INFRARED SINGULARITIES IN LOCAL FIELD THEORY

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The first term and an estimate of the following term in the Green's function expansion for a charged particle in the infrared region are obtained in a local field theory in which quantum electrodynamics in all orders of e is included. A similar expansion is obtained for the Compton scattering amplitude with a fixed value of the transferred momentum. The method is based on application of the dispersion relations and expansions of the current matrix elements in terms of the soft-photon momenta.

1. INTRODUCTION

THE infrared asymptotic behavior of the Green's function of a charged particle has been the subject of many investigations. It was shown by various methods—the method of renormalization group [1], approximate solution of functional equations [2], solution of integral equations in the ladder approximation [3], or functional integration [4]—that the first term of the expansion of the Green's function in the infrared region is of the form $(1 - r^2m^{-2})^{\gamma-1}$, where γ is generally speaking a series in e^2, in which the first term is obtained. It was shown with the aid of a renormalization group that there is no term of order e^4 in γ[5].

This raises the question of finding an exact expression for the exponent γ and of finding the next terms of the Green's-function expansion in the infrared region $r^2 \to m^2$. This question was dealt with by Milekhin [6], who used the method of functional integration [7], but estimated the next higher terms of the expansion by perturbation theory. Analogous results were obtained recently by Barbashov [8] with the aid of a new improved method of functional integration.

In this paper, without using perturbation theory, we obtain the first term and estimate the second term of the Green's-function expansion in the infrared region. An analogous expansion was obtained for the amplitude of Compton scattering at fixed momentum transfer. We start from the general premises of the local field theory [9], including quantum electrodynamics in all orders in e, i.e., we assume in addition to the general axioms [9], that the continuous energy spectrum can start with zero and that the theory should be gauge-invariant, and use an indefinite metric for the time-dependent photons. In order to avoid divergences in the spectrality condition, connected with the zero energy of the photons, we introduce during intermediate stages of the calculations an effective photon mass λ, which we then let approach zero ahead of all the physical variables.

The method employed consists in using dispersion relations and expanding the matrix elements of the currents in the momenta of soft photons. These expansions are a generalization of the results obtained in lower order in e by Low [10] and of Bilen'kiĭ and Ryndin [11]. To obtain these expansions, we use the general equations of local theory [9], which were investigated by Medvedev and Polivanov [12] in connection with an axiomatic construction of the theory of strong interactions, and we consider them with allowance for the electromagnetic interaction. We shall show that a natural mathematical apparatus for the theory of dispersion relations with allowance of electromagnetic interaction is the apparatus of generalized power functions, developed in the book of Gel'fand and Shilov [13].

In addition to an analysis of the Green's function and the amplitude of the Compton scattering, we consider also the application of the obtained expansions of the matrix elements of the current to inelastic processes and find the first terms of the expansion of the cross section of bremsstrahlung in the radiation energy. In the last section of the paper we consider the question of factorization of the infrared divergences. We show that this factorization is a simple consequence of the spectrality and of the physically justified requirement that there be no infrared singularities if the momenta of the charged particles remain unchanged.

To simplify the exposition we consider only processes in which only one spinless charged particle (meson) exists besides an arbitrary number of neutral particles. Generalization to the case of a charged particle with spin $1/2$, and also to the case of an arbitrary number of charged particles, does not entail any difficulties in principle.

The system of units and metric employed are such that $\hbar = c = 1$ and $ab = a^0b^0 = \mathbf{a}\cdot\mathbf{b}$. The amplitude of the state of the particle with momentum k is normalized by the condition $\langle k' \mid k\rangle = (2\pi)^3 2k^0\delta(\mathbf{k}' - \mathbf{k})$, and accordingly we introduce the notation

$$\tilde{d}\mathbf{k} = [(2\pi)^3 2k^0]^{-1}d\mathbf{k}. \tag{1.1}$$

If some quantity $F(k_i)$ depends on the photon momentum k_i (and on its polarization vector), then for the product of such quantities corresponding to n photons we used the symbol

$$(F(k))_n = \prod_{i=1}^{n} F(k_i). \tag{1.2}$$

2. MATRIX ELEMENTS OF NEUTRAL CURRENTS

We consider the matrix element

$$\langle k_1, \ldots k_n, p|J|r\rangle = \langle n, p|J|r\rangle, \tag{2.1}$$

where $J = J(0)$, $J(x) = i(\delta S/\delta\chi(x))S^+$ is the neutral-current operator and $\chi(x)$ is an arbitrary local electrically-neutral field. If χ is an electromagnetic field, then we shall also write j in place of J; p and r are the momenta of the charged meson with mass m, and k_i are the momenta of soft photons with real polarization vectors ϵ_i. We put $t = (p - r)^2$.

