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THEORY OF MULTIPLE SCATTERING. II

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Multiple scattering of particles or quanta from a plane layer of matter is considered for the case of anisotropic scattering by an individual force center. An explicit analytic solution of the transport equation is obtained for cases when the expansion of the cross section in Legendre polynomials has a finite number of terms and when the cross section has a sharp peak at small angles (Coulomb scattering).

INTRODUCTION

In our earlier paper[1] we derived an equation for multiple scattering of particles passing through a body of finite dimensions. We used the approximation $N_0|a|^2\lambda \ll 1$, where N_0 is the concentration of the scattering centers arranged randomly within the body, a the scattering amplitude, and λ the wavelength of the incident particle. The scattering was assumed to be purely elastic. All the inelastic processes were regarded as absorption. For the case when the scattering cross section is isotropic (for example, scattering of slow neutrons by nuclei) and the scattering body or target is spherical, the derived equation was solved and an explicit expression obtained for the distribution of the scattered particles outside and inside the body.

In this paper we consider the case when the cross section for the scattering by an individual force center is anisotropic, and the scattering body has the form of a flat layer. This case is of interest to the solution of many problems in atomic and nuclear physics, astrophysics, and physics of turbid bodies.

Coulomb scattering, which usually plays the main role in problems involving the passage of charged particles to various bodies, is highly anisotropic. When light is scattered by atoms, molecules or dust particles the anisotropy is not so strongly pronounced, but noticeable errors can result if it is neglected. It is equally important to take into account the anisotropy in the case of multiple nuclear scattering of fast nucleons, etc.

In this paper we derive an explicit expression for the distribution outside and inside a body of finite dimensions, for particles multiply and anisotropically scattered by an individual force center. We use the same approximations as before[1]. It is assumed for concreteness that the scattering body-target has the form of a flat layer, although the equation itself and the method of its solution are valid for any geometry of the body. We note that the problem of multiple scattering with an anisotropic cross section was solved so far only by numerical means[2]. Approximate analytic expressions were obtained only for Coulomb scattering through small angles[3].

1. STATEMENT OF THE PROBLEM

1. We consider a scatterer in the form of a flat layer, with infinite x and y dimensions, with thickness 2L in the z direction. The coordinate plane (xy) coincides with the plane on which the particles are incident. The number of particles leaving a unit surface of the layer at an angle ϑ_1 to the z axis and striking a unit area located perpendicular to their momentum is equal to

$$I(\mathbf{p}_1\mathbf{p}_0) = 2I_0N_0\sigma_s K(\mathbf{n}_1, \mathbf{n}_0; is_1, is_0) \times |\sec\vartheta_1| \exp[-|s_1|L - |s_0|L];$$

$$s_1 = \alpha\sec\vartheta_1, \quad s_0 = \alpha\sec\vartheta_0, \quad \alpha = N_0\sigma_0, \quad \mathbf{p}_1 = p_0\mathbf{n}_1, \quad \mathbf{p}_0 = p_0\mathbf{n}_0, \tag{1}$$

$$K(\mathbf{n}_1\mathbf{n}_0; uw) = \varkappa(\mathbf{n}_1\mathbf{n}_0)f(u-w) + \pi^{-1}\alpha\varepsilon\int d\mathbf{n}\varkappa(\mathbf{n}_1\mathbf{n}) \times \int_{-\infty}^{+\infty} dp\,[\alpha + ip\cos\vartheta]^{-1}K(\mathbf{n}\mathbf{n}_0; pw)f(u-p), \tag{2}$$

$$\varkappa(\mathbf{n}_1\mathbf{n}_0) = \sigma(\mathbf{n}_1\mathbf{n}_0)\sigma_s^{-1}, \quad \sigma_s = \int d\mathbf{n}\sigma(\mathbf{n}\mathbf{n}_0),$$

$$\cos\vartheta_0 = \mathbf{n}_0\mathbf{n}_2, \quad \cos\vartheta_1 = \mathbf{n}_1\mathbf{n}_2, \quad \cos\vartheta = \mathbf{n}\mathbf{n}_2, \quad \cos\theta = \mathbf{n}_1\mathbf{n}_2,$$

$$f(u-p) = (u-p)^{-1}\sin(u-p)L, \quad \varepsilon = \sigma_s\sigma_0^{-1}. \tag{3}$$

Here $\mathbf{n}_2$ is a unit vector along the z axis; $\mathbf{p}_0$ and $\mathbf{p}_1$

are the momenta of the incident and scattered particles. We put $|\mathbf{p}_1| = |\mathbf{p}_0|$, since we are interested in particles passing through matter without a change in their energy. A particle losing part of its energy will be regarded as absorbed, $\sigma(\mathbf{n}_1\mathbf{n}_0)$ is the differential scattering cross section and σ_0 is the total cross section, including both elastic and inelastic processes; I_0 is the number of particles incident on a unit surface of the layer.

