FUNCTIONAL INTEGRALS IN QUANTUM ELECTRODYNAMICS AND THE INFRARED LIMIT OF THE GREEN'S FUNCTIONS

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Submitted to JETP July 31, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) **48**, 607-621 (February, 1965)

The Green's functions for the Klein-Gordon and Dirac equations in an arbitrary external field are written in the form of a functional integral. It is shown that the functional quadratures can be carried out for fields which admit an exact solution of the equations (constant field and plane-wave field). The form of the solutions obtained allows the construction of quantum mechanical Green's functions by the method of functional averaging over the external fields. Their asymptotic forms in the infrared region can be investigated. Corrections to the Bloch-Nordsieck formula are found for a simple field theory model.

INTRODUCTION

THE difficulties in finding exact solutions by the method of functional integration in relativistic quantum field theory are connected, first, with the solution of the quantum equations for a particle in an arbitrary external field and, second, with performing the functional averaging of these solutions over the external fields with account of the contributions from vacuum polarization.

In the present paper we propose a method of a formal solution of the equations of field theory in the form of a functional integral. In mathematics and physics such solutions of differential equations are known from the work of Feynman,[1] but these were obtained and justified for equations of the type of the heat conduction equation and for the Schrödinger equation. Here we propose a different approach based on the Weierstrass transformation[2] in function space. By this method we obtain the Green's functions for the Klein-Gordon and Dirac equations in an external field. If the external field admits a solution in closed form, the functional quadratures can be carried out. Besides the now well-known Gaussian functional integrals, we shall also discuss other forms, which arise in the solution of the Dirac equation in constant and plane wave fields.[1)]

The Green's functions for the Klein-Gordon and Dirac equations obtained by this method allow one easily to carry out the functional averaging over the external fields without performing the original functional quadratures arising in the solution of the equations, and thus to find the single-particle quantum mechanical Green's functions.

The problem of the approximate computation of the integrals occurring is discussed on a simple example. A method of approximation is proposed which is good in the infrared region. With its help we are able to construct corrections to the Bloch-Nordsieck formula. The infrared limit has been investigated along different lines, but still within the functional method, in the papers of Blank,[4] Fradkin,[5] and Milekhin.[6] In the last paper, corrections to the Bloch-Nordsieck formula were found, but these depended only linearly on $(p^2 - m^2)/m^2$, owing to an incorrect account of the contribution from the large momenta of the virtual particles in the infrared region. It will turn out that the corrections to the Bloch-Nordsieck formula contain also terms of the form $[(p^2 - m^2)/m^2] \ln [(p^2 - m^2)/m^2]$ (cf. also[5] on this point).

1. THE GREEN'S FUNCTION OF THE DIRAC EQUATION IN AN EXTERNAL FIELD

Our method of writing the solutions of differential equations in the form of functional integrals can be applied to second-order differential equations in which the differential operator can be written as the product of two operators of lower order. The transformation used is not new; it was successfully employed in statistical physics first by Stratonovich,[7] and then by Hubbard[8] and Edwards.[9]

[1)]The possibility of an exact solution of the Dirac equation in a plane wave field has been shown by Volkov.[1]

Let us consider the Dirac equation for the Green's function in the electromagnetic field $A_\mu(x)$, $\partial A_\mu/\partial x_\mu = 0$:

$$[i\gamma_\mu\partial_\mu - m + e\gamma_\mu A_\mu(x)]G(x, y|A) = -\delta(x-y). \quad (1)$$

As usual, we set

$$G(x, y|A) = [i\gamma_\mu\partial_\mu + m + e\gamma_\mu A_\mu(x)]\mathcal{G}(x, y\, A). \quad (2)$$

Then we have for $\mathcal{G}(x, y|A)$

$$[(i\partial_\mu + eA_\mu(x))^2 - m^2 + e\sigma_{\mu\nu}\partial A_\nu(x)/\partial x_\mu]\mathcal{G}(x, y|A) = -\delta(x-y). \quad (3)$$

Writing the inversion operator in exponential form, as proposed by Fock[10] and Feynman,[1] we express the solution of Eq. (3) in operator form:

$$\mathcal{G}(x, y|A) = i\int_0^\infty ds \exp\Big\{i\int_0^s d\xi(i\partial_\mu(\xi) + eA_\mu(x, \xi))^2 - ism^2 + i^2e\int_0^s d\xi\sigma_{\mu\nu}(\xi)\frac{\partial A_\mu}{\partial x_\mu}(x, \xi)\Big\}\delta(x-y). \quad (4)$$

In this way of writing, the exponential, whose exponent contains non-commuting operators, is, according to Feynman,[1] understood as a T exponential, where the ordering index s has the meaning of a proper time and all operators are regarded as commuting functions of the variable ξ.

Let us now perform a transformation in function space of the type of the Weierstrass transformation in the one-dimensional case[2] (cf. also[7-9]):

$$\exp\Big\{i\int_0 d\xi(i\partial_\mu(\xi) + eA_\mu(x, \xi))^2\Big\} = C\int\delta^4\nu\exp\Big\{-i\int_0^s\nu_\mu^2(\xi)d\xi + 2i\int_0^s d\xi\,\nu_\mu(\xi)(i\partial_\mu(\xi) + eA_\mu(x, \xi))\Big\}. \quad (5)$$

Here the functional integral is extended over the space of the four functions $\nu_\mu(\xi)$ with a Gaussian measure. The constant C is determined by the condition

$$C\int\delta^4\nu\exp\Big\{-i\int_0^s\nu_\mu^2(\xi)d\xi\Big\} = 1.$$

Substituting (5) in (4), we can now, according to the Feynman rules, "unscramble" the operator

$$\exp\Big\{-2\int_0^s\nu_\mu(\xi)\partial_\mu(\xi)d\xi\Big\}$$

and obtain a solution to Eq. (2) in the form of a functional integral. As a result we have [2)]

$$\mathcal{G}(x, y|A) = i\int_0^\infty ds\, e^{-im^2s}C\int\delta^4\nu \times\exp\Big\{-i\int_0^s d\xi\Big[\nu_\mu^2(\xi) - e(2\nu_\mu(\xi) + \sigma_{\mu\nu}(\xi)i\partial_\nu) \times A_\mu\Big(x - 2\int_\xi^s\nu(\eta)d\eta\Big)\Big]\Big\}\delta^4\Big(x - y - 2\int_0^s\nu(\eta)d\eta\Big). \quad (6)$$