Let us investigate the dependence of (2.1) on $k_1, \ldots, k_n$. To this end we use a system of equations which follows from the principal axioms of the local field theory [9,12] when account is taken of the electromagnetic interaction ($n = 1, 2, \ldots$):

$$\langle n, p|J|r\rangle = -(2\pi)^3 \times \sum_N \Big[\frac{\delta(\mathbf{r}_N - \mathbf{p} - \mathbf{K}_n)\langle n-1, p|\varepsilon_n j|N\rangle\langle N|J|r\rangle}{r_N^0 - p^0 - K_n^0 - i0} + \frac{\delta(\mathbf{r}_N - \mathbf{r} + \mathbf{k}_n)\langle n-1, p|J|N\rangle\langle N|\varepsilon_n j|r\rangle}{r_N^0 - r^0 + k_n^0 - i0}\Big]; \tag{2.2}$$

$$K_n = k_1 + \ldots + k_n. \tag{2.3}$$

In local theory, Eq. (2.2) is defined accurate to a polynomial in k_n. However, we seek only those terms of the matrix element (2.1) which are singular when any of the photon momenta tends to zero. Therefore any arbitrariness in (2.2), other than that connected with the introduction of terms that are singular as $k_n \to 0$, is of no importance to us. Equally unimportant are the possible divergences arising in the integration over large momenta of intermediate particles. In order to avoid divergences in the integration over small momenta of intermediate photons, we assign to them a small mass λ. We must assume here that λ is much smaller than all the momenta $|\mathbf{k}_i|$. In those expressions which are finite when $k_i \neq 0$, we shall put $\lambda = 0$.

In Eq. (2.2) the terms singular as $k_n \to 0$ come only from those intermediate states which cause non-integrable vanishing of the denominators when we set in them $\lambda = 0$ and $k_n = 0$. In the first term of the right side of (2.2), such states should contain a meson with momentum r_1 close to p, $n-1$ photons with momenta q_i close to k_i ($i = 1, \ldots, n-1$), and an arbitrary number of photons with momenta close to zero. A nonintegrable contribution is made only by those terms in $\langle n-1, p|\epsilon_n j|N\rangle$, which contain $(\delta(k-q))_{n-1}$, that is, which correspond to non-connective diagrams. In the second term, a contribution of interest to us is made only by those states which contain a meson with momentum r_1 close to r, and an arbitrary number of photons with momenta close to zero.

We write out the system of equations which give a singular dependence on k_n as $k_n \to 0$ not for the matrix elements (2.1) themselves, but for the amplitudes $J(a, p; r, b)$, defined by[1)]

$$\langle a, p|J|r, b\rangle = (m/\lambda)^{\beta} J(a, p; r, b), \tag{2.4}$$

where a and b are the quantum number of the electrically neutral particles,

$$\beta = \beta(t) = \frac{\alpha}{\pi}\Big(1 - pr\int_0^1 dx p_x^{-2}\Big),$$

$$p_x = px + r(1-x) \tag{2.5}$$

and α is the fine-structure constant. The physical meaning of the function β consists in the fact that it represents a relativistic generalization of the Coulomb Regge trajectory [15-19]. We obtain ($n = 1, 2, \ldots$)

$$J(n, p; r) = -(2\pi)^3 \sum_{l=0}^{\infty} \frac{1}{l!}\Big(\Sigma\eta\int_g \tilde{d}\mathbf{q}\Big)_l \Big[\int_{g_1} \tilde{d}\mathbf{r}_1\Big(\frac{m}{\lambda}\Big)^b \times \frac{\delta(\mathbf{r}_1 + \mathbf{Q}_l - \mathbf{p} - \mathbf{k}_n)\varepsilon_n j(p; r_1, l) J(l, n-1, r_1; r)}{r_1^0 + Q_l^0 - p^0 - k_n^0 - i0}$$