Equation (2) is a particular form of a more general equation, obtained earlier[1] and suitable for arbitrary geometry of the scattering body:

$$K(\mathbf{n}_1\mathbf{n}_0; \mathbf{uw}) = \varkappa(\mathbf{n}_1\mathbf{n}_0)F(\mathbf{u}-\mathbf{w}) + (2\pi)^{-3}\alpha\varepsilon\int d\mathbf{n}\varkappa(\mathbf{n}_1\mathbf{n})$$

$$\times\int d\mathbf{p}[\alpha+i\mathbf{pn}]^{-1}K(\mathbf{nn}_0; \mathbf{pw})F(\mathbf{u}-\mathbf{p}). \qquad (4)$$

The dependence on the geometry is determined completely by the functions $F(\mathbf{a}-\mathbf{b})$. For a plane layer

$$F(\mathbf{a}-\mathbf{b}) = \omega(\mathbf{a}, \mathbf{b})f(a_z-b_z),$$

$$\omega(\mathbf{a}, \mathbf{b}) = 8\pi^2\delta(a_x-b_x)\delta(a_y-b_y)\exp[ib_zL-ia_zL]. \quad (5)$$

Substituting (5) in (4) and putting

$$K(\mathbf{n}_1\mathbf{n}_0; \mathbf{uw}) = \omega(\mathbf{uw})K(\mathbf{n}_1\mathbf{n}_0; u_zw_z),$$

we obtain (2). Using then formulas (20), (24), (32), and (35) from[1], we can check the correctness of (1) and (2).

2. If the scattering is from a sphere of radius R whose center is at the origin, then the number of particles scattered through an angle ϑ to the direction of the incident beam is equal to

$$I(\mathbf{p}_1\mathbf{p}_0) = I_0N_0\sigma_sK(\mathbf{n}_1\mathbf{n}_0; i\alpha\mathbf{n}_1i\alpha\mathbf{n}_0)e^{-2\alpha R}. \qquad (6)$$

In this case

$$F(\mathbf{a}-\mathbf{b}) = 4\pi R\gamma^{-2}[(\gamma R)^{-1}\sin\gamma R-\cos\gamma R],$$

$$\gamma = |\mathbf{a}-\mathbf{b}|. \qquad (7)$$

3. To describe multiple scattering of light passing through a plane layer of substance V. A. Ambartsumyan[4], V. V. Sobolev[5], Chandrasekhar[2], and others used a kinetic equation in the form

$$dI(\tau, \vartheta) = [-I(\tau, \vartheta)+B(\tau, \vartheta)]\sec\vartheta d\tau, \qquad (8)$$

where τ is the optical thickness of the layer. The quantity $B(\tau, \vartheta)$ satisfies an integral equation with a resolvent expressed in terms of two functions, which in turn are solutions of a system of two nonlinear integral equations of a single variable. These functions were found by numerical means and tabulated for many values of the argument.

The quantity $B(\tau, \vartheta)$ is connected with (2) by the simple relation

$$K(\mathbf{n}_1\mathbf{n}_0; uw) = (2\varepsilon I_0)^{-1}e^{i(u-w)L}\int_0^{2\alpha L}d\tau B(\tau, \vartheta)e^{-iu\alpha^{-1}\tau}. \quad (9)$$

An advantage of our Eq. (4) is not that it can be used directly for an arbitrary geometry of the scattering body, but primarily that it yields a solution in explicit analytic form, with sufficient accuracy for most physical applications. We shall describe the method of solving (4), since it is not trivial and is of interest in itself, all the more because similar equations are encountered in many physical problems, for example in problems of wave diffraction by bodies of different shapes.

