}}{\omega_{Le}}. \qquad (1.22)$$

A comparison with the growth rate for the usual two-stream instability shows that the present expression contains a factor $J_0^{2/3}$ which reduces the growth rate in this region when an external rf field is applied.

2. KINETIC THEORY FOR OSCILLATIONS OF AN UNMAGNETIZED PLASMA

In the present section we derive a dispersion equation and consider certain spectra of the irrotational oscillations of a plasma in an rf field; however, in contrast with the preceeding section here we shall take account of the thermal motion of the particles. To describe irrotational oscillations we use the kinetic equation with the self-consistent interaction:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}_\alpha\frac{\partial f_\alpha}{\partial \mathbf{r}_\alpha} - \frac{\partial f_\alpha}{\partial \mathbf{p}_\alpha}\frac{\partial}{\partial \mathbf{r}_\alpha}\sum_\beta\int d\mathbf{r}_\beta d\mathbf{p}_\beta\frac{e_\alpha e_\beta}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|}f_\beta = -\frac{\partial f_\alpha}{\partial \mathbf{p}_\alpha}e_\alpha\mathbf{E}(t). \qquad (2.1)$$

Here, f_α is the distribution function for particles of type α and $\mathbf{E}(t)$ is the time-varying external electric field.

The equilibrium distribution function for a uniform plasma in an external electric field is

$$f_\alpha^{(0)}(\mathbf{p}_\alpha, t) = f_{\alpha 0}\left(\mathbf{p}_\alpha - e_\alpha\int_{-\infty}^{t} dt'\mathbf{E}(t')\right). \qquad (2.2)$$

Our problem is now reduced to that of analyzing the oscillations representing weak perturbations of the state described by (2.2). Thus, writing the distribution function in the form $f_\alpha = f_\alpha^{(0)} + \delta f_\alpha(\mathbf{p}_\alpha, t)e^{-i\mathbf{k}\cdot\mathbf{r}_\alpha}$ and linearizing (2.1) we obtain the initial equation for the problem

$$\frac{\partial \delta f_\alpha}{\partial t} + i\mathbf{k}\mathbf{v}_\alpha\delta f_\alpha - i\mathbf{k}\frac{\partial f_\alpha^{(0)}(\mathbf{p}_\alpha, t)}{\partial \mathbf{p}_\alpha}\sum_\beta\frac{4\pi e_\alpha e_\beta}{k^2}\int d\mathbf{p}_\beta\delta f_\beta = -\frac{\partial \delta f_\alpha}{\partial \mathbf{p}_\alpha}e_\alpha\mathbf{E}(t). \qquad (2.3)$$

It will be convenient to introduce the function

$$\psi_\alpha(t, \mathbf{p}_\alpha) = \exp\left\{i\frac{e_\alpha}{m_\alpha}\mathbf{k}\int_{-\infty}^{t} dt'\int_{-\infty}^{t'} dt''\mathbf{E}(t'')\right\} \times \delta f_\alpha\left(t, \mathbf{p}_\alpha + e_\alpha\int_{-\infty}^{t} dt'\mathbf{E}(t')\right), \qquad (2.4)$$

which, as is easily shown, is governed by the equation

$$\frac{\partial \psi_\alpha(t, \mathbf{p}_\alpha)}{\partial t} + i\mathbf{k}\mathbf{v}_\alpha\psi_\alpha - i\mathbf{k}\frac{\partial f_{\alpha 0}(\mathbf{p}_\alpha)}{\partial \mathbf{p}_\alpha}\sum_\beta\frac{4\pi e_\alpha e_\beta}{k^2}\int d\mathbf{p}_\beta\psi_\beta(t, \mathbf{p}_\beta) \times \exp\left\{i\left(\frac{e_\alpha}{m_\alpha} - \frac{e_\beta}{m_\beta}\right)\mathbf{k}\int_{-\infty} dt'\int_{-\infty} dt''\mathbf{E}(t'')\right\} = 0. \qquad (2.5)$$