This expression can be further transformed in two ways. By virtue of the presence of the δ function in (6), we can at once obtain the Fourier transform:

$$\mathcal{G}(x, p|A) = \int\frac{d^4(x-y)}{(2\pi)^4}\mathcal{G}(x, y|A)e^{-ip(x-y)}.$$

Introducing further the substitution $\nu_\mu(\xi) = \omega_\mu(\xi) + p_\mu$, we find

$$\mathcal{G}(x, p|A) = i\int_0^\infty ds e^{is(p^2-m^2)}C\int\delta^4\omega\exp\Big\{-i\int_0^s d\xi\Big[\omega_\mu^2(\xi) - e(2p_\mu + 2\omega_\mu(\xi) + \sigma_{\mu\nu}i\partial_\nu) \times A_\mu\Big(x - 2p(s-\xi) - 2\int_\xi^s\omega(\eta)d\eta\Big)\Big]\Big\} \quad (7)$$

with the condition

$$\int_0^s\omega_\mu(\eta)d\eta + sp_\mu = x_\mu - y_\mu,$$

which follows from the δ function in (6).

In the other method, the δ function is used in the functional integration over $\nu_\mu(\xi)$. To this end we make the substitution

$$\nu_\mu(\xi) = \chi_\mu(\xi) + a_\mu/s, \qquad a_\mu = \int_0^s\dot{\nu}_\mu(\xi)d\xi.$$

It follows that

$$\int_0^s\chi_\mu(\xi)d\xi = 0.$$

As a result we have now an integral over all $\chi_\mu(\xi)$ and a four-dimensional integral over a_μ:

$$\mathcal{G}(x, y|A) = i\int_0^\infty ds\int_{-\infty}^\infty d^4a\,\Phi(s, a)\delta^4(x - y - 2a), \quad (8)$$

$$\Phi(s, a) = \frac{\exp(-ims^2 - ia_\mu^2/s)}{i\pi^2s^2}C_1\int\delta^4\chi\exp\Big\{-i\int_0^s\chi_\mu^2(\xi)d\xi + ie\int_0^s d\xi\Big[2\frac{a_\mu}{s} + 2\chi_\mu(\xi) + \sigma_{\mu\nu}(\xi)i\partial_\nu\Big] \times A_\mu\Big(x - 2\frac{s-\xi}{s}a + 2\int_0^\xi\chi(\eta)d\eta\Big)\Big\}.$$

Performing the integration over the a_μ, we obtain finally

$$\mathcal{G}(x, y|A) = i\int_0^\infty ds\Phi\Big(s, \frac{x-y}{2}\Big). \quad (8')$$

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2)It must be noted that the γ_μ matrices in the exponents of (6) and (7) remain "scrambled" and therefore must still carry the ordering index ξ.

Formulas (7) and (8′) give, for e = 0, the Green's functions for the free equations in p and x space, respectively:

$$\overline{\mathcal{G}}_0(p) = i\int_0^\infty ds\, e^{is(p^2-m^2)},$$

$$\mathcal{G}_0(x-y) = i\int_0^\infty \frac{ds}{i\pi^2 s^2}\exp\left\{-ism^2 - i\frac{(x-y)^2}{4s}\right\}. \quad (9)$$

The Green's function for the Dirac equation (1) is found according to (2).

Let us now consider some specific fields $A_\mu(x)$, for which the Dirac equation is exactly soluble; this means in our treatment that the functional quadratures can be carried out.

The simplest case of constant fields $F_{\mu\nu}$, so that the $A_\mu(x)$ are linear functions, is easily handled, since (7) and (8′) involve Gaussian quadratures, which are easily carried out.

More interesting is the case when $F_{\mu\nu} = 0$ and $A_\mu(x) = \partial f(x)/\partial x_\mu$. This example is related to the problem of the change of the fermion Green's function under gauge transformations of the field, $A_\mu \to A_\mu + \partial f/\partial x_\mu$. Setting $A_\mu = \partial f/\partial x_\mu$ in (8′), we have

$$\mathcal{G}(x, y|f) = i\int_0^\infty ds\Phi'\left(s, \frac{x-y}{2}\right), \quad (10)$$

where Φ' differs from Φ in (8) in that A_μ is replaced by $\partial f/\partial x_\mu$ and the term $\sigma_{\mu\nu} i\partial_\nu$ is omitted, since $\sigma_{\mu\nu}\partial^2 f/\partial x_\mu \partial x_\nu = 0$.

The integrand of the integral in the exponent in (10) is a total differential,

$$\frac{d}{d\xi} f = \left[\frac{x_\mu - y_\mu}{s} - 2\chi_\mu(\xi)\right]$$
$$\times \frac{\partial}{\partial x_\mu} f\left(x - \frac{s-\xi}{s}(x-y) + 2\int_0^\xi \chi d\eta\right),$$

so that we can carry out the integration over ξ. Using

$$\int_0^s \chi_\mu(\eta)\, d\eta = 0, \qquad C_1\int \delta^4\chi \exp\left\{-i\int_0^s \chi_\mu^2(\xi)\, d\xi\right\} = 1,$$

we obtain finally

$$\mathcal{G}(x, y|f) = \exp\{ie[f(x) - f(y)]\}$$
$$\times\, i\int_0 \frac{ds}{i\pi^2 s^2}\exp\left\{-ism^2 - i\frac{(x-y)^2}{4s}\right\}$$

$$G(x, y|f) = \exp\{ie[f(x) - f(y)]\}(i\gamma_\mu\partial_\mu + m)\mathcal{G}_0(x-y). \quad (11)$$

This solution has been obtained by a different method by Fradkin,[11] Svidzinskiĭ,[12] Schwinger,[13] and others.