1)These amplitudes are finite when $\lambda = 0$ (see Sec. 6 and [14]).

$$+\int_{g_2} d\mathbf{r}_1 \left(\frac{m}{\lambda}\right)^b \times \frac{\delta(\mathbf{r}_1+\mathbf{Q}_l-\mathbf{r}+\mathbf{k}_n) J(n-1, p; r_1, l)\, \varepsilon_n j(l, r_1; r)}{r_1^0+Q_l^0-r^0+k_n^0-i0} \Bigg] \quad (2.6)$$

Here

$$b = -\beta((p-r)^2) + \beta((p-r_1)^2) + \beta((r_1-r)^2). \quad (2.7)$$

The argument l defining J and j denotes dependence on the photon momenta $q_1, \ldots, q_l$, the sum of which is equal to Q_l; Σ denotes summation over the polarizations of these photons; $\eta = -1$ for temporal polarization and $\eta = +1$ for spatial polarization[2]; the regions of integration are defined by the equations $|q| < h$ (region g), $||r_1| - |p|| < h_1$ (region g_1), and $||r_1| - |r|| < h_2$ (region g_2), where h, h_1, and h_2 are arbitrarily small, with $|k_n| \ll h$.

The system (2.6) has the following solution:

$$J(n, p; r) = (F(k))_n J(p; r) + O(k_i(k^{-1})_n), \quad (2.8)$$

where

$$F(k) = e\left(\frac{p\varepsilon}{pk} - \frac{r\varepsilon}{rk}\right)\Big[1 + 2\Big(pk \ln\frac{m^2}{rk} - rk\ln\frac{m^2}{pk}\Big) \times \beta'((p-r)^2)\Big] \quad (2.9)$$

and O(x) denotes a quantity of order x; $\beta'(t)$ is the derivative of the function (2.5). Thus, the coefficient in the logarithmic dependence on k is determined by the slope of the Coulomb Regge trajectory.

Besides the amplitudes (2.8), the right side of (2.6) contains also amplitudes with photons in the initial and final states. However, they are connected with (2.8) by a symmetry condition. In fact, the matrix element $\langle k | J | q \rangle$ (k and q are the photon momenta) is obtained when $k \neq q$ from $\langle k, q | J \rangle$ by substituting $-q$ for q. Choosing $h < |k_i|$ ($i = 1, \ldots, n-1$), we find that if (2.8) is satisfied, then $J(n-1, p; r_1, l)$, as a function of all the photon momenta, satisfies an analogous relation.

In the case of an electromagnetic current, it follows from the gauge invariance that the product of (2.1) by $k = K_n + p - r$ should vanish. Expression (2.8) satisfies this condition, if we can neglect the term K_n in k, that is, if $p \neq r$. However, (2.6) contains the amplitudes $j(n, p; r)$ for arbitrarily small values of $p - r$. Expression (2.8) is not gauge-invariant for these amplitudes, but can be made gauge-invariant by adding a term of order of a constant for each of the k_i;

$$j(n, p; r) = (F(k))_n\Big[j(p; r) + \sum_{i=1}^{n} B(k_i) f((p-r)^2)\Big], \quad (2.10)$$

where

$$j(p; r) = (p+r) f((p-r)^2), \qquad f(0) = e, \quad (2.11)$$

$$B(k) = \Big[k\Big(\frac{p\varepsilon}{pk} + \frac{r\varepsilon}{rk}\Big) - 2\varepsilon\Big]\Big(\frac{p\varepsilon}{pk} - \frac{r\varepsilon}{rk}\Big)^{-1}. \quad (2.12)$$

We now substitute expression (2.8) and its corollary (2.10) in the right side of (2.6). Using the equation

$$\frac{\delta(\mathbf{a})}{a^0 - i0} = \frac{i}{(2\pi)^3}\int_{x^0>0} d^4x e^{-iax}, \qquad a^0 \to a^0 - i0, \quad (2.13)$$

we can sum over l and go over to the limit as $\lambda \to 0$. We then obtain an expression which does not contain an explicit dependence on λ and has a small parameter $r_1 - p$ or $r_1 - r$. Expanding in this parameter and estimating each term of the expansion, we obtain expression (2.8). This expression is symmetrical in all the k_i, and therefore the limitation $|k_n| \ll |k_i|$ ($i = 1, \ldots, n-1$), which was used during the course of the calculations, is immaterial.