For a monochromatic external field $\mathbf{E} = \mathbf{E}_0 \sin\omega_0 t$ the relation in (2.5) becomes

$$\frac{\partial \psi_\alpha}{\partial t} + i\mathbf{k}\mathbf{v}_\alpha\psi_\alpha - i\mathbf{k}\frac{\partial f_{\alpha 0}(\mathbf{p}_\alpha)}{\partial \mathbf{p}_\alpha}\sum_\beta\frac{4\pi e_\alpha e_\beta}{k^2} \times \int d\mathbf{p}_\beta\psi_\beta(t, \mathbf{p}_\beta)\sum_{l=-\infty}^{\infty}e^{-il\omega_0 t}J_l(a_{\alpha\beta}) = 0, \qquad (2.6)$$

where

$$a_{\alpha\beta} = \frac{\mathbf{kE}}{\omega_0}\left(\frac{e_\alpha}{m_\alpha} - \frac{e_\beta}{m_\beta}\right).$$

As in the preceding section we assume that the frequency of the external field ω_0 is much higher than the frequency of the characteristic plasma oscillations. We also assume that $\omega_0^2 \gg (\mathbf{k}\cdot\mathbf{v}_\alpha)^2$. The relatively slow motions of the plasma can then be described by the function $\langle\psi_\alpha\rangle$ which represents the result of averaging the function ψ_α over the period of the high-frequency external field. In this case we have from (2.6)

$$\frac{\partial\langle\psi_\alpha\rangle}{\partial t} + i\mathbf{k}\mathbf{v}_\alpha\langle\psi_\alpha\rangle - i\mathbf{k}\frac{\partial f_{\alpha 0}(\mathbf{p}_\alpha)}{\partial \mathbf{p}_\alpha} \times \sum_\beta\frac{4\pi e_\alpha e_\beta}{k^2}\int d\mathbf{p}_\beta\langle\psi_\beta\rangle J_0(a_{\alpha\beta}) = 0. \qquad (2.7)$$

Equation (2.7) can be solved by the usual method used in the kinetic investigation of characteristic plasma oscillations.[7] The dispersion equation for the irrotational oscillations can be written in the form

$$\left|\delta_{\alpha\beta} + \frac{4\pi e_\alpha e_\beta}{k^2}J_0(a_{\alpha\beta})\int\frac{d\mathbf{p}_\alpha}{\omega + i0 - \mathbf{k}\mathbf{v}_\alpha}\mathbf{k}\frac{\partial f_\alpha}{\partial \mathbf{p}_\alpha}\right| = 0. \qquad (2.8)$$

The number of rows (and columns) of the determinant on the left side of (2.8) is equal to the number of charged-particle species in the plasma. If the plasma consists of electrons and one ion species (2.8) becomes

$$1+\delta\varepsilon_e(\omega,\mathbf{k})+\delta\varepsilon_i(\omega,\mathbf{k})+[1-J_0^2(a)]\times\delta\varepsilon_e(\omega,\mathbf{k})\,\delta\varepsilon_i(\omega,\mathbf{k})=0, \qquad (2.9)$$

where

$$\delta\varepsilon_\alpha(\omega,\mathbf{k})=\frac{4\pi e_\alpha^2}{k^2}\int\frac{d\mathbf{p}_\alpha}{\omega+i0-\mathbf{k}\mathbf{v}_\alpha}\,\mathbf{k}\frac{\partial f}{\partial\mathbf{p}_\alpha}. \qquad (2.10)$$

We now use (2.9) for a plasma with a Maxwellian particle distribution. In other words, it is assumed that the momentum distribution function is Maxwellian in the coordinate system in which particles of a given species do not oscillate under the effect of the external field. Then (2.9) becomes

$$1+\frac{\omega_{Le}^2}{k^2v_{Te}^2}\left[1-J_+\left(\frac{\omega}{kv_{Te}}\right)\right]+\frac{\omega_{Li}^2}{k^2v_{Ti}^2}\left[1-J_+\left(\frac{\omega}{kv_{Ti}}\right)\right]+[1-J_0^2(a)]\frac{\omega_{Le}^2}{k^2v_{Te}^2}\left[1-J_+\left(\frac{\omega}{kv_{Te}}\right)\right]\frac{\omega_{Li}^2}{k^2v_{Ti}^2}\times\left[1-J_+\left(\frac{\omega}{kv_{Ti}}\right)\right]=0, \qquad (2.11)$$

