NONLINEAR INTERACTION LAGRANGIANS

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A mechanism for removing ultraviolet divergences in a local nonlinear quantum field theory is investigated within a model of scalar mesons interacting with a fixed point source. It is proved that the perturbation theory series, considered as expansions of the stationary states of the total Hamiltonian in terms of the stationary states of the free Hamiltonian, are always divergent in the case of nonlinear interactions. The functional methods which permit one to construct a finite S-matrix from a nonlinear Lagrangian, are in their essence methods for summing divergent series. The problem of uniqueness remains open for the time being, even if unitarity and causality are satisfied.

1. INTRODUCTION

IT has been attempted recently[1-4] to construct a finite local quantum field theory for a scalar field by introducing an essentially nonlinear interaction Lagrangian, satisfying certain requirements. It turned out to be possible to construct the S-matrix in terms of powers of the interaction Lagrangian, so that no ultraviolet divergences appear in any order. Unitarity has been verified only in second order of perturbation theory[4] and in third order for the Green's function[3].

In previous work[1-3] formal mathematical operations were employed to obtain the radiative operators of the S-matrix, so that the mechanism which led to the elimination of the ultraviolet divergences was not completely clarified, in our opinion. In the present paper it is attempted to understand what happens when an essentially nonlinear interaction Lagrangian is introduced, on the basis of a model of scalar mesons interacting with a fixed point source.

The model is described by the Hamiltonian

$$H = H_0 + H_I, \tag{1.1}$$

$$H_0 = \frac{1}{2}\int d\mathbf{x} :[\pi^2(\mathbf{x}) + (\nabla\varphi(\mathbf{x}))^2 + \mu^2\varphi^2(\mathbf{x})]:, \tag{1.2}$$

$$H_I = g\int d\mathbf{x}\,\delta(\mathbf{x}) :U(\varphi(\mathbf{x})): = g:U(\varphi(0)):, \tag{1.3}$$

where $\varphi(\mathbf{x})$ and $\pi(\mathbf{x})$ are the boson field operators, $U(\varphi)$ is some function of φ. If $U(\varphi) = \varphi^n$, with $n \geq 2$, the Hamiltonian (1.1) describes a nontrivial theory, which includes scattering and other physical processes. We shall be mainly interested in the presence of ultraviolet divergences for a point interaction with $n \geq 2$. We note that models with $n \geq 3$ are nonrenormalizable theories.

The present model is attractive because it allows us to establish a complete correspondence between a quantum field theoretical problem and an ordinary Schrödinger equation with a potential, for which the complete analysis is known.

Our problem consists in the following: is it possible to find such functions $U(\varphi)$ for which there are no ultraviolet divergences in each order of perturbation theory? Intuitively it might seem that the difficulties of the theory arise in the region where the fields φ are large. It would seem that if one could choose the interaction in a form which would make it small for large φ, such theories should contain no difficulties connected with large energies. And the existence of a finite perturbation theory would speak in favor of the existence of an expansion of the stationary states of the total Hamiltonian H in terms of the stationary states of H_0.

However, it turns out that it is not so. The effect of the infinite number of degrees of freedom of the quantized field turns out to be more important. In this respect our result is in agreement with the result of van Hove[5], who has proved that for a model with $U(\varphi) = \varphi$ the state vector spaces of the total Hamiltonian and the free Hamiltonian are mutually orthogonal subspaces of one Hilbert space.

On the other hand, the previously developed methods[1-3] allow one to construct, within the proposed model, for a certain class of interaction functions $U(\varphi)$, a perturbation series for the complete S-matrix which is free of divergences. At the same time it is impossible to construct for this class of Lagrangians a quantum-field-theoretical Schrödinger equation with a potential, since, firstly, the interaction cannot be a potential, and secondly, the perturbation theory series are strongly divergent. This means that the mentioned methods[1,2] are essentially methods of summation for divergent series. Therefore it is of prime importance to check such properties of the S-matrix as unitarity and causality. This agrees with the result of the axiomatic approach to quantum field theory, when the existence of a unitary S-matrix mapping the asymptotic fields φ_{in} onto φ_{out} is possible, whereas the existence of the "halved" unitary matrix S(t, $-\infty$) (evolution matrix) for finite t, is forbidden by Haag's theorem.[6]