4. We put $K = K_0 + K_1$. We choose for K_0 the solution of an equation that differs from (4) only in that $F(\mathbf{u}-\mathbf{p})$ is replaced by $(2\pi)^3\delta(\mathbf{u}-\mathbf{p})$ under the integral sign. K_0 is an exact solution of the equation for an unbounded medium:

$$K_0(\mathbf{n}_1\mathbf{n}_0; \mathbf{uw}) = F(\mathbf{u}-\mathbf{w})\Gamma(\mathbf{n}_1\mathbf{n}_0; \mathbf{u}), \qquad (10)$$

$$\Gamma(\mathbf{n}_1\mathbf{n}_0; \mathbf{u}) = \varkappa(\mathbf{n}_1\mathbf{n}_0)$$

$$+\alpha\varepsilon\int d\mathbf{n}\varkappa(\mathbf{n}_1\mathbf{n})[\alpha+i\mathbf{un}]^{-1}\Gamma(\mathbf{nn}_0; \mathbf{u}). \qquad (11)$$

Substituting (10) in (4) we verify that the resolvent of the equation for K_1 is the function K itself. We obtain for the equation

$$K(\mathbf{n}_1\mathbf{n}_0; \mathbf{uw}) = \varkappa(\mathbf{n}_1\mathbf{n}_0)F(\mathbf{u}-\mathbf{w})$$

$$+(2\pi)^{-3}\varepsilon\int d\mathbf{p}F(\mathbf{u}-\mathbf{p})F(\mathbf{p}-\mathbf{w})g(\mathbf{n}_1\mathbf{n}_0; \mathbf{p})-(2\pi)^{-6}\alpha\varepsilon^2$$

$$\times\int d\mathbf{n}\int d\mathbf{p}(\alpha+i\mathbf{pn})^{-1}K(\mathbf{n}_1\mathbf{n}; \mathbf{up})\int d\mathbf{q}[(2\pi)^3\delta(\mathbf{p}-\mathbf{q})$$

$$-F(\mathbf{p}-\mathbf{q})]g(\mathbf{nn}_0; \mathbf{q})F(\mathbf{q}-\mathbf{w}), \qquad (12)$$

$$\varepsilon g(\mathbf{n}_1\mathbf{n}_0; \mathbf{p}) = \Gamma(\mathbf{n}_1\mathbf{n}_0; \mathbf{p})-\varkappa(\mathbf{n}_1\mathbf{n}_0). \qquad (13)$$

The equation for a flat layer is obtained from (12) in which u and w are replaced by u_z and w_z, $F(\mathbf{a}-\mathbf{b})$ by $f(a_z-b)$, $(2\pi)^{-3}\epsilon$ by $\pi^{-1}\epsilon$, $(2\pi)^3\delta(\mathbf{a}-\mathbf{b})$ by $\pi\delta(a_z-b_z)$, and $\int d\mathbf{p}\int d\mathbf{q}$ by $\int dp_z\int dq_z$. The subscript z will be left out from now on.

We expand all the angle-dependent quantities in Legendre polynomials

$$\varkappa(\mathbf{n}_1\mathbf{n}_0) = (4\pi)^{-1}\sum_l\varkappa_lP_l(\mathbf{n}_1\mathbf{n}_0) \qquad (14)$$

etc, and confine ourselves only to the case when the particles are incident along the z axis, that is, $\mathbf{n}_2 = \mathbf{n}_0$. Denoting the expansion coefficient $K(\mathbf{n}_1\mathbf{n}_0; uw)$ by $(4\pi)^{-1}K_l(u, w)$, we obtain for them the equation

$$K_l(u, w) = K_l^{(0)}(u, w)+\pi^{-2}\varepsilon^2\sum_{Lnl_1}[C_{n0l_10}^{L0}]^2\int_{-\infty}^{+\infty}dpG_L(p)K_{l_1}(u, p)$$

$$\times \int_{-\infty}^{+\infty} dq[f(p-q)-\pi\delta(p-q)]g_{nl}(q)f(q-w); \quad (15)$$

$$K_l^0(u,w)=\varkappa_l f(u-w)+\pi^{-1}\varepsilon\int_{-\infty}^{+\infty} dpg_l(p)f(u-p)f(p-w), \quad (16)$$

$$G_L=\frac{i\alpha}{p}Q_L\left(\frac{i\alpha}{p}\right),\quad \varepsilon g_{lk}(u)=\Gamma_{lk}(u)-\varkappa_l\delta_{lk},\quad g_l=\sum_k g_{kl}, \quad (17)$$

where $Q_L(x)$ is a Legendre polynomial of the second kind and C^{L0}_{n0l0} are Clebsch-Gordan coefficients. We have interchanged the vectors $\mathbf{n}_1$ and $\mathbf{n}_0$, since Eqs. (1) and (2) are invariant relative to such an interchange (reciprocity theorem).