where $v_{Te}=(T_e/m_e)^{1/2}$ and $v_{Ti}=(T_i/m_i)^{1/2}$ are the electron and ion thermal velocities, while

$$J_+(x)=xe^{-x^2/2}\int_{i\infty}^{x}d\tau e^{\tau^2/2}.$$

If the phase velocity of the wave is large compared with the particle thermal velocities (2.11) yields the following expressions for the real frequency ω and the damping of the high-frequency oscillations:

$$\omega^2=\omega_{Le}^2+3v_{Te}^2k^2+\omega_{Li}^2J_0^2(a), \qquad (2.12)$$

$$\gamma=\sqrt{\frac{\pi}{8}}\frac{\omega_{Le}^4}{k^3v_{Te}^3}\exp\left\{-\frac{\omega^2}{2k^2v_{Te}^2}\right\}. \qquad (2.13)$$

The expression in (2.13) is of the same form as the damping rate for Langmuir oscillations in the absence of an external rf field, the sole difference being that in the right side of the present formula the frequency is determined by (2.12). Both of these expressions apply when the wavelength is large compared with the electron Debye radius $r_{De}=(T_e/4\pi e^2n_e^{(0)})^{1/2}$. According to (2.12) the effect of the external field on the rf plasma oscillations can be neglected at wavelengths smaller than $[(3m_i/m_e)(T_e/4\pi e_i^2n_i^{(0)})]^{1/2}$. If those wavelengths correspond to large values of the argument of the Bessel function J_0 the external rf field can be neglected at even higher wavelengths.

The other rf spectrum appearing as a generalization of (1.14) and also corresponding to the case of phase velocities appreciably greater than the particle-thermal velocities is given by

$$\omega^2=\omega_{Li}^2[1-J_0^2(a)]+3k^2[v_{Ti}^2+v_s^2], \qquad (2.14)$$

$$\gamma=\sqrt{\frac{\pi}{2}}\frac{\omega^4}{k^3}\left\{\frac{1}{v_{Te}^3}\exp\left[-\frac{\omega^2}{2k^2v_{Te}^2}\right]+\frac{1}{2v_{Ti}^3}\exp\left[-\frac{\omega^2}{2k^2v_{Ti}^2}\right]\right\}, \qquad (2.15)$$

where $v_s^2=|e_i/e|T_e/m_i$. These formulas apply when

$$\omega_{Li}^2[1-J_0^2(a)]\gg k^2v_{Te}^2,\quad k^2v_{Ti}^2. \qquad (2.16)$$

Thus, (2.14) differs from (1.14) only by small corrections. The damping of low-frequency oscillations described by (2.15) is specifically a kinetic phenomenon, since it stems from the inverse Cerenkov effect.

Another important nonhydrodynamic effect is the excitation of ion-acoustic oscillations when the electron temperature is appreciably greater than the ion temperature. By means of (2.9) we can delineate the effect of the external rf electric field on ion-acoustic waves whose phase velocity is large compared with the ion thermal velocity, but much smaller than the electron thermal velocity. From (2.9) we find

$$\omega^2=\omega_{Li}^2\left\{1-\frac{1}{1+(kr_{De})^2}J_0^2(a)\right\}+3k^2v_{Ti}^2, \qquad (2.17)$$

$$\gamma=\sqrt{\frac{\pi}{8}}\frac{\omega^4}{k^3}\left\{\frac{1}{v_{Ti}^3}\exp\left[-\frac{\omega^2}{2k^2v_{Ti}^2}\right]+\frac{\omega_{Le}^2\omega_{Li}^2}{\omega^4v_{Te}^3}\frac{J_0^2(a)}{[1+(kr_{De})^{-2}]^2}\right\}. \qquad (2.18)$$

In the limit of wavelengths larger than the electron Debye radius we have from (2.17)

$$\omega^2=\omega_{Li}^2\{1-J_0^2(a)\}+k^2\{v_s^2J_0^2(a)+3v_{Ti}^2\}. \qquad (2.19)$$

If the distance traversed by an electron in one oscillation period under the effect of the external rf field is large compared with the electron Debye radius (at angles between $\mathbf{k}$ and $\mathbf{E}_0$ which are not too close to $\pi/2$) the most important term in (2.19) is the first term. In other words, the spectrum of ion-acoustic oscillations is similar to the low-frequency oscillations in (2.14), which also obviously correspond to the joint oscillations of electrons and ions. However, the conditions for applicability of these formulas are generally different.