It follows from all that was said that the construction of a finite S-matrix by perturbation theory in terms of a nonlinear interaction Lagrangian has little in common with the ideology of perturbation theory in the classical Lagrangian formulation of the theory, when it is assumed that a small perturbation produces a small change in the states of the free field. There exists only a method for constructing a finite S-matrix, as a unitary operator mapping the Hilbert space of the in-states into the Hilbert space of the out-states, in terms of powers of the coupling constant, and unitarity and causality play an essential role in the verification of the method.

2. THE QUANTUM-FIELD SCHRÖDINGER EQUATION

Let us now formulate the problem. We assume that the system is enclosed in a cubic box of volume V. The momentum vector k of the bosons has components which are integral multiples of $2\pi V^{-1/3}$. We introduce the creation and annihilation operators a_k^+ and a_k for free bosons, by means of the Fourier expansion

$$\varphi(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} (a_{\mathbf{k}} e^{i\mathbf{kx}} + a_{\mathbf{k}}^+ e^{-i\mathbf{kx}}), \qquad (2.1)$$

where

$$\omega_{\mathbf{k}} = (\mu^2 + \mathbf{k}^2)^{1/2}, \qquad [a_{\mathbf{k}}, a_{\mathbf{k}'}^+] = \delta_{\mathbf{k},\mathbf{k}'}.$$

Then we obtain for H_0 and H_I

$$H_0 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}}^+ a_{\mathbf{k}},$$

$$H_I = g : U\left(\frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} (a_{\mathbf{k}} + a_{\mathbf{k}}^+)\right) : . \qquad (2.2)$$

The interaction Hamiltonian has been taken in the normal product form. We shall discuss below what implications are involved by this for functions $U(\varphi)$ of a complicated form.

The equation $a_k = (q_k + ip_k)/(2)^{1/2}$ defines the Hermitean operators q_k and p_k which are subject to the commutation relations

$$[q_{\mathbf{k}}, q_{\mathbf{k}'}] = [p_{\mathbf{k}}, p_{\mathbf{k}'}] = 0, \qquad [q_{\mathbf{k}}, p_{\mathbf{k}'}] = i\delta_{\mathbf{k},\mathbf{k}'}.$$

In the representation in which the q_k are diagonal, one can put

$$p_{\mathbf{k}} = -i\partial/\partial q_{\mathbf{k}}, \qquad (2.3)$$

and the state vector of the field will be a function $\Phi = \Phi(q_k)$ with various (infinitely many) numerical q_k. In terms of p_k and q_k Eqs. (2.2) become

$$H_0 = \sum_{\mathbf{k}} \frac{1}{2} \omega_{\mathbf{k}} (p_{\mathbf{k}}^2 + q_{\mathbf{k}}^2 - 1), \qquad (2.4)$$

$$H_I = g : U\left(\frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \frac{q_{\mathbf{k}}}{\sqrt{\omega_{\mathbf{k}}}}\right) : . \qquad (2.5)$$

The quantum-field Schrödinger equation for the stationary states of the interacting system can be written in the form

$$\left\{\frac{1}{2} \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(-\frac{\partial^2}{\partial q_{\mathbf{k}}^2} + q_{\mathbf{k}}^2 - 1\right) + g : U\left(\frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \frac{q_{\mathbf{k}}}{\sqrt{\omega_{\mathbf{k}}}}\right) : \right\} \Phi = E\Phi. \qquad (2.6)$$

This equation is a complete analogue of the ordinary Schrödinger equation, but has an infinite number of degrees of freedom. Our problem is to solve Eq. (2.6) by means of perturbation theory (expanding in the constant g) and trying to determine the class of functions $U(\varphi)$ for which a finite limit exists as $V \to \infty$.