Analogously, the expansion of $\Gamma(\mathbf{n}_1\mathbf{n}_0;\, u)$ yields

$$\Gamma_l(u)=\varkappa_l+\frac{\alpha\varepsilon}{2}\varkappa_l\sum_{n=0}\int_{-1}^{+1} dy\frac{P_l(y)P_n(y)}{\alpha+iyu}\Gamma_n(u). \quad (18)$$

2. SOLUTION OF THE EQUATION FOR MULTIPLE SCATTERING

1. We consider first the case when we can neglect all the terms of the expansion (14) starting with some term, so that all the $\kappa_l \approx 0$ when $l \geq l_0$. Then (18) is a finite system of algebraic equations. Its solution (for the imaginary $u \to iu$ of interest to us) is

$$\Gamma_l(iu)=\frac{b_l(u)}{\Delta(u)}+\frac{\alpha\varepsilon}{2u}\frac{b(u)}{\Delta^2(u)}\left[1-\frac{\alpha\varepsilon}{2u}\frac{h(u)}{\Delta(u)}\ln\frac{\alpha+u}{\alpha-u}\right]^{-1}$$
$$\times h_l(u)\ln\frac{\alpha+u}{\alpha-u},$$
$$b_l(u)=\sum_n \varkappa_n A_{nl}(u)P_n(\mathbf{n}_0\mathbf{n}_2),\qquad b(u)=\sum_l b_l(u)P_l(\alpha/u),$$
$$h_l(u)=\sum_n \varkappa_n A_{nl}(u)P_n(\alpha/u),\qquad h(u)=\sum_l h_l(u)P_l(\alpha/u), \quad (19)$$

where $\Delta(u)$ is the determinant of the algebraic system, consisting of the elements

$$\Delta_{nl}(u)=\delta_{nl}+\frac{\alpha\varepsilon}{u}\varkappa_n\sum_L[C^{L0}_{n0l0}]^2\sum_{m=1}^{L}\frac{1}{m}P_{m-1}\left(\frac{\alpha}{u}\right)P_{L-m}\left(\frac{\alpha}{u}\right),$$

and $A_{nl}(u)$ is the cofactor of the corresponding element of the determinant. The arguments of the Legendre polynomials α/u can be larger than unity.

Since (19) has a logarithmic singularity at $u = \pm\alpha$, we represent it in the form of a Cauchy integral, choosing a contour consisting of a large circle with cuts from $i\alpha$ to $i\infty$ and from $-i\alpha$ to $-i\infty$, and two small circles around the poles at the points $iu = \pm i\beta$. Then $\Gamma_l(iu)$ is uniquely determined and can be written in the form

$$\Gamma_l(iu)=\varkappa_l-\sum_{\mu=\pm1}\left\{\frac{\xi_{l\mu}}{\mu u-\beta}+\frac{\varepsilon}{2}\alpha\int_\alpha^\infty\frac{dx}{x}\frac{\eta_{l\mu}(x)}{\mu u-x}\right\}. \quad (20)$$

Here $\xi_{l,\pm1}$ is the residue of $\Gamma_l(iu)$ at the pole $u = \pm\beta$, where β is the root of the equation

$$2\beta\Delta(\beta)=\alpha\varepsilon h(\beta)[\ln(\alpha+\beta)-\ln(\alpha-\beta)];$$

$$\xi_{l\mu}\equiv\beta h_l(\mu\beta)\xi(\mu\beta) = \beta h_l(\mu\beta)(\alpha^2-\beta^2)$$
$$\times\{[\beta^2+\alpha^2(\varepsilon\delta_0(\beta)-1)+\beta(\alpha^2-\beta^2)\delta_1(\beta)]h(\beta)\Delta(\beta)\}^{-1}$$
$$\times\sum_k h_k(\mu\beta)\equiv\sum_k\xi_{k\mu}^{(l)}, \quad (21)$$