3. LOW-FREQUENCY IRROTATIONAL PLASMA OSCILLATIONS IN A CONSTANT MAGNETIC FIELD

In a constant magnetic field **B** the motion of the charged particles is characterized by the cyclotron frequencies $\Omega_\alpha = e_\alpha B/m_\alpha c$. We consider the case in which the frequency of the external electric field is much higher than the cyclotron frequency, following the approach used in the preceding section; the dispersion equation for the irrotational oscillations is found to be:

$$\left|\delta_{\alpha\beta} + \frac{4\pi e_\alpha e_\beta}{k^2} J_0(a_{\alpha\beta}) \int d\mathbf{p}_\alpha \sum_{n=-\infty}^{\infty} J_n^2\left(\frac{k_\perp v_{\perp\alpha}}{\Omega_\alpha}\right) \times \frac{k_z \partial f_{\alpha 0}/\partial p_{\alpha z} + n\Omega_\alpha \partial f_{\alpha 0}/v_{\perp\alpha}\partial p_{\perp\alpha}}{\omega + i0 - n\Omega_\alpha - k_z v_{z\alpha}}\right| = 0. \quad (3.1)$$

The z axis is along the magnetic field while $v_\perp^2 = v_x^2 + v_y^2$.

In a plasma consisting of electrons and one ion species (3.1) becomes

$$1 + \delta\varepsilon_e(\omega, \mathbf{k}) + \delta\varepsilon_i(\omega, \mathbf{k}) + [1 - J_0^2(a)]\delta\varepsilon_e(\omega, \mathbf{k})\delta\varepsilon_i(\omega, \mathbf{k}) = 0, \quad (3.2)$$

where

$$\delta\varepsilon_\alpha(\omega, \mathbf{k}) = \frac{4\pi e_\alpha^2}{k^2} \sum_{n=-\infty}^{\infty} \int d\mathbf{p}_\alpha J_n^2\left(\frac{k_\perp v_{\perp\alpha}}{\Omega_\alpha}\right) \times \frac{k_z \partial f_{\alpha 0}/\partial p_{\alpha z} + n\Omega_\alpha \partial f_{\alpha 0}/v_{\perp\alpha}\partial p_{\perp\alpha}}{\omega + i0 - n\Omega_\alpha - k_z v_{z\alpha}}. \quad (3.3)$$

If $f_{\alpha 0}$ is taken to be a Maxwellian function, after integration over momentum we have from (3.3)[11]

$$\delta\varepsilon_\alpha(\omega, \mathbf{k}) = \frac{\omega_{L\alpha}^2}{k^2 v_{T\alpha}^2}\left\{1 - \sum_{n=-\infty}^{\infty} \frac{\omega}{\omega - n\Omega_\alpha} A_{|n|}(z_\alpha) J_+(\beta_n^\alpha)\right\}. \quad (3.4)$$

Here

$$A_n(z_\alpha) = e^{-z_\alpha} I_n(z_\alpha), \quad J_+(\beta_n^\alpha) = \beta_n^\alpha e^{-(\beta_n^\alpha)^2/2} \int_{i\infty}^{\beta_n^\alpha} d\tau e^{\tau^2/2},$$

$$z_\alpha = \frac{k_\perp^2 v_{T\alpha}^2}{\Omega_\alpha^2}, \quad \beta_n^\alpha = \frac{\omega - n\Omega_\alpha}{|k_z| v_{T\alpha}}, \quad (3.5)$$

where $I_n(z_\alpha)$ is the Bessel function of imaginary argument.