A less trivial problem is the determination of the Green's function for the Dirac equation in a plane wave field.[3] Let us consider a plane wave of arbitrary form:

$$A_\mu(x) = \varepsilon_\mu\varphi(kx); \qquad \varepsilon_\mu k_\mu = 0, \quad k_\mu^2 = 0. \quad (12)$$

We start from formula (7). We have

$$\overline{\mathcal{G}}(x, p|A) = i\int_0^\infty ds e^{is(p^2-m^2)} C\int \delta^4\omega \exp\left\{-i\int_0^s \omega_\mu^2(\xi)\, d\xi\right\}$$
$$\times \exp\left\{ie\int_0^s d\xi\left[2p\varepsilon + 2\varepsilon\omega(\xi) + \sigma_{\mu\nu}\varepsilon_\mu k_\nu i\frac{\partial}{\partial(\mathbf{kx})}\right]\right.$$
$$\left.\times\, \varphi\left(\mathbf{kx} - 2\mathbf{kp}(s-\xi) - 2\int_0 k\omega(\eta)\, d\eta\right)\right\}. \quad (13)$$

The integral over ω is clearly not of the Gaussian type, since $\omega_\mu(\eta)$ enters in the argument of the arbitrary function φ. Let us introduce the infinite-dimensional δ function

$$\prod_\xi \delta(\psi(\xi) - 2\mathbf{k}\omega(\xi))$$
$$= C_2\int \delta\alpha \exp\left\{i\int_0^s d\xi\, \alpha(\xi)[\psi(\xi) - 2\mathbf{k}\omega(\xi)]\right\}, \quad (14)$$

where the normalization constant C_2 is equal to

$$C_2 = \prod_\xi \frac{1}{2\pi}.$$

With the help of (14) we can write the integral over $\omega_\mu(\xi)$ in (13) in a form which can be integrated over ω:

$$C\int \delta^4\omega \exp\left\{-i\int_0^s \omega_\mu^2(\xi)\, d\xi\right\}$$
$$\times\, C_2C_3\int \delta\alpha\delta\psi \exp\left\{i\int_0^s d\xi\alpha(\xi)[\psi(\xi) - 2\mathbf{k}\omega(\xi)]\right\}$$
$$\times \exp\left\{ie\int_0 d\xi\left[2\varepsilon\mathbf{p} + 2\varepsilon\omega(\xi) + \sigma_{\mu\nu}(\xi)\varepsilon_\mu k_\nu i\frac{\partial}{\partial(\mathbf{kx})}\right]\right.$$
$$\left.\times\, \varphi\left(\mathbf{kx} - 2\mathbf{kp}(s-\xi) - \int_\xi^s \psi(\eta)\, d\eta\right)\right\}$$
$$= C_2C_3\int \delta\alpha\delta\psi \exp\left\{i\int_0^s \alpha(\xi)\psi(\xi)\, d\xi\right\}\exp\left\{ie\int_0^s d\xi\left[2\varepsilon\mathbf{p}\right.\right.$$
$$\left.\left. + \sigma_{\mu\nu}(\xi)\varepsilon_\mu k_\nu i\frac{\partial}{\partial(\mathbf{kx})}\right]\varphi\right\}\exp\left\{i\int_0^s d\xi\left[k_\mu\alpha(\xi) - \varepsilon_\mu e\varphi\right]^2\right\}. \quad (15)$$

Taking into account that, owing to (12), $[k_\mu\alpha(\xi) - \epsilon_\mu e\varphi]^2 = \epsilon_\mu^2 e^2\varphi^2$ is independent of $\alpha(\xi)$, we can first integrate over $\alpha(\xi)$ in (15), which gives

$$C_2\int \delta\alpha \exp\left\{i\int_0^s \alpha(\xi)\psi(\xi)\, d\xi\right\} = \prod_\xi \delta(\psi(\xi)),$$

and then integrate over $\psi(\xi)$, i.e., set $\psi(\xi) = 0$. Finally, we have

$$\overline{\mathcal{G}}(x, p|A) = i\int_0^\infty ds e^{is(p^2-m^2)} \exp\Big\{ ie\int_0^s \Big(2p_\mu + \sigma_{\mu\nu} ik_\nu \frac{\partial}{\partial(\mathbf{kx})}\Big) \times A_\mu(\mathbf{kx}-2\mathbf{kp}\xi) + ie^2\int_0^s d\xi A_\mu^2(\mathbf{kx}-2\mathbf{kp}\xi)\Big\}. \quad (16)$$

The Green's function for an electron in a plane wave field is obtained by acting on (16) with the operator $[i\gamma_\mu\partial_\mu + \gamma_\mu p_\mu + m + ieA_\mu(x)]$.

2. THE INFRARED LIMIT OF THE GREEN'S FUNCTION IN THE MODEL $L_{int} = g\psi^2(x)\varphi(x)$

We shall use the technique of obtaining the solutions of the equations in an external field, as discussed in Sec. 1. On the example of a simple relativistically invariant model of two scalar fields interacting through $L_{int} = g\psi^2(x)\varphi(x)$ without account of vacuum polarization, we shall explain a method of approximating the quantum Green's function in the infrared region.

We assume in analogy to electrodynamics that the mass of the field $\varphi(x)$ is equal to zero. The Green's function for the particle corresponding to the field ψ in a classical external field $\varphi(x)$ satisfies the equation

$$[i^2\partial_\mu^2 - m^2 + g\varphi(x)]G(x, y|\varphi) = -\delta(x-y). \quad (17)$$

Repeating the procedure of Sec. 1, we have

$$G(x, y|\varphi) = i\int_0^\infty ds \exp\Big\{-im^2 s + i\int_0^s d\xi[i^2\partial_\mu^2(\xi) + g\varphi(x,\xi)]\Big\} \times \delta(x-y). \quad (18)$$

Analogously, we rewrite (5) in the form

$$\exp\Big\{i\int_0^s d\xi i^2\partial_\mu^2(\xi)\Big\} = C\int \delta^4\nu \exp\Big\{-i\int_0^s d\xi[\nu_\mu^2(\xi) - 2i\nu_\mu(\xi)\partial_\mu(\xi)]\Big\}. \quad (19)$$