$\delta_0(x) = h(x)[\Delta(x)]^{-1}$; $\delta_1(\beta)$ is the derivative of $\delta_0(x)$ at the point $x = \beta$, divided by $\delta_0(\beta)$; $\eta_{l\pm1}(x)$ is the discontinuity of $\Gamma_l(ix)$ on the x-plane cuts from $\pm i\alpha$ to $\pm i\infty$:

$$\eta_{l\pm1}(x)\equiv\sum_k\eta^{(l)}_{k\pm1}(x)=h_l(\pm x)\sum_k h_k(\pm x)[\Delta(x)]^{-2}\theta(x), \quad (22)$$
$$\theta(x)=\left[\left(1+\frac{\alpha\varepsilon}{2x}\frac{h(x)}{\Delta(x)}\ln\frac{x-\alpha}{x+\alpha}\right)^2+\left(\frac{\alpha\pi\varepsilon h(x)}{2x\Delta(x)}\right)^2\right]^{-1}.$$

2. We substitute (16)—(22) in (15), calculate the integrals, and replace u and w by iu and iw, since (1) contains K for imaginary values of u and w. We obtain

$$K_l(iu,iw)=K_l^0(iu,iw)-\alpha\varepsilon\sum_{\mu=\pm1,\,n}\int_\alpha^\infty\frac{dx}{x}K_n(iu,i\mu x)$$
$$\times\left[N_\mu(x,\beta,w)r_{l\mu}^{(n)}(x)+\frac{\varepsilon}{2}\int_\alpha^\infty dyN_\mu(x,y,w)S_{l\mu}^{(n)}(x,y)\right];$$
$$N_\mu(x,y,z)=(x+y)^{-1}e^{-(x+y)L}f(iy+i\mu z), \quad (23)$$
$$r_{l\mu}^{(k)}(y)=\sum_{Ln}[C^{L0}_{n0k0}]^2\xi^{(l)}_{n-\mu}P_L\left(\mu\frac{\alpha}{y}\right),$$
$$S_{l\mu}^{(k)}(x,y)=\frac{\alpha}{y}\sum_{Ln}[C^{L0}_{n0k0}]^2\eta^{(l)}_{n-\mu}(y)P_L\left(\mu\frac{\alpha}{x}\right). \quad (24)$$

If the scattering from the individual force center is isotropic, then K_l differs from zero only when $l = 0$, with

$$r_{0\mu}(x) = \beta\xi(\mu\beta),\qquad S_{0\mu}(x,y) = \alpha y^{-1}\theta(y).$$

3. Let us explain the physical meaning of the different terms of (23). To this end we imagine an infinite medium and pass through it the surfaces corresponding to the boundaries of our layer. On the layer boundary z = 0 we specify the flux of the incident particles. Scattering and absorption in the layer and in the surrounding medium results in a definite particle distribution. From each point of

the medium, the particles can enter the layer either directly, without being scattered, or after a certain number of collisions, that is, as a result of diffusion. The free term K_l^0 determines the number of particles that cross the boundary of the layer in the $\mathbf{n}_1$ direction. In order to describe the scattering by a layer of matter situated in vacuum, it is necessary to subtract from K_l^0 the contribution due to the particles coming from the medium. The first term in the integral of (23) compensates for the contribution from diffusion. This term contains a degenerate kernel and will be evaluated exactly. The second term compensates for the contribution from the particles entering the layer without being scattered, and will be evaluated by successive approximations. It turns out that high accuracy is attained even in the first approximation. The terms with $\mu = +1$ and $\mu = -1$ in (23) are connected with the particles crossing the surfaces z = 0 and z = 2L, respectively.

4. Regarding the free term K_l^0 and the term with the nondegenerate kernel as a new free term, we can solve (23) formally. The obtained "solution" will again be an equation in K_l, but it will be easier to solve than the initial equation. We thus obtain

$$K_l(iu, iw) = \tilde{K}_l(iu, iw) - \frac{\alpha\varepsilon^2}{2}\sum_{\mu=\pm 1,\, n}\int_\alpha^\infty \frac{dx}{x}K_n(iu, i\mu x)M_{l\mu}^{(n)}(ix, iw), \tag{25}$$

$$M_{l\mu}^{(n)}(ix, iw) = R_{l\mu}^{(n)}(ix, iw) - \frac{\alpha\varepsilon^2}{2}\sum_{\nu=\pm 1,\, k}\int_\alpha^\infty \frac{dy}{y}R_{k\mu}^{(n)}(ix, i\nu y)M_{l\nu}^{(k)}(iy, iw). \tag{26}$$