Let us now investigate plasma oscillations that arise under these conditions in the frequency region $|\omega| \ll \Omega_i$. In this case we only retain the n = 0 term in the series in (3.4) and obtain the following dispersion equation:

$$1 + \frac{\omega_{Le}^2}{k^2 v_{Te}^2}[1 - A_0(z_e) J_+(\beta_0^e)] + \frac{\omega_{Li}^2}{k^2 v_{Ti}^2}[1 - A_0(z_i) J_+(\beta_0^i)] + [1 - J_0^2(a)] \frac{\omega_{Le}^2}{k^2 v_{Te}^2}[1 - A_0(z_e) J_+(\beta_0^e)] \frac{\omega_{Li}^2}{k^2 v_{Ti}^2} \times [1 - A_0(z_i) J_+(\beta_0^i)] = 0. \quad (3.6)$$

The expression in (3.6) generalizes the dispersion equation we have given earlier (2.11) to the case of an external magnetic field. If the waves propagate along the magnetic field ($k_\perp = 0$) it is evident that the earlier result (2.11) applies.

For wavelengths somewhat greater than the mean Larmor radius of the particles ($k_\perp v_{T\alpha}/\Omega_\alpha \ll 1$) and propagating at angles not too close to $\pi/2$, we have from (3.6)

$$1 + \frac{\omega_{Le}^2}{k^2 v_{Te}^2}\left[1 - J_+\left(\frac{\omega}{|k_z| v_{Te}}\right)\right] + \frac{\omega_{Li}^2}{k^2 v_{Ti}^2}\left[1 - J_+\left(\frac{\omega}{|k_z| v_{Ti}}\right)\right] + [1 - J_0^2(a)] \frac{\omega_{Le}^2}{k^2 v_{Te}^2}\left[1 - J_+\left(\frac{\omega}{|k_z| v_{Te}}\right)\right] \times \frac{\omega_{Li}^2}{k^2 v_{Ti}^2}\left[1 - J_+\left(\frac{\omega}{|k_z| v_{Ti}}\right)\right] = 0. \quad (3.7)$$

This expression differs from (2.11) only in that the quantity k is replaced by $|k_z|$ in the argument of the J_+ functions.

Using (3.7) we can obtain the oscillation spectrum and the damping rate in the presence of an external magnetic field. Thus, if the phase velocity is much greater than the particle thermal velocities (2.12) and (2.13) are replaced by

$$\omega^2 = \cos^2\theta\{\omega_{Le}^2 + \omega_{Li}^2 J_0^2(a) + 3v_{Te}^2 k^2\}, \quad (3.8)$$

$$\gamma = \sqrt{\frac{\pi}{8}} \frac{\omega_{Le}^4}{k_z^3 v_{Te}^3} \exp\left\{-\frac{\omega^2}{2k_z^2 v_{Te}^2}\right\}, \quad (3.9)$$

where θ is the angle between **k** and **B**.

In the frequency region $|k_z| v_{Ti} \ll |\omega| \ll |k_z| v_{Te}$ (when $T_e \gg T_i$) we find propagation is possible for ion-acoustic waves such that the frequency

$$\omega^2 = \cos^2\theta\left\{\omega_{Li}^2\left[1 - \frac{J_0^2(a)}{1 + (kr_{De})^2}\right] + 3k^2 v_{Ti}^2\right\} \quad (3.10)$$

and the damping rate

$$\gamma = \sqrt{\frac{\pi}{8}} \frac{\omega^4}{k_z^3}\left\{\frac{1}{v_{Ti}^3} \exp\left[-\frac{\omega^2}{2k_z^2 v_{Ti}^2}\right] + \frac{\omega_{Li}^2 \omega_{Le}^2 J_0^2 \cos^4\theta}{\omega^4 v_{Te}^3 [1 + (kr_{De})^{-2}]^2}\right\}. \quad (3.11)$$

Finally, in the limit of wavelengths larger than the electron Debye radius, using (3.10) we obtain an expression which differs from that obtained earlier in the absence of the magnetic field (2.19) by the factor $\cos^2\theta$.