The uniqueness of (2.8) follows from the fact that the principal (pole) term in k_n gives a photonless intermediate state in (2.6), and also from the uniqueness of (2.10).

Expression (2.8) gives the first two terms of the expansion of the matrix element (2.1) and (2.5) in the photon momenta, which can be written in the form

$$J(n, p; r) = \Big(e\Big(\frac{p\varepsilon}{pk} - \frac{r\varepsilon}{rk}\Big)\Big)_n \Big[1 + \sum_{i=1}^{n} 2(p-r)k_i \ln\frac{m^2}{rk_i} \times \beta'((p-r)^2)\Big] J(p; r) + O(k_i(k^{-1})_n). \quad (2.14)$$

This expansion is valid also for a matrix element of more general form:

$$\langle k_1, \ldots, k_n, a, p | J | r, b, q_1, \ldots q_m \rangle,$$

and also for

$$\langle k_1, \ldots, k_n, a | J | \bar{p}, r, b, q_1, \ldots, q_m \rangle,$$

where a and b are quantum numbers or neutral rigid particles, $\bar{p}$ is the momentum of the antimeson, and q_i are the momenta of the soft photons. The momenta $\bar{p}$ and q_i will enter in the

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[2]It is convenient to use a Feynman gauge in which the vacuum does not contain temporal or longitudinal photons. The complete system of state amplitudes should include in this case states with these photons, and the summation over the polarizations of the intermediate photons is carried out by the formula $\Sigma \eta \epsilon_m \epsilon_n = -g_{mn}$ (g = metric tensor). Equation (2.6) does not depend on the gauge.

expansion (2.14) with a minus sign. The next term in the expansion (2.14) is discussed in [20].

The expansions obtained for one photon contain terms of order k^{-1}, $e^2 \ln k$, and k^0. In [10,11] the terms of order k^{-1} and k^0 were obtained in the lower order in e. We see that as $k \to 0$ the term of order $e^2 \ln k$, due to the electromagnetic interaction, is theoretically more important than the term of order k^0. However, the coefficient e^2 greatly reduces its value when $k \neq 0$.

Let us apply the obtained expansion to bremsstrahlung. If the photons are fixed, then we can use the expansion (2.14) directly. Let us consider the process of interaction of a charged meson with a neutral rigid particle, in which an arbitrary number of soft photons is emitted. Let the total energy w of these photons be fixed. Let also the square of the momentum transferred to the meson, $t = (p - r)^2$, be fixed. Then the cross section of this process, summed over all the soft photons, is equal to

$$d\sigma(w) = \left(\frac{2w}{m}\right)^d \left(\frac{1}{w} + \frac{g}{d}\ln\frac{m}{w}\right)\frac{\exp(-Cd+D)}{\Gamma(d)}\,d\sigma' + O\left(\left(\frac{w}{m}\right)^d\right). \tag{2.15}$$

Here $d = -2\beta(t)$,

$$g = e^2\beta'(t)\int\frac{d\mathbf{n}}{(2\pi)^3}\left(\frac{p}{pn} - \frac{r}{rn}\right)^2 2(r-p)n,$$

$$n = \{1, \mathbf{n}\},\ |\mathbf{n}| = 1. \tag{2.16}$$

C—Euler's constant, Γ—gamma function,

$$D = \varphi(p,p) - 2\varphi(p,r) + \varphi(r,r),$$

$$\varphi(p,r) = \frac{\alpha}{\pi}pr\int_0^1\frac{dx}{p_x^2}\frac{1}{2a}\ln\frac{1+a}{1-a},\quad a = \frac{|\mathbf{p}_x|}{p_x^0}, \tag{2.17}$$

and $d\sigma'$ is the cross section of the nonradiative process, determined by the amplitude J (2.4).