The free Hamiltonian represents an ensemble of uncoupled harmonic oscillators. The eigenfunctions of H_0 are infinite products

$$\Phi(\{n_{\mathbf{k}}\}) = \prod_{\mathbf{k}'} h_{n_{\mathbf{k}'}}(q_{\mathbf{k}'}), \qquad (2.7)$$

where

$$h_n(q) = H_n(q) e^{-q^2/2} / (2^n n! \sqrt{\pi})^{1/2}, \qquad (2.8)$$

h_n are the orthonormal Hermite functions. The eigenfunctions $\Phi(\{n_k\})$ depend on the set of integers $\{n_k\}$; to each momentum there is associated the positive integer n_k representing the number of bosons with momentum k. The energy eigenvalue corresponding to $\Phi(\{n_k\})$ is $E = \Sigma_k n_k \omega_k$. For the ground state (the boson vacuum) $n_k = 0$.

We now consider the interaction Hamiltonian (2.5). In our representation it is a function of the numerical variables q_k, and thus can be considered as an ordinary potential. We would like to consider H_I as a small perturbation. It is reasonable to require a priori that since H_0 represents an ensemble of oscillators, the perturbation should not increase for $q_k \to \infty$ faster than q_k^2. We shall also assume that

$$\lim :U(\varphi):=0. \qquad (2.9)$$

It is also necessary to consider the normal product form of the interaction Hamiltonian. Usually, when one deals with an $H_I(\varphi)$ which is only a finite sum of the form

$$H_I(\varphi)=\sum_{m=1}^{N} c_m\varphi^m,$$

one means by normal product $:\varphi_m:$ such a rearrangement of the creation operators a_k^+ and annihilation operators a_k, that all operators a_k are to the right of the operators a_k^+. The following definition will be equivalent to this. Consider the matrix element

$$\langle 0|:H_I(\varphi):|n\rangle;$$

here $|0\rangle = \Phi(\{0\})$ and $|n\rangle = \Phi(\{n_k\})$, where $n = \Sigma_k n_k$; then the normal form $:H_I(\varphi):$ of H_I is that form in which for any n the following equality holds:

$$\langle 0|:H_I(\varphi):|n\rangle - e_n\langle 0|:\varphi^n:|n\rangle = c_n\langle 0|\varphi^n|n\rangle. \qquad (2.10)$$

Let the function $U(\varphi)$ be given, and assume it can be expanded in Taylor series in the neighborhood of the point $\varphi = 0$:

$$U(\varphi)=\sum_{m=1}^{\infty}\frac{u_m}{m!}\varphi^m. \qquad (2.11)$$

Then the normal form of the operator $U(\varphi)$ will be the function $:U(\varphi):$ satisfying the condition

$$\langle 0|:U(\varphi):|n\rangle = \frac{u_n}{n!}\langle 0|:\varphi^n:|n\rangle. \qquad (2.12)$$

Substituting into (2.12) the explicit expressions for $|0\rangle$ and $|n\rangle$, given in (2.7) and (2.8), and carrying out the integration, we obtain

$$\frac{1}{\sqrt{2\pi D}}\int_{-\infty}^{\infty} dv:U(v):\frac{H_n(v/\sqrt{2D})\,e^{-v^2/2D}}{(2D)^{n/2}} = u_n, \qquad (2.13)$$

where

$$D=\frac{1}{V}\sum_{\mathbf{k}}\frac{1}{2\omega_{\mathbf{k}}},$$

and at this stage of the computation the quantity D is considered finite and should automatically drop out of the expressions for physical amplitudes.