The quantity K_l differs from the exact solution in that it contains an additional contribution from the particles entering the layer from the surrounding medium without being scattered. This contribution is compensated for by the integral term in (25):

$$\tilde{K}_l(iu, iw) = \rho_{l1}(u, w)\Gamma_l(iu) + \sum_{\mu=\pm 1}\Big\{\frac{\varepsilon}{2}A_{l\mu}(u, w) + D_{l_0}\sum_n \xi_{n\mu}H_{l\mu}^{(n)}(\beta, w)[\mu u - \beta]^{-1}e^{(\mu u-\beta)L}\Big\}, \tag{27}$$

$$\Gamma_l\rho_{l\mu}(y, w) = \Gamma_l f(i\mu y - iw) - \varepsilon D_{l_0}\sum_{\lambda k n} X_{\lambda-\lambda}^{kn}(\mu y)H_{l-\lambda}^{(k)}(\beta, w)\Gamma_n, \tag{28}$$

$$H_{l\mu}^{(k)}(\beta, w) = f(i\beta - iw)D_{-\mu-1}^{kl}(\beta) + f(i\beta + iw)D_{-\mu 1}^{kl}(\beta), \tag{29}$$

where $D_{l_0}^{-1}$ is the determinant of the system of algebraic equations obtained in the solution of (23) as an equation with degenerate kernel, while $D_{\mu\lambda}^{kl}$ is the corresponding cofactor:

$$D_{l_0}{}^{-1} = \|\, \delta_{ln}\delta_{\mu\nu} + \varepsilon X_{\mu\nu}{}^{ln}(\mu\beta)\,\|,$$

$$X_{\mu\lambda}{}^{ln}(z, \beta) \equiv X_{\mu\lambda}{}^{ln}(z) = \alpha\int_\alpha^\infty \frac{dy}{y} r_{l\mu}^{(n)}(y)N_\lambda(\beta, y, z),$$

$$X_{\mu\lambda}^{(l)} = \sum_n X_{\mu\lambda}{}^{ln}. \tag{30}$$

The quantities $X_{\mu\lambda}^l$, $R_{l\mu}^{(n)}$, and $A_{l\mu}$ cannot be expressed in terms of elementary functions, and we shall therefore represent them henceforth in the form of rapidly converging series:

$$R_{l\mu}^{(n)}(ix, iw) = \int_\alpha^\infty (x+y)^{-1}e^{-(x+y)L}S_{l\mu}^{(n)}(x, y)\rho_{l-\mu}(y, w), \tag{31}$$

$$A_{l\mu}(u, w) = \alpha\int_\alpha^\infty \frac{dx}{x}\frac{e^{(\mu u - x)L}}{\mu u - x}\eta_{l\mu}(x)\rho_{l\mu}(x, w). \tag{32}$$

Solving (26) by the method of successive approximations, we can easily verify that when $\epsilon \le 0.7$ the integral term does not exceed $0.3(\epsilon/2)^2$ of the first term. The small numerical factor is a result of the fact that the integrand decreases like y^{-3} with increasing y. On the other hand, if $\epsilon \to 1$, then we see directly from (28) that the quantities $\rho_{l\mu}$ and $R_{l\mu}^{(n)}$ decrease because of the increase of the diffusion terms. In this case an estimate of the integral term (26) yields $\lesssim 0.1(\epsilon/2)^2$. We see that in every case we can neglect with good accuracy the integral term (25). This makes (25) not an equation, but an expression for K_l (iu, iw). This solves our problem and yields an expression for K_l in terms of quadratures. Multiplying K_l (iu, iw) by $P_l(\mathbf{n}_1\mathbf{n}_0)$, summing over l, putting $w = \alpha \sec\vartheta_1$ and $u = \alpha\sec\vartheta_0 = \alpha$, and substituting in (1) we obtain the sought-for expression for the intensity of scattering by a plane layer.