If w is not fixed and can vary from 0 to ΔE (ΔE—energy resolution), then

$$d\sigma(\Delta E) = \left(\frac{2\Delta E}{m}\right)^d\left(1 + \frac{g}{1+d}\Delta E\ln\frac{m}{\Delta E}\right) \times \frac{\exp(-Cd+D)}{\Gamma(1+d)}\,d\sigma' + O\left(\left(\frac{\Delta E}{m}\right)^{1+d}\right). \tag{2.18}$$

This formula gives the first two terms of the asymptotic expansion of the cross section in ΔE. The first term of this expansion is well known [14], and the second is a consequence of the expansion (2.14).

3. MATRIX ELEMENTS OF CHARGED CURRENTS AND FIELDS

We now consider the operator of the charged-meson current

$$I(x) = i(\delta S / \delta\varphi^+(x))S^+. \tag{3.1}$$

It is convenient to consider the matrix element not of the operator I itself, but of the field operator

$$\Phi(x) = \varphi(x) + \int D^a(x-y)I(y)\,dy, \tag{3.2}$$

where D^a is the advanced Green's function of the Klein-Gordon equation. We consider the matrix element

$$\langle 0|\Phi|r, k_1, \ldots k_n\rangle = \langle 0|\Phi|r, n\rangle, \tag{3.3}$$

where $\Phi = \Phi(0)$. It is not defined by (3.2) when $n = 0$. From (3.2) we have

$$\langle 0|\Phi|r,n\rangle = -i\int\int dx\,dy\,e^{-ik_nx}D^a(-y) \times \langle 0|[T_a(j(x)I(y)) + \Lambda(x,y)]|r, n-1\rangle\,\varepsilon_n, \tag{3.4}$$

where Λ is a quasilocal operator. It is easy to verify that as $k_i \to 0$ it gives a singular dependence on each k_i only in the case when $n = 1$.

Assume first that $n \geq 2$. Then the equation defining the singular dependence of the matrix element (3.3) on k_n as $k_n \to 0$ is of the form

$$\langle 0|\Phi|r,n\rangle = -(2\pi)^3\sum_N\Bigg[\frac{\delta(\mathbf{r}_N - \mathbf{k}_n)\langle 0|\varepsilon_n j|N\rangle\langle N|\Phi|r, n-1\rangle}{r_N^0 + k_n^0 + i0} \times\frac{(r_N - r - K_{n-1})^2 - m^2}{(r+K_n)^2 - m^2} + \frac{\delta(\mathbf{r}_N - \mathbf{r} - \mathbf{K}_n)\langle 0|\Phi|N\rangle\langle N|\varepsilon_n j|r, n-1\rangle}{r_N^0 - r^0 - K_n^0 + i0} \times\frac{r_N^2 - m^2}{(r+K_n)^2 - m^2}\Bigg] \tag{3.5}$$

As in the preceding section, we can verify that, with the accuracy of interest to us, this equation can be rewritten in the form $(n = 2, 3, \ldots)$

$$\langle 0|\Phi|r,n\rangle = -(2\pi)^3\int_{g_2}\tilde{d}\mathbf{r}_1\sum_l\frac{1}{l!}\left(\Sigma\eta\int_g\tilde{d}\mathbf{q}\right)_l \times\frac{\delta(\mathbf{r}_1 + \mathbf{Q}_l - \mathbf{r} - \mathbf{k}_n)\langle 0|\Phi|r_1, n-1, l\rangle\langle l, r_1|\varepsilon_n j|r\rangle}{r_1^0 + Q_l^0 - r^0 - k_n^0 + i0}. \tag{3.6}$$