The function $:U(\varphi):$ can be determined from (2.13) by making use of the circumstance that the Hermite functions form a complete orthonormal system:

$$:U(\varphi):=\pi^{1/4}\sum_{m=1}^{\infty}\frac{u_m}{\sqrt{m!}}D^{m/2}\,h_m\left(\frac{\varphi}{\sqrt{2D}}\right)e^{\varphi^2/4D}. \qquad (2.14)$$

Since according to our formulation of the problem $:U(\varphi):$ is to be considered a function of φ, the series (2.14) must converge for arbitrary D. From the theory of orthogonal series (cf. e.g.[7]) it follows that it is necessary that

$$\lim_{m\to\infty}\frac{u_m}{\sqrt{m!}}(\sqrt{D})^m=0 \qquad (2.15)$$

which implies that u_m must satisfy the inequality

$$|u_m|<A^m m^{\sigma m}, \qquad (2.16)$$

where $0 < \sigma < 1/2$ and A is some constant. This means that the function $U(\varphi)$ in (2.11) is an entire analytic function in the complex φ-plane and belongs to the class S^σ in the Gel'fand-Shilov classification.[8]

An example of such a function, which in addition satisfies also (2.9) is

$$U(\varphi)=\int_0^{\infty} d\alpha e^{-\alpha}\sin(\varphi\alpha^\sigma), \qquad 0<\sigma<\frac{1}{2}. \qquad (2.17)$$

It is easy to show that

$$:U(\varphi):=\int_0^{\infty} d\alpha\exp\left\{-\alpha+\frac{1}{2}D\alpha^{2\sigma}\right\}\sin(\varphi\alpha^\sigma). \qquad (2.18)$$

It is important to note that for the interaction Lagrangians considered in[1-3], for which the function $U(\varphi)$ had cuts in the complex φ-plane, the condition (2.16) is not satisfied, since in this case $u_m \sim m!$ as $m \to \infty$, and therefore the series (2.14) for $:U(\varphi):$ diverges everywhere, and consequently does not represent any function of φ.

We compute the correction to the ground state energy of the source by means of perturbation theory. The first nonvanishing correction will be in the second order and is given by the well-known formula

$$E_0^{(2)}=\sum_{\{n_{\mathbf{k}}\}}{}' |(\Phi(\{0\}),\,H_I\Phi(\{n_{\mathbf{k}}\}))|^2\left[E_0^{(0)}-\sum_{\mathbf{k}} n_{\mathbf{k}}\omega_{\mathbf{k}}\right]^{-1}, \qquad (2.19)$$

where the accent on the summation sign denotes the omission of the term with the quantum numbers of the vacuum (all $n_k = 0$) in the sum over the intermediate states $\{n_k\}$. We choose $E_0^{(0)} = 0$. Making use of (2.7), (2.8), (2.11) and (2.12), we obtain

$$(\Phi(\{0\}),\,H_I\Phi(\{n_{\mathbf{k}}\}))=gu_n\prod_{\mathbf{k}}[n_{\mathbf{k}}!\,(2V\omega_{\mathbf{k}})^{n_{\mathbf{k}}}]^{-1/2}, \qquad (2.20)$$

where

$$n=\sum_{\mathbf{k}} n_{\mathbf{k}}.$$

Substituting (2.20) into (2.19) and carrying out some simple transformations one can obtain

$$E_0^{(2)}=-g^2\int_0^{\infty} dt\,R(t), \qquad (2.21)$$

$$R(t)=\sum_{n=1}^{\infty}\frac{u_n^2}{n!}\Delta^n(t), \qquad (2.22)$$

where

$$\Delta(t)=\frac{1}{V}\sum_{\mathbf{k}}\frac{e^{-t\omega_{\mathbf{k}}}}{2\omega_{\mathbf{k}}}\xrightarrow[V\to\infty]{}\frac{1}{(2\pi)^3}\int\frac{d\mathbf{k}e^{-t\omega}}{2\omega}\underset{t\to 0}{\sim}\frac{1}{t^2}.$$

The series representing the function R(t) converges for all t, by virtue of (2.16), but as $t\to 0$ it increases very rapidly: thus, for the interaction Hamiltonian (2.17) we have

$$R(t)\underset{t\to 0}{\sim}\exp\{a(1/t)^{(1-2\sigma)^{-1}}\},$$

where a is some positive constant.

Thus, for the class of Lagrangians for which $:U(\varphi):$ is a function of φ, the ultraviolet divergences of perturbation theory for a point interaction are not only not eliminated, but are substantially stronger than in the usual nonrenormalizable theories in which $H_I(\varphi)$ has the form of a polynomial.