Substituting (19) in (18) and "unscrambling" the differential operator, we find

$$G(\mathbf{x}, \mathbf{y}|\varphi) = i\int_0^\infty ds e^{-ism^2} C\int \delta^4\nu \exp\Big\{-i\int_0^s d\xi\Big[\nu_\mu^2(\xi) - g\varphi\Big(x - 2\int_\xi^s \nu(\eta)d\eta\Big)\Big]\Big\}\delta^4\Big(\mathbf{x}-\mathbf{y}-2\int_0^s \mathbf{v}(\xi)d\xi\Big).$$

We further need the Fourier transform G with respect to the difference x—y [cf. (7)]:

$$\overline{G}(\mathbf{x}, \mathbf{p}|\varphi) = i\int_0^\infty ds e^{is(p^2-m^2)} C\int \delta^4\nu \exp\Big\{-i\int_0^s d\xi\Big[\nu_\mu^2(\xi) - g\varphi\Big(\mathbf{x} - 2\mathbf{p}(s-\xi) - 2\int_\xi^s \mathbf{v}(\eta)d\eta\Big)\Big]\Big\} \quad (20)$$

Now that we have the Green's function in the presence of the classical field $\varphi(x)$ in the form (20), it is easy to perform the functional averaging over this field with the weight function

$$\exp\Big\{-\frac{i}{2}\int d^4q D(\mathbf{q})\varphi(\mathbf{q})\varphi(-\mathbf{q})\Big\},$$

where $D(q) = 1/(q^2 + i\epsilon)$ is the causal function of the field $\varphi(q)$. We thus obtain for the quantum mechanical Green's function

$$G(p) = i\int_0^\infty ds e^{is(p^2-m^2)} C\int \delta^4\nu \exp\Big\{-i\int_0^s d\xi\nu_\mu^2(\xi)\Big\} \times \exp\Big\{-i\frac{g_1^2}{2}\int_0^s\int_0^s d\xi_1 d\xi_2 \Delta(\xi_1, \xi_2|\nu)\Big\}, \quad (21)$$

where

$$\Delta(\xi_1, \xi_2|\nu) = \int d^4qD(q)\exp\Big\{-2\mathbf{pq}|\xi_1-\xi_2| - 2i\int_{\xi_2}^{\xi_1} \mathbf{qv}(\eta)d\eta\Big\},$$

$$g_1 = g/(2\pi)^2. \quad (21')$$

Since the integral over $\nu(\xi)$ in (21) can not be evaluated explicitly, we shall compute it approximately.

If we make an expansion in powers of g_1^2, we regain the perturbation series for the Green's function; here the functional integrals are easily computed, since they involve expressions of the type

$$\Delta_1(\xi_1-\xi_2) = C\int \delta^4\nu \exp\Big\{-i\int_0^s \nu_\mu^2(\xi)d\xi\Big\}\Delta(\xi_1, \xi_2|\nu) = \int d^4qD(\mathbf{q})\exp\{i(\mathbf{q}^2 - 2\mathbf{pq})|\xi_1-\xi_2|\}. \quad (22)$$

It is seen from (22) that the functional argument in $\Delta(\xi_1, \xi_2|\nu)$ leads, after integration, to a quadratic dependence on the momenta q of the field quanta. If we are interested in the low energy region, we can therefore neglect the dependence on ν in $\Delta(\xi_1, \xi_2|\nu)$. In this approximation, the results for the infrared region were obtained by a different method by Fradkin[11] and Milekhin.[6] However, this approximation significantly alters the behavior at large values of the momentum and leads, in particular, to stronger divergences of the unrenormalized quantities. We shall not, therefore, neglect the ν dependence, and instead use a different approximation procedure.

For this purpose, let us consider the series expansion of the quantity

$$L = C\int \delta^4\nu \exp\Big\{-i\int_0^s \nu_\mu^2(\xi)d\xi\Big\}$$

$$\times\exp\left\{-\frac{ig_1^2}{2}\int\limits_0^s\int d\xi_1\,d\xi_2\Delta(\xi_1,\xi_2|\nu)\right\}$$

$$=1-\frac{ig_1^2}{2}\int\limits_0^s\int d\xi_1 d\xi_2\Delta_1(\xi_1-\xi_2)+\frac{1}{2}\left(\frac{ig_1^2}{2}\right)^2$$

$$\times\int\int\limits_0^s\int\int d\xi_1\ldots d\xi_4\ \Delta_2(\xi_1,\xi_2,\xi_3,\xi_4)+\ldots, \tag{23}$$

where $\Delta_1(\xi_1-\xi_2)$ is defined by (22), and

$$\Delta_2(\xi_1,\xi_2,\xi_3,\xi_4)$$

$$=C\int\delta^4\nu\exp\left\{-i\int\limits_0^s\nu_\mu^2(\xi)\,d\xi\right\}\Delta(\xi_1,\xi_2|\nu)\Delta(\xi_1,\xi_2|\nu)$$

$$=\int d^4q_1d^4q_2D(\mathbf{q}_1)D(\mathbf{q}_2)\exp\{i(\mathbf{q}_1^2-2\mathbf{pq}_1)|\xi_1-\xi_2|$$

$$+i(\mathbf{q}_2^2-2\mathbf{pq}_2)|\xi_3-\xi_4|+2i\mathbf{q}_1\mathbf{q}_2\theta(\xi_1,\xi_2,\xi_3,\xi_4)\},$$

$$\theta(\xi_1,\xi_2,\xi_3,\xi_4)=[\theta(\xi_3-\xi_1)-\theta(\xi_4-\xi_1)]\xi_1$$

$$+[\theta(\xi_4-\xi_2)-\theta(\xi_3-\xi_2)]\xi_2$$

$$+[\theta(\xi_1-\xi_3)-\theta(\xi_2-\xi_3)]\xi_3$$

$$+[\theta(\xi_2-\xi_4)-\theta(\xi_3-\xi_4)]\xi_4,$$

$$\theta(x)=1\ \text{for}\quad x>0,\quad \theta(x)=0\ \text{for}\quad x<0. \tag{24}$$

Let us now reorder the series (23). To this end we add and subtract to the exponent of the second exponential in (23) the first term after the unit in the expansion of (23) and expand in the difference

$$-\frac{ig_1^2}{2}\int\limits_0^s\int d\xi_1\,d\xi_2\,[\Delta(\xi_1,\xi_2|\nu)-\Delta_1(\xi_1-\xi_2)].$$