If the thermal motion of the plasma particles is neglected, i.e., if a hydrodynamic analysis similar to that in the first section is used, in place of (3.1) we find

$$\left|\delta_{\alpha\beta}-\frac{\omega_{L\alpha}}{\omega^2}\left(\cos^2\theta+\sin^2\theta\frac{\omega^2}{\omega^2-\Omega\alpha^2}\right)J_0(a)\right|=0. \quad (3.12)$$

In a plasma consisting of electrons and one ion species (3.12) assumes the form

$$1-\frac{\omega_{Le}^2+\omega_{Li}^2}{\omega^2}\cos^2\theta-\left[\frac{\omega_{Le}^2}{\omega^2-\Omega_e^2}+\frac{\omega_{Li}^2}{\omega^2-\Omega_i^2}\right]\sin^2\theta$$
$$+[1-J_0^2(a)]\frac{\omega_{Le}^2\omega_{Li}^2}{\omega^4}\left(\cos^2\theta+\sin^2\theta\frac{\omega^2}{\omega^2-\Omega_e^2}\right)$$
$$\times\left(\cos^2\theta+\sin^2\theta\frac{\omega^2}{\omega^2-\Omega_i^2}\right)=0. \quad (3.13)$$

The magnetic field does not have an effect on the oscillation spectrum for strictly longitudinal propagation ($\theta = 0$) and we obtain the earlier expressions (1.13) and (1.14). In propagation of waves directly across the magnetic field the oscillation frequency is given by

$$\omega^2=\omega_{Le}^2+\omega_{Li}^2+\Omega_e^2+\Omega_i^2$$
$$-\frac{\Omega_e^2(\Omega_i^2+\omega_{Li}^2)+(1-J_0^2(a))\omega_{Le}^2\omega_{Li}^2}{\Omega_e^2+\omega_{Le}^2}, \quad (3.14)$$

$$\omega^2=\frac{\Omega_e^2(\Omega_i^2+\omega_{Li}^2)+(1-J_0^2(a))\omega_{Le}^2\omega_{Li}^2}{\Omega_e^2+\omega_{Le}^2}. \quad (3.15)$$

Let us now assume that the magnetic field is not large so that the inequality $\omega_{Le}^2/\Omega_e^2 \gg 1$ is satisfied. From (3.14) and (3.15) we then have

$$\omega^2\approx\omega_{Le}^2+\omega_{Li}^2J_0^2(a), \quad (3.16)$$

$$\omega^2\approx\omega_{Li}^2\left(\frac{\Omega_e^2}{\omega_{Le}^2}+1-J_0^2(a)\right). \quad (3.17)$$

It is evident from (3.17) that the external rf field will have an important effect on the plasma oscillation spectrum when

$$1-J_0^2(a)\gg\frac{\Omega_e^2}{\omega_{Le}^2}. \quad (3.18)$$

Under these conditions the oscillation spectrum (3.17) becomes similar to the characteristic spectrum (1.14) described in detail in the first section of the present work. Finally, we present expressions for the oscillation spectrum for waves propagating at arbitrary angles (but not close to $\pi/2$ or 0).

In the frequency region $|\omega| \ll \Omega_i$ we obtain the electron branch

$$\omega^2=\cos^2\theta(\omega_{Le}^2+\omega_{Li}^2J_0^2(a)). \quad (3.19)$$

In general, when one takes account of spatial dispersion due to the thermal motion of particles the small thermal corrections to the frequency completely determine the velocity of propagation of a wave; similarly, here [(and also in (3.20)], we keep the small term $\omega_{Li}^2J_0^2$, which completely determines the spatial dispersion of the cold plasma.

Evidently (3.19) differs from the spectrum obtained earlier (1.13) only in the presence of the factor $\cos^2\theta$. Correspondingly, in the ion branch of the oscillations the difference from (1.14) (no magnetic field) reduces to the appearance of the factor $\cos^2\theta$.

In the intermediate frequency region where $\Omega_i \ll |\omega| \ll |\Omega_e|$ propagation of electron oscillations with frequency

$$\omega^2=\omega_{Le}^2\cos^2\theta+\omega_{Li}^2J_0^2(a) \quad (3.20)$$

becomes possible almost over the entire angular region if $\Omega_i^2 \ll \omega_{Le}^2 \ll \Omega_e^2$. In this case the magnetic field has no effect on the spectrum of ion oscillations and the earlier result (1.14) is recovered.

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