This system is closed. Using for the matrix elements of the electromagnetic current the expansion (2.10) generalized to the case when $q\epsilon \neq 0$ (that is, for temporal and longitudinal photons), we find its solution for fixed k_1 (in the Feynman gauge)

$$\langle 0|\Phi|r, k_1, \ldots k_n\rangle = \prod_{i=2}^{n} L(k_i)\langle 0|\Phi|r, k_1\rangle + O\Big(k_j \prod_{i=2} k_i^{-1}\Big), \quad (3.7)$$

where $j = 2, \ldots n$ and

$$L(k) = e\frac{r\varepsilon}{rk}. \quad (3.8)$$

Since expression (3.7) should be symmetrical in all k_i, $i = 1, \ldots n$, we conclude that

$$\langle 0|\Phi|r, k_1, \ldots k_n\rangle = (L(k))_n Z + O(k_i(k^{-1})_n), \quad (3.9)$$

where Z does not depend on k_i and is consequently a constant. It can be called the matrix element (3.3) for $n = 0$:

$$\langle 0|\Phi|r\rangle = Z. \quad (3.10)$$

The expression for the matrix element of the current I is obtained from (3.9) with the aid of the equation

$$\langle 0|I|r, n\rangle = [(r + K_n)^2 - m^2]\langle 0|\Phi|r, n\rangle. \quad (3.11)$$

The constant Z is then equal to

$$Z = \lim_{k\to 0} (2er\varepsilon)^{-1}\langle 0|I|r, k\rangle. \quad (3.12)$$

From the point of view of perturbations, the constant Z is determined by the corrections to the external meson line and contains infrared divergences.

4. GREEN'S FUNCTION

With the aid of expression (3.9) we can investigate the behavior of the Green's function of a charged meson in the infrared region. We write the spectral representation of this function in the following form:

$$G(r^2) = \int_{m^2}^{m^2+h} \frac{g(q^2)\,dq^2}{q^2 - r^2 - i0} + R(r^2), \quad (4.1)$$

where the function $R(r^2)$ is regular in the infrared region $r^2 \to m^2$. The spectral function in this expression can be written in the form

$$g(r^2) = (2\pi)^3 \sum_N \delta(r - r_N)\langle 0|\Phi|N\rangle\langle N|\Phi^+|0\rangle, \quad (4.2)$$

where the state N consists of a meson with momentum r_1 close to r and of an arbitrary number of soft photons when h is sufficiently close to zero. We use (3.9) and replace the δ-function by its Fourier integral. This allows us to sum over all the intermediate photons. Further, in the coordinate system where $r = 0$, we can expand the integrand in (4.2) in terms of r_i and estimate each term of the expansion.

As a result we obtain, in a Feynman gauge, the following expression:

$$g(r^2) = Z_1^2 e^{-C\gamma} \frac{x^{\gamma-1}}{\Gamma(\gamma)} (1 + O(x)), \quad (4.3)$$

where

$$x = r^2 m^{-2} - 1,\ \gamma = -\alpha/\pi,\ Z_1^2 = Z^2 m^{-2}(m/\lambda e)^\gamma. \quad (4.4)$$

We see that, since $\gamma < 0$, the spectral function (4.3) is not integrable in the usual sense near $x = 0$. It is natural to generalize the concept of the integral in (4.1) or, what is the same, to regard the spectral function near $x = 0$ as a generalized power function

$$f(x, a) = \frac{x_+^a}{\Gamma(a+1)}, \qquad x_+^a = \begin{cases} x^a, & x > 0 \\ 0, & x \leqslant 0 \end{cases}, \quad (4.5)$$

defined by Gel'fand and Shilov [13]. The integral

$$\int_{-\infty}^{\infty} f(x, a)\varphi(x)\,dx,$$

where $\varphi(x)$ is regular at zero, is determined when $\mathrm{Re}\, a < -1$ with the aid of a suitably chosen regularization, which is equivalent to analytic continuation as $a_0 \to a$ of the integral

$$\int_{-\infty}^{\infty} f(x, a_0)\varphi(x)\,dx, \qquad \mathrm{Re}\, a_0 > -1.$$