Thus, a negative answer has been obtained to the question whether it is possible within the framework of quantum field theory to consider $H_I(\varphi)$ as the analogue of an ordinary potential in a Schrödinger equation. This means that the influence of the infinite number of degrees of freedom of the field is extraordinarily large and such that the stationary states of the total Hamiltonian H cannot be expanded in terms of the stationary states of H_0.

In the case of the Hamiltonians which were considered in[1-3], when $u_m\sim m!$ as $m\to\infty$, the series (2.22) for R(t) diverges for any t, but it turns out that the series is summable to a function which decreases as $t\to 0$ so that the correction $E_0^{(2)}$ to the energy turns out to be finite.

3. FUNCTIONAL METHODS AND "FINITE" INTERACTION LAGRANGIANS

In this section we apply to the model under consideration the functional methods of the preceding papers[1-3], where it has been shown that it is possible to construct a perturbation theory without ultraviolet divergences for a class of interaction Hamiltonians $H_I(\varphi)=gU(\varphi)$, where $U(\alpha)$, considered as a function of the complex variable α possesses the following properties:

1) $U(\alpha)$ is analytic in the complex α-plane with a finite number of cuts and the integral of $|U(\alpha)|^2$ exists for any bounded domain.

2) $U(\alpha)$ is real and has no singularities on the real axis and can be expanded in a Taylor series (2.11) around the point $\alpha=0$.

3) At infinity $U(\alpha)$ satisfies the condition

$$\lim_{|\alpha|\to\infty}\frac{U(\alpha)}{\alpha^2}=0.$$

It follows from the considerations given above that for this class of Lagrangians there does not exist a function $:U(\varphi):$, i.e., the Schrödinger equation (2.6) does not contain a potential $:U(\varphi):$ but a divergent formal series (2.14).

We consider the expression for the S-matrix in the interaction picture

$$S=T\exp\left\{-ig\int_{-\infty}^{\infty}dtU(\varphi(t))\right\}, \qquad (3.1)$$

where

$$\varphi(t)=\frac{1}{(2\pi)^{3/2}}\int\frac{d\mathbf{k}}{\sqrt{2\omega}}(a_{\mathbf{k}}e^{-i\omega t}+a_{\mathbf{k}}^{+}e^{i\omega t}),\ [a_{\mathbf{k}},a_{\mathbf{k}'}^{+}]=\delta(\mathbf{k}-\mathbf{k}').$$

It is assumed that the interaction Lagrangian $U(\varphi)$ is in normal product form. We shall see below how to realize this requirement.

The basic problem of the theory is to find the S-matrix expanded in a series of normal products of the field operator $\varphi(t)$, i.e.

$$S=\sum_m\frac{1}{m!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_mS_m(t_1,\ldots,t_m):\varphi(t_1)\ldots\varphi(t_m):. \qquad (3.2)$$

If the expansion coefficients $S_m(t_1,\ldots,t_m)$ are known, the amplitudes for the various physical processes will be the Fourier transforms of these functions:

$$\mathcal{F}_m(E_1,\ldots,E_m)=\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_m\exp\{i(E_1t_1+\ldots+E_mt_m)\}$$
$$\times S_m(t_1,\ldots,t_m). \qquad (3.3)$$

The transition to normal ordering in (3.1) can be realized by means of Wick's theorem written in functional form[9]:

$$S=\exp\left\{\frac{1}{2}\int\int_{-\infty}^{\infty}dt_1dt_2\Delta_c(t_1-t_2)\frac{\delta^2}{\delta\varphi(t_1)\delta\varphi(t_2)}\right\}$$
$$\times\exp\left\{-ig\int_{-\infty}^{\infty}dtU(\varphi(t))\right\}, \qquad (3.4)$$

where

$$\Delta_c(t_1-t_2)=\langle 0|T(\varphi(t_1)\varphi(t_2))|0\rangle$$
$$=\frac{1}{(2\pi)^3}\int\frac{d\mathbf{k}}{2\omega}\exp\{-i\omega|t_1-t_2|\}. \qquad (3.5)$$