After the integration we have the following expansion for L:

$$L=\exp\left\{-\frac{ig_1^2}{2}\int\limits_0^s\int d\xi_1 d\xi_2\Delta_1(\xi_1-\xi_2)\right\}C\int\delta^4\nu$$

$$\times\exp\left\{-i\int\limits_0^s\nu^2(\xi)\,d\xi-\frac{ig_1^2}{2}\int\limits_0^s\int d\xi_1 d\xi_2[\Delta(\xi_1,\xi_2|\nu)\right.$$

$$\left.-\Delta_1(\xi_1-\xi_2)]\right\}=\exp\left\{-\frac{ig_1^2}{2}\int\limits_0^s\int d\xi_1\,d\xi_2\,\Delta_1(\xi_1-\xi_2)\right\}$$

$$\times\left\{1+\frac{1}{2}\left(\frac{ig_1^2}{2}\right)^2\int\int\limits_0^s\int\int d\xi_1\ldots d\xi_4[\Delta_2(\xi_1,\xi_2,\xi_3,\xi_4)\right.$$

$$\left.-\Delta_1(\xi_1-\xi_2)\Delta_1(\xi_3-\xi_4)]+\ldots\right\}. \tag{25}$$

It should be noted that the series (23) can be summed further in the infrared region by adding and subtracting in the exponent of the second exponential the quantity

$$-\frac{ig_1^2}{2}\int\limits_0^s\int d\xi_1 d\xi_2\Delta_1(\xi_1-\xi_2)$$

$$+\frac{1}{2}\left(\frac{ig_1^2}{2}\right)^2\int\int\limits_0^s\int\int d\xi_1\ldots d\xi_4[\Delta_2(\xi_1,\xi_2,\xi_4,\xi_3)$$

$$-\Delta_1(\xi_1-\xi_2)\Delta_1(\xi_3-\xi_4)]. \tag{26}$$

Instead of (25) we shall then have an exponential with the exponent (26) as a common factor and a series starting with g_1^6. However, in the infrared region this does not alter the result, since the second term in (26) vanishes for $s\to\infty$ and the first term is the determining one, as it behaves like $-\ln p^2s$ (see below).

If we substitute (25) in the expression (21) for the Green's function, we obtain an expansion in powers of g_1^2 with the common factor

$$\exp\left\{-\frac{ig_1^2}{2}\int\limits_0^s\int\Delta_1(\xi_1-\xi_2)\,d\xi_1\,d\xi_2\right\},$$

which, as will be seen in the following, includes all infrared divergences of the Green's function. Each term in the series in g_1^2 is a difference of an expression obtained by ordinary perturbation theory and the same expression without the nondiagonal quadratic terms in the photon momenta, $\mathbf{q}_i\mathbf{q}_j$, as can be seen on the example $\Delta_2(\xi_1,\xi_2,\xi_3,\xi_4)-\Delta_1(\xi_1-\xi_2)\Delta_1(\xi_3-\xi_4)$. It is easy to verify that this difference contains no infrared divergences. This result confirms the theorem of Yennie, Frautschi, and Suura[14] on the factorization of the infrared divergences in quantum electrodynamics.

Let us elucidate this by an example. We calculate the first two terms of this expansion for the Green's function

$$G(p)=i\int\limits_0^\infty ds\exp[is(p^2-m^2)]$$

$$\times\exp\left\{-\frac{ig_1^2}{2}\int\limits_0^s\int d\xi_1 d\xi_2\int d^4qD(q)\right.$$

$$\left.\times\exp[i(q^2-2pq)|\xi_1-\xi_2|]\right\}$$

$$\times\left[1+\frac{1}{2}\left(\frac{ig_1^2}{2}\right)^2\int\int\limits_0^s\int\int d\xi_1\ldots d\xi_4\right.$$

$$\times\int\frac{d^4q_1d^4q_2}{\mathbf{q}_1^2\mathbf{q}_2^2}\exp\{i(\mathbf{q}_1^2-2\mathbf{pq}_1)|\xi_1-\xi_2|\}$$

$$\times\exp\{i(\mathbf{q}_2^2-2\mathbf{pq}_2)|\xi_3-\xi_4|\}$$

$$\left.\times(\exp\{2i\mathbf{q}_1\mathbf{q}_2\theta(\xi_1,\xi_2,\xi_3,\xi_4)\}-1)\right]. \tag{27}$$

Dividing the fourfold integral over ξ_1, ξ_2, ξ_3, and ξ_4 into integrals over the regions $\xi_i>\xi_j>\xi_k>\xi_l$ corresponds to taking all Feynman graphs of the given order of g_1^2 into account. The region in which

$\theta(\xi_1, \ldots, \xi_4) = 0$ corresponds to a graph in which there are no terms $\mathbf{q}_1 \cdot \mathbf{q}_2$ (Fig. 1). These graphs give no contribution to (27), since there θ enters in the form $\exp(2i\mathbf{q}_1 \cdot \mathbf{q}_2 \theta) - 1$. The regions in which $\theta \neq 0$ correspond to the two graphs shown in Fig. 2.

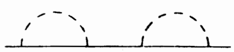

FIG. 1

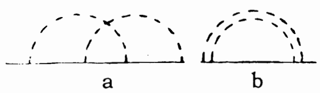

FIG. 2

From these we subtract, according to (27), the expressions for the same graphs without the nondiagonal term $\mathbf{q}_1 \cdot \mathbf{q}_2$. This difference contains no infrared divergences, as already noted. For example, for the graph of Fig. 2

$$\int \frac{d^4q_1 d^4q_2}{\mathbf{q}_1^2\mathbf{q}_2^2(\mathbf{p}^2 - m^2)^2[m^2 - (\mathbf{p}+\mathbf{q})^2]^2}\left[\frac{1}{m^2 - (\mathbf{p}+\mathbf{q}_1+\mathbf{q}_2)^2} - \frac{1}{m^2 - (\mathbf{p}+\mathbf{q}_1+\mathbf{q}_2)^2 + 2\mathbf{q}_1\mathbf{q}_2}\right]$$

$$= \int \frac{d^4q_1 d^4q_2}{\mathbf{q}_1^2\mathbf{q}_2^2(m^2 - p^2)^2[m^2 - (\mathbf{p}+\mathbf{q}_1+\mathbf{q}_2)^2][m^2 - (\mathbf{p}+\mathbf{q}_1)^2]^2} \times \frac{2q_1q_2}{m^2 - (\mathbf{p}+\mathbf{q}_1+\mathbf{q}_2)^2 + 2\mathbf{q}_1\mathbf{q}_2}. \tag{28}$$

We see from this that the subtraction increases the power of the numerator under the integral sign in the region of small $\mathbf{q}$.