The function (4.5) is an analytic function of a (for regular φ it defines a functional analytic in a). When $a = -n$ (n–positive integer) we have

$$f(x, -n) = \delta^{(n)}(x). \quad (4.6)$$

Its expansion in a Taylor series near $a = -1$ is of the form

$$e^{-\gamma C} f(x, \gamma - 1) = \delta(x) + \gamma x_+^{-1} + \gamma^2 x_+^{-1} \ln x_+ + \ldots, \quad (4.7)$$

where x_+^{-1} is another generalized function, defined by the condition

$$x_+^{-1} = x^{-1} \text{ for } x > 0, \quad x_+^{-1} = 0 \text{ for } x \leqslant 0;$$

$$\int_{-\infty}^{\infty} x_+^{-1}\varphi(x)\,dx = \int_{-\infty}^{\infty} x^{-1}[\varphi(x) - \theta(1-x)\varphi(0)]\,dx. \quad (4.8)$$

Substituting (4.3) in (4.1) we have

$$G(r^2) = Z_2^2(-x)^{\gamma-1}(1 + O(x)) + \text{const}, \quad (4.9)$$

where

$$Z_2^2 = Z_1^2 \frac{\pi \exp(-C\gamma)}{\Gamma(\gamma)\sin \pi\gamma}. \quad (4.10)$$

Formula (4.9) gives the first term of the expansion of the Green's function in the infrared region and an estimate of the next term.

Let us compare formulas (4.3), (4.7), and (4.9) with the results of perturbation theory. Expanding the spectral function (4.3) in a series in γ with the aid of (4.7), we obtain

$$g(r^2) = m^{-2}[\delta(x) + \gamma(x^{-1} + O + \ldots) + \gamma^2(x^{-1}\ln x + O\ln x + \ldots) + \ldots]. \quad (4.11)$$

Comparing with the result in second order perturbation theory

$$g(r^2) = m^{-2}\left[\delta(x) + \gamma\frac{2+x}{2x(1+x)}\right], \quad (4.12)$$

we find that in the lowest order of perturbation theory the coefficient O is equal to

$$O(x) = Ox, \quad O = -1/2.$$

We see that the coefficient of $\gamma^2 \ln x$ in fourth order perturbation theory should be $-1/2$.

The perturbation theory series for the Green's function itself is more complicated. From (4.9) we have $(x < 0)$

$$G(r^2) = -m^2\{x^{-1} + O + \gamma[x^{-1}\ln^2(-x) + O\ln(-x) + \ldots] + \gamma^2[{}^1/_2 x^{-1}\ln^2(-x) + {}^1/_2 O\ln^2(-x) + \ldots] + \ldots\} + \text{const}. \quad (4.13)$$

We emphasize that for the term of order const in (4.9) and (4.13), a subtraction procedure is essential, that is, the region of large r^2.

5. COMPTON SCATTERING

Let us consider the amplitude of elastic scattering of a photon by a meson, ascribing to the photons in the intermediate states a small mass λ. The amplitude is equal to the matrix elements

$$\langle k_2, p|j\varepsilon_1|r\rangle = (m/\lambda)^\beta \varepsilon_{2a}\varepsilon_{1b}T^{ab}, \quad (5.1)$$

where $k_1 = k_2 + p - r$ and ϵ_1 are the momentum and polarization vector of the initial photon. We have separated, as in Sec. 2, the λ-dependent factor.

Let us investigate the dependence of T on $s = (r + k_1)^2$ near the threshold $s = m^2$ for fixed $t < 0$ and $\lambda = 0$.

Assume first that $\lambda \neq 0$. We put in T [9]

$$k_1^2 = k_2^2 = \tau < t/2. \quad (5.2)$$

We then obtain a function T_τ, satisfying the dispersion relations[3)]:

$$T_\tau(s, t) = \int_{m^2}^{m^2+h} \frac{T_{1\tau}(s', t)}{s' - s - i0} + R, \quad (5.3)$$

where R is finite at $s = m^2$.

[3)]This equation stands for several relations for invariant amplitudes.

As in Sec. 4, the absorptive part $T_{1\tau}$ in (5.3) is expressed by a sum over intermediate states containing a meson and an arbitrary number of low-energy photons. It is equal to

$$T_{1\tau}(s, t) = -\frac{\exp(Cb_0 + D_0)}{m^2\Gamma(-b_0)}(sm^{-2} - 1)^{-b_0-1}M^{ab}(\tau) + O((sm^{-2} - 1)^{-b_0}), \quad (5.4)$$

where

$$b_0 = 2\beta(\tau) - \beta(t),$$

$$M^{ab}(\tau) = (2p + k_2)^a(2r + k_1)^b f^2(\tau), \quad f(0) = e$$

and V_0 is a known function of t and τ.