We expand the S-matrix in a power series in g, since we aim at constructing a finite perturbation theory. We have

$$S=\sum_{n=0}^{\infty}\frac{(-ig)^n}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nR_n(t_1,\ldots,t_n), \qquad (3.6)$$

where

$$R_n(t_1, \ldots, t_n) = \exp\left\{\frac{1}{2}\int\int_{-\infty}^{\infty} d\tau_1 d\tau_2 \Delta_c(\tau_1 - \tau_2)\frac{\delta^2}{\delta\varphi(\tau_1)\delta\varphi(\tau_2)}\right\}$$

$$\times U(\varphi(t_1))\ldots U(\varphi(t_n))$$

$$= \exp\left\{\frac{1}{2}\sum_{i,j=1}^{n}\Delta_c(t_i - t_j)\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}\right\}$$

$$\times U(\alpha_1)\ldots U(\alpha_n)\Big|_{\alpha_j=\varphi(t_j)}. \quad (3.7)$$

The diagonal terms of the sum in the exponent

$$\sum_{j=1}^{n}\Delta_c(0)\frac{\partial^2}{\partial\alpha_j^2}$$

are left out. At this point the fact that $U(\varphi)$ is in normal form is formally taken into account. In the language of ordinary perturbation theory the contractions between operators $\varphi(t)$ at the same instants of time are eliminated. Finally

$$R_n(t_1, \ldots, t_n) = \exp\left\{\sum_{1\leqslant i<j\leqslant n}\Delta_c(t_i - t_j)\frac{\partial^2}{\partial\alpha_i\cdot\partial\alpha_j}\right\}$$

$$\times U(\alpha_1)\ldots U(\alpha_n)\Big|_{\alpha_j=\varphi(t_j)} \quad (3.8)$$

The problem is reduced to finding the radiative operators $F^{(n)}_{m_1\ldots m_n}$ in the expansion

$$R_n(t_1, \ldots, t_n) = \sum_{m_1,\ldots,m_n} F^{(n)}_{m_1\ldots m_n}(\Delta_c(t_i - t_j))$$

$$\times :\frac{\varphi^{m_1}(t_1)\ldots\varphi^{m_n}(t_n)}{m_1!\ldots m_n!}:. \quad (3.9)$$

In the following computations it is extremely important to consider that the functions $\Delta_c(t_i - t_j)$ are real and positive, whereas the causal function $\Delta_c(t)$ is a complex function of rather complicated structure (3.5). Fortunately, there exists a unique possibility for which the function $\Delta_c(t)$ becomes real and positive: this is the transition to a "Euclidean metric," based on the fact that for any amplitude $\mathcal{F}_n(E_1, \ldots, E_n)$ of any physical process there always exists a domain for the variables $E_1, \ldots, E_n$ where this amplitude is real. In the relativistic theory this is called the Euclidean domain. In the model under consideration the amplitudes are defined by Fourier integrals of the radiative operators $F^{(n)}_{m_1\ldots m_n}(\Delta_c(t_i - t_j))$ in (3.9):

$$\mathcal{F}^{(n)}_{m_1\ldots m_n}(E_1, \ldots, E_n) = -2\pi(-i)^{n+1}\delta(E_1 + \ldots + E_n)$$

$$\times f^{n}_{m_1\ldots m_n}(E_1, \ldots, E_n), \quad (3.10)$$

$$f^{(n)}_{m_1\ldots m_n}(E_1, \ldots, E_n) = ni^{n-1}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_n\delta(t_1 + \ldots + t_n)$$

$$\times e^{i(E_1t_1+\ldots+E_nt_n)}F^{(n)}_{m_1\ldots m_n}(\Delta_c(t_i - t_j)). \quad (3.11)$$

The "Euclidean" domain is in this case the region of sufficiently small E_j. The transformation to the real expression is realized by means of the substitution $t_j \to -it_j$ and

$$f^{(n)}_{m_1\ldots m_n}(E_1, \ldots, E_n) = n\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_n\delta(t_1 + \ldots + t_n)$$

$$\times e^{E_1t_1+\ldots+E_nt_n}F^{(n)}_{m_1\ldots m_n}(\Delta(t_i - t_j)), \quad (3.12)$$

$$\Delta(t) = \frac{1}{(2\pi)^3}\int\frac{d\mathbf{k}}{2\omega}e^{-\omega|t|}, \quad (3.13)$$

where $\Delta(t)$ is real and positive.