After integrating over $\mathbf{q}$ and ξ in (26) and renormalization, we have

$$G_r(p) = i\int_0^\infty ds \exp\left\{is(p^2 - m_r^2) + \frac{g_1^2}{p^2}\pi^2 I(p^2, s)\right\} \times \left[1 + \left(\frac{g_1^2\pi^2}{p^2}\right)^2 F(p^2, s)\right]. \tag{29}$$

$I(p^2, s) \to 0$ for $s \to 0$ and $I(p^2, s) \to \ln p^2 s$ for $s \to \infty$. As $s \to \infty$ the function $F(p^2, s)$ goes over into the expression (28) obtained by perturbation theory. Since we are interested in the infrared region, and the main contribution for $p^2 \approx m_r^2$ comes from the integration over large values of s in (29), we can replace $I(p^2, s)$ and $F(p^2, s)$ by their asymptotic values for $s \to \infty$. As a result we have, up to order g_1^4,

$$G(p) = \frac{|1 - p^2/m^2|^{g^2/16\pi^2m^2}}{p^2 - m^2}\left\{N_1(p) + \left(\frac{g^2}{16\pi^2m^2}\right)^2 \times \left[a_1\frac{p^2 - m^2}{m^2}\ln\frac{p^2 - m^2}{m^2} + a_2\frac{p^2 - m^2}{m^2}\right]N_2(p)\right\}, \tag{30}$$

where

$$N_1(p) = \varepsilon(p^2 - m^2)\int_0^\infty dx\, e^{ix\varepsilon(p^2 - m^2)}\, x^{-g^2/16\pi^2m^2},$$

$$N_2(p) = \iint_0^\infty dx_1\, dx_2\, e^{i(x_1+x_2)\varepsilon(p^2-m^2)}(x_1 - x_2)^{-g^2/16\pi^2m^2};$$

$$a_1, a_2 = \text{const}; \qquad \varepsilon(x) = 1,\ x > 0,\quad \varepsilon(x) = -1,\ x < 0.$$

It should be noted that (30) is valid if $g^2/16\pi^2m^2 < 1$. In the opposite case, the function $G(p)$ approaches a constant for $p^2 \approx m^2$.[6]

Milekhin[6] has considered the infrared limit for this model using a different method, in which the terms quadratic in the photon momenta $\mathbf{q}$ were not subtracted. Since the behavior at large values of the momentum is essential in the calculation of the corrections in g^2 to the basic formula (as was already noted by Milekhin), it is not surprising that our result (30) differs from that of Milekhin.

The dependence of the corrections on $[(p^2 - m^2)/m^2]\ln[(p^2 - m^2)/m^2]$ and not only on $(p^2 - m^2)/m^2$ was also mentioned in the paper of Solov'ev.[5]

3. QUANTUM ELECTRODYNAMICS

We shall start with formula (7) for the Green's function for the squared Dirac equation. $A_\mu(x)$ is taken in an arbitrary gauge. The quantum mechanical Green's function is obtained by integrating over $A_\mu(x)$ according to the formula

$$G(p) = \int \delta^4 A \exp\left\{-\frac{i}{2}\int d^4q D_{\mu\nu}^{-1}(q)A_\mu(q)A_\nu(-q)\right\} \times G(x, p|A)S_0(A)\left[\int \delta^4 A \exp\left\{-\frac{i}{2}\int d^4q D_{\mu\nu}^{-1}(q)A_\mu(q) \times A_\nu(-q)\right\}S_0(A)\right]^{-1}. \tag{31}$$

Here

$$D_{\mu\nu}(q) = \frac{1}{-q^2}\left[\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}(1 - d_l)\right]$$

is the propagation function for the photon in an arbitrary gauge; $S_0(A)$ is the contribution from the polarization of the electron-positron vacuum. According to Salam and Matthews[15] the last quantity can be written as

$$S_0(A) = \exp\left\{i\sum_{n=1}^\infty \frac{e^{2n}}{2n}\sigma_{2n}(A)\right\}; \tag{32}$$

$$\sigma_2(A) = \int d^4q A_\mu(q)A_\nu(-q)\Pi_{\mu\nu}(q),$$

$$\sigma_4(A) = \int d^4q_1\, dq_2\, d^4q_3\, A_\mu(q_1)A_\nu(q_2)A_\lambda(q_3)A_\sigma(q_4) \times M_{\mu\nu\lambda\sigma}(q_1 q_2 q_3)\delta^4\left(\sum_1^4 q_i\right), \tag{33}$$

where

$$\Pi_{\mu\nu}(q) = (\delta_{\mu\nu} - q_\mu q_\nu/q^2)I(q^2)$$

is the polarization operator in the e^2 approximation; $M_{\mu\nu\lambda\sigma}(q_1q_2q_3)$ is the expression corresponding to the graph for scattering of light by light in the e^4 approximation.

In the sum (32) we restrict ourselves to the first term $\sigma_2(A)$, since we are interested in the infrared region, where $\sigma_2 \sim q_3^2$, $\sigma_4 \sim q^4$. It will be clear from the following that the account of $\sigma_2(A)$ does not alter the infrared limit of $G(p)$. In order to perform the integration in (31), we write the Green's function with the help of (7) in the form

$$G(p,x|A)=i\int_0^\infty ds e^{is(p^2-m^2)}C\int \delta^4\omega \exp\left\{-i\int_0^s \omega_\mu{}^2(\xi)\,d\xi\right\}$$
$$\times\left[i\gamma\partial+\gamma p+m-i\gamma_\mu\int d^4y\frac{\delta}{\delta K_\mu(q,s|\omega)}\right]$$
$$\times\exp\left\{ie_1\int d^4qA_\mu(q)K_\mu(q,s|\omega)e^{iqx}\right\},\tag{34}$$

where

$$K_\mu(q,s|\omega)=\int_0^s d\xi[2p_\mu+2\omega_\mu(\xi)-\delta_{\mu\nu}(\xi)q_\nu]$$
$$\times\exp\left\{-2i\mathbf{pq}|s-\xi|-2i\int_\xi^s \mathbf{q}\boldsymbol{\omega}(\eta)\,d\eta\right\},$$

$e_1=e/2\pi$.