For $b_0 > -1$, the absorptive part of (5.4) becomes infinite when $s = m^2$. However, $b_0 < 0$ under condition (5.2), and therefore the absorptive part of (5.4) is integrable, in accord with the general theory. However, when $\tau = 0$ we have $b_0 = -\beta(t) > 0$ (for $t < 0$), and the absorptive part is not integrable in the usual sense. We must therefore substitute (5.4) in (5.3) for $\tau < t/2$, that is, carry out the integration for $b_0 < 0$, and only then can we put $\tau = 0$. This is equivalent to defining the absorptive part of (5.4) at $\tau = 0$

$$T_1(s, t) = -\frac{\exp(-C\beta + \delta)}{m^2\Gamma(\beta)}(sm^{-2} - 1)_+^{\beta-1}M^{ab}(0) + O((sm^{-2} - 1)_+^{\beta}) \quad (5.5)$$

as a generalization of the power function (5.4). The function δ is equal to

$$\delta = \delta(t) = -\frac{\alpha}{\pi}\left(1 + pr\int_0^1 p_x^{-2}\ln|2x - 1|dx\right).$$

In the vicinity of $s = m^2$ the amplitude T is equal to

$$T(s, t) = -\frac{\pi\exp(-C\beta+\delta)}{m^2\Gamma(\beta)\sin\pi\beta} \times (1 + sm^{-2})^{\beta-1}M^{ab}(0) + O((1 - sm^{-2})^\beta) + \text{const}. \quad (5.6)$$

For $t < 0$ we have $\beta(t) < 0$, and the amplitude has a singularity stronger than a pole.

Expression (5.6) is a generalization of the result obtained in [21,19].

6. ON THE FACTORIZATION OF THE INFRARED DIVERGENCES

We have considered above the matrix elements (2.4), which describe processes with a fixed number of particles, by introducing a fictitious photon mass λ and then separating the λ-dependence in the form of a factor. In those cases when the amplitudes remaining after this separation were calculated to the end in some region, as in Sec. 5, they turned out to be finite for $\lambda = 0$. We shall show that this is not an accident, and that the amplitude J in (2.4) is finite when $\lambda = 0$ [14].

In fact, it is practically obvious that the infrared singularities should be nonexistent if the momenta of the charged particles remain unchanged. In this case there is no radiation, and the elastic process is observable. Therefore the matrix element $\langle a, p \,|\, JJ \,|\, p, a \rangle$ (where J is the neutral current, p the charged-particle momentum, and a are quantum numbers of the neutral particles) should be finite when $\lambda = 0$. Let us expand the product of the currents in this expression in a complete system of amplitudes. From the independence of the individual expansion terms corresponding to physically different intermediate states it follows that each such term should be finite when $\lambda = 0$.

We consider the term

$$\sum_{n=0}^{\infty} \frac{1}{n!} \Big(\Sigma\eta \int_{g} \widetilde{d}\mathbf{q}\Big)_n \langle a, p \,|\, J \,|\, r, b, n\rangle \langle n, b, r \,|\, J \,|\, p, a\rangle, \quad (6.1)$$

where r is the momentum of the charged meson, b are quantum numbers of neutral rigid particles with energies larger than h, and n is the number of soft photons with momenta q_i, $q_i^0 < h$, where h is arbitrarily small.

Substituting here expression (2.4) and confining ourselves to the first term of expansion in q_i, we find that, accurate to terms that are negligibly small for small h, this expression is equal to

$$\exp\{2\beta((p-r)^2)\ln(m/2h) + D\} \times J(a, p; r, b) J(b, r; p, a), \quad (6.2)$$

where D is given by formulas (2.16) and (2.17). From the fact that (6.1), and consequently also (6.2), is finite when $\lambda = 0$, it follows that the amplitude $J(a, p; r, b)$ is finite when $\lambda = 0$.

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