It is essential that Eq. (3.12) cannot be obtained from (3.11) by means of a displacement of the contour $t_j \to -it_j'$ if $F^{(n)}_{m_1\ldots m_n}(\Delta_{ij})$ have essential singularities for $\Delta_{ij} = 0$. Therefore we shall consider (3.12) as the starting expression for the amplitude, and the transition to the physical region will be realized by means of analytic continuation in the variables E_j.

Thus we see that the transformation to the "Euclidean" metric is not only a question of convenience, but is intrinsically inherent in the method of treating nonlinear interactions under consideration.

Thus, we shall assume that in (3.8) the causal functions $\Delta_c(t)$ have been replaced by the functions $\Delta(t)$, defined in (3.13). In order to obtain the expansion (3.9) from (3.8) we make use of the operator identity:

$$\exp\left\{\Delta_{ij}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}\right\}$$

$$= \frac{1}{\pi}\int\int_{-\infty}^{\infty}d\sigma_{ij}d\sigma_{ji}\exp\left\{-\sigma_{ij}^2 - \sigma_{ji}^2 + (\Delta(t_i - t_j))^{1/2}\right.$$

$$\left.\times\left[(\sigma_{ij} + i\sigma_{ij})\frac{\partial}{\partial\alpha_i}(\sigma_{ij} - i\sigma_{ij})\frac{\partial}{\partial\alpha_j}\right]\right\}. \quad (3.14)$$

Substituting (3.14) into (3.8) and taking into account that the integrand in (3.14) contains translation operators in the variables α_j, we obtain

$$R_n(t_1, \ldots, t_n)$$

$$= \pi^{-n(n-1)/2}\int_{-\infty}^{\infty}\ldots\int\prod_{1\leqslant i\neq i\leqslant n}d\sigma_{ij}\exp\left\{-\sum_{1\leqslant i\neq j\leqslant n}\sigma_{ij}^2\right\}$$

$$\times\prod_{l=1}U\Big(\alpha_l + \sum_{1\leqslant j<l}(\Delta(t_l - t_j))^{1/2}(\sigma_{jl} + i\sigma_{lj})$$

$$+ \sum_{l<j\leqslant n}(\Delta(t_j - t_l))^{1/2}(\sigma_{lj} - i\sigma_{jl})\Big)\Big|_{\alpha_l=\varphi(t_l)}. \quad (3.15)$$

This expression is sufficiently complicated; however, one can convince oneself that the properties 1) and 3) guarantee the convergence of this integral for any n. Expanding R_n in powers of $\alpha_j = \varphi(t_j)$ one can obtain the radiative operators $F^{(n)}_{m_1 \ldots m_n}(\Delta(t_i - t_j))$. There will be no ultraviolet divergences either, due to condition 3). Thus the problem of expanding the S-matrix in a series with respect to normal products of the field operator $\varphi(t)$ has been solved within the framework of perturbation theory.

It is however essential to note that the operators (3.14) are not defined on the class of functions $U(\alpha)$ under consideration, since the operation of translation which has (3.15) as a consequence is mathematically inadmissible. This means that the series expansions for the radiative operators $F^{(n)}_{m_1 \ldots m_n}(\Delta(t_i - t_j))$ in powers of $\Delta(t_i - t_j)$ diverge for all values of $\Delta(t_i - t_j)$, i.e., are asymptotic expansions. The proposed functional method is simply a method of summation for these divergent series. It is known[10] that any procedure of summation of a divergent series is non-unique, and different methods may lead to different functions.[1)] One can reestablish uniqueness by imposing supplementary conditions on the functions obtained from the summation. This is an essential problem, and it is not clear a priori whether the unitarity and causality conditions are sufficient conditions for the unique determination of the radiative operators $F^{(n)}_{m_1 \ldots m_n}$.