Under our assumptions on the contribution from vacuum polarization in the infrared region, (31) contains an integral over A_μ of the Gaussian type:

$$\int\delta^4A\exp\left\{-\frac{i}{2}\int d^4q\,[D_{\mu\nu}{}^{-1}(q)+e_1{}^2\Pi_{\mu\nu}(q)]A_\mu(q)A_\nu(-q)\right.$$
$$\left.+ie_1\int d^4qK_\mu(q,\ s|\omega)A_\mu(q)\right\}$$
$$\times\left[\int\delta^4A\exp\left\{-\frac{i}{2}\int d^4q\,[D_{\mu\nu}{}^{-1}(q)\right.\right.$$
$$\left.\left.+e_1{}^2\Pi_{\mu\nu}(q)]A_\mu(q)A_\nu(-q)\right\}\right]^{-1}$$
$$=\exp\left\{-\frac{ie_1{}^2}{2}\int d^4q\,[D_{\mu\nu}{}^{-1}(q)\right.$$
$$\left.+e_1{}^2\Pi_{\mu\nu}(q)]^{-1}K_\mu(q,s|\omega)K_\nu(q,s|\omega)\right\},\tag{35}$$

where the inversion operator is

$$[D_{\mu\nu}(q)+e_1{}^2\Pi_{\mu\nu}(q)]^{-1}$$
$$=-q^{-2}[\delta_{\mu\nu}-q_\mu q_\nu q^{-2}(d_{tr}(q^2)-d_l)],$$
$$d_{tr}(q^2)=1/(1-e_1{}^2I(q^2)).\tag{36}$$

Since, in computing (35), we kept only terms up to order e_1^4 in the exponent, we shall also restrict ourselves to this order in the final result. We therefore expand the denominator in (36) up to e_1^2. This is justified in the infrared region, since $I(q^2)\to(1/15)q^2/m^2$ for $q^2\ll m^2$. We can then neglect the contribution from $I(q^2)$, i.e., set $d_{tr}(q^2)=1$. [As is well known, $I(q^2)\to\ln(q^2/m^2)$ for $q^2\to\infty$, and one must include the remaining terms in (32), since σ_2 alone leads to an unphysical pole in $d_{tr}(q^2)$.[11]] The expression proportional to $q_\nu q_\mu$ in the exponent of (35) can be calculated exactly. This is a consequence of the fact that the problem with the potential $A_\mu=\partial f/\partial x_\mu$ has an exact solution:

$$ie_1{}^2\int d^4q\frac{q_\mu q_\nu(1-d_l)}{q^2q^2}\int_0^s\!\!\int[2\omega_\mu(\xi_1)+2p_\mu-\sigma_{\mu\lambda}(\xi_1)q_\lambda]$$
$$\times[2\omega_\nu(\xi_2)-2p_\nu+\sigma_{\nu\varkappa}q_\varkappa]$$
$$\times\exp\left\{-2pqi|\xi_1-\xi_2|-2i\int_{\xi_2}^{\xi_1}q\omega(\eta)\,d\eta\right\}d\xi_1\,d\xi_2$$
$$=ie_1{}^2\int d^4q\frac{1-d_l}{q^2q^2}2\left(1-\exp\left\{-2ipqs-2i\int_0^s q\omega d\eta\right\}\right)$$
$$=\frac{e_1{}^2}{2}(1-d_l)\left(\ln\frac{M}{m}+\ln m|x-y|\right).\tag{37}$$

Here we have used

$$\int_0 d\xi(p_\mu+\omega_\mu(\xi))=x_\mu-y_\mu.$$

Thus the term proportional to $q_\mu q_\nu$ in the propagation function $D_{\mu\nu}(q)$ leads to the following factor in $G(p)$:

$$Z_2'(m|x-y|)^{e_1^2(1-d_l)/2},\qquad Z_2'=\exp\left\{\frac{e_1^2}{2}(1-d_l)\ln\frac{M}{m}\right\},$$

which is unimportant in the infrared region. We see from this result that the choice of a gauge, i.e., of d_l, does not affect the infrared limit in quantum electrodynamics. Leaving out the factor just mentioned, we obtain

$$G(p)=i\int_0^\infty dse^{is(p^2-m^2)}C\int\delta^4\omega\exp\left\{-i\int_0^s\omega_\mu{}^2(\xi)\,d\xi\right\}\left[\gamma p+m\right.$$
$$\left.+e_1{}^2\int\frac{d^4q}{q^2}\gamma_\mu K_\mu(-q,\ s|\omega)\right]\exp\left\{-\frac{ie_1{}^2}{2}\int\frac{d^4q}{q^2}\right.$$
$$\times\int_0^s\!\!\int d\xi_1d\xi_2[4(p_\mu+\omega_\mu(\xi_1))(p_\mu+\omega_\mu(\xi_2))$$
$$+4(p_\mu+\omega_\mu(\xi_1))q_\nu\sigma_{\nu\mu}(\xi_2)-\sigma_{\mu\nu}(\xi_1)\sigma_{\nu\rho}(\xi_2)q_\mu q_\rho]$$
$$\left.\times\exp\left[2ipq|\xi_1-\xi_2|-2i\int_{\xi_2}^{\xi_1}q\omega(\eta)\,d\eta\right]\right\}.\tag{38}$$

Let us further omit the terms proportional to the photon momentum q in the exponent of (38). These terms contain γ matrices, and we are thus neglecting spin effects in the infrared region (cf.[16] on this point).