As an example we treat the amplitude for scattering of a boson by a source in second order of perturbation theory, for the interaction

$$H_I = g \int dx \delta(x) \frac{\varphi(x)}{(1 + f\varphi^2(x))^{1/2}}. \qquad (3.16)$$

According to (3.12) we obtain

$$f_{11}^{(2)}(E) = \int_{-\infty}^{\infty} dt e^{Et} F_{11}^{(2)}(\Delta(t)). \qquad (3.17)$$

An explicit expression for the radiative operator $F^{(2)}_{11}(\Delta(t))$ follows from earlier results[1]:

$$F_{11}^{(2)}(\Delta) = g^2 \int_0^\infty \frac{dx J_0(x)[1 - 2x^2 f^2 \Delta^2(t)]}{(1 + x^2 f^2 \Delta^2(t))^{5/2}}, \qquad (3.18)$$

$$f_{11}^{(2)}(E) = g^2 \int_0^\infty dt \operatorname{ch} Et \int_0^\infty \frac{dx J_0(x)[1 - 2x^2 f^2 \Delta^2(t)]}{[1 + x^2 f^2 \Delta^2(t)]^{5/2}}. \qquad (3.19)$$

1)Expanding the operator (3.8) by means of a diagonalization [2] of

$$\sum_{i<j} \Delta_{ij} \frac{\partial^2}{\partial\alpha_i \partial\alpha_j} = \sum_j \lambda_j \frac{\partial^2}{\partial\beta_j^2}$$

and subsequent use of Eqs. (3.14), may in general have the consequence that the functions $F^{(n)}_{m_1 \ldots m_n}$ obtained in this way differ from (3.15)

This integral converges for $|E| < 2\mu$, i.e., the amplitude is real up to the threshold for the production of a second meson. The analytic continuation in E can be carried out as in the relativistic case[4], and one can obtain for the imaginary part of $f^{(2)}_{11}(E)$

$$\operatorname{Im} f_{11}^{(2)}(E) = g^2 \sum_{n=1}^{[E/2]} \frac{u_{n+1}^2}{n!} \Omega_n(E),$$

where

$$\Omega_n(E) = \frac{1}{(2\pi)^{3n}} \int \frac{d\mathbf{k}_1}{2\omega_1} \ldots \int \frac{d\mathbf{k}_{2n}}{2\omega_{2n}} \delta(E - \omega_1 - \ldots - \omega_{2n}).$$

Unitarity is satisfied in this order and the asymptotic behavior of the imaginary part can be derived by means of the method used in[4]:

$$\operatorname{Im} f_{11}^{(2)}(E) \sim e^{(E/2)\ln E} a(E), \qquad E \ln E \gg 1,$$

where a(E) is a function of slower increase and E is expressed in units of the mass μ.

CONCLUSION

The above analysis has shown that Haag's theorem is true within the framework of the present model, as well as in relativistic theory, i.e., there does not exist a matrix $S(t, -\infty)$ for finite t and the stationary states of the total Hamiltonian cannot be expanded in series with respect to the stationary states of the free Hamiltonian. This however does not contradict the existence of a complete S-matrix mapping the asymptotic free states. It turned out to be possible to construct a finite S-matrix as the power series of the interaction for a certain class of Lagrangians although unitarity has not been proved in higher orders. The mathematical methods employed are not rigorous. It is very likely that one could obtain different finite S-matrices from the same Lagrangian, by using different methods of computation. The uniqueness problem is extremely important. However we considered it essential to show that there exists a possibility of constructing a finite S-matrix from a nonlinear Lagrangian within the framework of some method, such that the S-matrix satisfies in all orders of perturbation theory the causality and unitarity conditions.

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