We now apply the procedure of Sec. 2 for an approximate calculation of the functional integral over $\omega_\mu(\xi)$. The quantity $\Delta(\xi_1, \xi_2 | \omega)$ of (21′) is approximated by

$$\Delta_1(\xi_1-\xi_2)=\int\frac{d^4q}{q^2}\exp\{i(q^2-2pq)|\xi_1-\xi_2|\}.$$

We shall construct an approximation in the difference so that there is no term proportional to e_1^2 in the expansion of the exponential, as shown in Sec. 2. Keeping only the first term of this approximation in (38), we have

$$G(p)=i\int_0^\infty ds e^{is(p^2-m^2)}[p\gamma+m]$$
$$\times C\int\delta^4\omega\exp\left\{-i\int_0^s\omega_\mu^2(\xi)d\xi\right\}$$
$$\times\exp\left\{-\frac{ie_1^2}{2}\int\int_0^s d\xi_1 d\xi_2\cdot 4[p^2+p\omega(\xi_1)\right.$$
$$\left.+\omega(\xi_1)\omega(\xi_2)]\Delta_1(\xi_1-\xi_2)\right\}. \tag{39}$$

To compute the integral over ω_μ, we must, according to [1], know the integral operator $N(\xi_1, \xi_2)$ defined by the equation

$$\int_0^s d\xi N(\xi_1,\xi)[\delta(\xi-\xi_2)+2e_1^2\Delta_1(\xi-\xi_2)]=\delta(\xi_1-\xi_2). \tag{40}$$

However, the exact solution of (40) is very difficult. We shall therefore take recourse to a method of successive approximations in the constant e_1^2. Keeping only the first three terms, we find

$$N(\xi_1,\xi_2)=\delta(\xi_1-\xi_2)-2e_1^2\Delta(\xi_1-\xi_2)$$
$$+(2e_1^2)^2\int_0\Delta(\xi_1-\xi)\Delta(\xi-\xi_2)d\xi+O(e_1^6). \tag{41}$$

After integration the Green's function is written in the form

$$G(p)=i\int_0^\infty ds e^{is(p^2-m^2)}[p\gamma+m]$$
$$\times\exp\left\{-2ie_1^2p^2\int\int_0^s d\xi_1 d\xi_2\Delta_1(\xi_1-\xi_2)\right.$$
$$+4ie_1^4p^2\int\int\int\int_0^s d\xi_1\dots d\xi_4\Delta_1(\xi_1-\xi_3)$$
$$\left.\times\Delta_1(\xi_2-\xi_4)+\dots\right\}J, \tag{42}$$

where

$$J=\int\delta^4\omega\exp\left\{-i\int\int_0^s d\xi_1 d\xi_2[\delta(\xi_1-\xi_2)+2e_1^2\Delta_1(\xi_1-\xi_2)]\right.$$
$$\left.\times\omega_\mu(\xi_1)\omega_\mu(\xi_2)\right\}=[\mathrm{Det}\,(\delta(\xi_1-\xi_2)+2e_1^2\Delta_1(\xi_1-\xi_2))]^{-2}$$
$$=\exp\left\{2i\sum_{n=1}^\infty\frac{(-2e_1^2)^n}{n}\sigma_n(\Delta_1)\right\}, \tag{43}$$

$$\sigma_n(\Delta_1)=\int_0^s\dots\int_0^s d\xi_1\dots d\xi_n\Delta_1(\xi_1-\xi_2)\Delta(\xi_2-\xi_3)$$
$$\dots\Delta(\xi_n-\xi_1). \tag{44}$$

If we now substitute (41) in (42), we obtain a perturbation series in e_1^2 in the exponent. The first term of this series, $-2ie_1^2p^2\sigma_1(\Delta_1)$, gives the main contribution in the infrared region, since it tends to $4\pi^2e_1^2\ln p^2s$ for $s\to\infty$, whereas the terms related to the expression

$$4ie_1^4p^2\int_0^s\dots\int_0^s d\xi_1\dots d\xi_4N(\xi_1,\xi_2)\Delta_1(\xi_1-\xi_3)\Delta_1(\xi_2-\xi_4)$$

and corresponding to the graphs of Fig. 3 vanish in the limit $s\to\infty$. For example, the first term is

$$\int\int_0^s\int d\xi_1 d\xi_2 d\xi_3\Delta_1(\xi_1-\xi_2)\Delta(\xi_2-\xi_3)\to\frac{\ln p^2s}{p^2s}.$$

The contribution of J to G(p) also has a vanishing limit for $s\to\infty$. The quantities σ_n in (44) correspond to the graphs of Fig. 4. Including $\sigma_1 = s\Delta(0)$ leads only to a mass renormalization; σ_2 also leads to mass renormalization and contributes a finite term which goes to zero as $s\to\infty$.

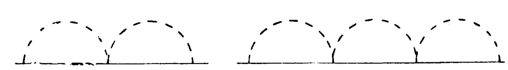

FIG. 3

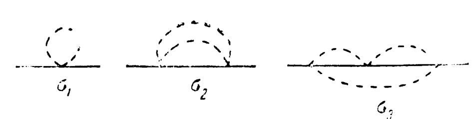

FIG. 4

The remaining σ_n have the same effect after renormalization.

Keeping only the first term in the exponent of (42), we thus arrive at the well known result that for $p^2\sim m^2$

$$G(p)=\frac{|1-p^2/m^2|^{e^2/4\pi^2}}{\gamma p-m}I(p), \tag{45}$$

where

$$I(p)=\varepsilon(p^2-m^2)\int_0^\infty dx e^{ix\varepsilon(p^2-m^2)}x^{e^2/4\pi^2}$$

$$\varepsilon(x)=1 \text{ for } x>0,\quad \varepsilon(x)=-1 \text{ for } x<0. \tag{46}$$

In conclusion we note that our method of writing the solutions of the Dirac equation in an external field and deriving the quantum mechanical quantities from them can be used not only to find the infrared limit, but also to investigate the region of high energies. However, in the latter case one must apply different methods of approximation for the calculation of the functional integrals, since the approach of the present paper is based on the assumption of the smallness of the contribution from the terms quadratic in the virtual momenta $q_i q_j$ as compared to terms proportional to q_i (cf.[17] on this point).

The author thanks Prof. D. I. Blokhintsev and L. D. Solov'ev for interesting discussions.

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Translated by R. Lipperheide
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