3. INTENSITY OF MULTIPLE SCATTERING IN THE CASE OF A WEAKLY ANISOTROPIC CROSS SECTION

1. The general course of the solution is quite complicated. We shall therefore consider two important cases: a) when the expansion of the cross section in Legendre polynomials contains not too many terms; b) when the scattering from an individual force center has a sharp maximum at small angles. Case a) covers a large group of problems involving the scattering of light by atoms and elec-

trons, the scattering of neutrons by nuclei, etc. Case b) is characteristic of the scattering of charged particles.

To simplify the final expressions even more, we consider separately the cases of small ($\epsilon \sim 1$) and large ($\epsilon \ll 1$) absorption, and replace the single formula (25) by two much simpler ones. It turns out that these two formulas cover the entire region of values of ϵ, and can be joined together with good accuracy at $\epsilon \sim 0.7$.

2. We consider first the case a) with $\epsilon < 0.7$. Discarding in (2.5) terms smaller than $\epsilon^2/4$, we obtain

$$K_l(i\alpha,\ is_1) = \Gamma_l(is_1) f(is_1 - i\alpha) + \sum_{\mu=\pm1} \Big\{ \frac{\varepsilon}{2} [A_{l\mu}(s_1,\ \alpha) - \varepsilon\varkappa_l B_{l\mu}(s_1,\ \alpha)] + f(i\mu\beta - i\alpha)[(\mu s_1 - \beta)^{-1} \xi_{l\mu} e^{(\mu s_1 - \beta)L} - \varepsilon\varkappa_l X^{(l)}_{-\mu\mu}(s_1)] \Big\}, \tag{33}$$

where

$$B_{l\mu}(s_1,\ \alpha) = \alpha \int_\alpha^\infty \frac{dx}{x} f(i\mu s_1 - x) R_{l\mu}(ix,\ i\alpha). \tag{34}$$

The explicit form of the quantities contained in (33) also simplifies appreciably. We note that when $\epsilon < 0.7$ the quantity $\alpha - \beta$ is of the order of $\alpha \exp(-2\epsilon^{-1})$. Inasmuch as the $X^{(l)}_{\lambda\mu}$ are proportional to $\alpha - \beta$, we can replace $\rho_{l\mu}(y,\ w)$ by $f(i\mu y - iw)$. With the same accuracy, $\beta = \alpha \tanh[(\epsilon\delta_0)^{-1}]$. Expressions (31) and (32) contain under the integral sign the function $\theta(x)$, which is equal to unity when x is large and to zero when $x = \alpha$, and which has a sharp maximum of the order of $4\pi^{-2}\epsilon^{-2}\delta_0^{-2}$ at the point $x_0 = \alpha^2\beta^{-1}$. Using the sharpness of $\theta(x) - 1$, we can take out the expressions outside the integral sign at the point x_0. This cannot be done if $(\alpha - \beta)L \gg 1$, for then in (31) $\rho_{l1}(x,\ w)$ becomes sharper than $\theta(x) - 1$. In this case, however, we can discard in general the terms containing η_{l1}, since they are smaller by a factor $\exp[(\beta - \alpha)L]$ than the terms with ξ_{l1}. The final formulas will contain the function

$$\mathcal{T}(\varepsilon\delta_0) = \int_\alpha^\infty \frac{dx}{x} [\theta(x) - 1] = \frac{\varepsilon}{1-\varepsilon} - \frac{2(\alpha^2 - \beta^2)}{\varepsilon\delta_0[\beta^2 + \alpha^2(\varepsilon\delta_0 - 1)]}. \tag{35}$$

In this approximation

$$A_{l\mu}(u, w) = -\alpha\mathcal{T}(\varepsilon\delta_0)\mathcal{L}_l(-\mu x_0)[\mu u + x_0]^{-1} e^{-(\mu u + x_0)L} \times f(i\mu x_0 + iw) + [2(u-w)]^{-1} e^{-\mu(u-w)L} \sum_n (-\mu)^{n+1} \times d_{ln}[\Phi_n(\mu w,\ \mu w) - \Phi_n(\mu u,\ \mu w)], \tag{36}$$

where d_{ln} is the coefficient of expansion of $\mathcal{L}_l$ in powers of x^{-1}:

$$\mathcal{L}_l(x) \equiv h_l(x) b(x) [\Delta(x)]^{-2} = \sum_n (\alpha x^{-1})^n d_{ln}. \tag{37}$$

The functions Φ_n are expressed in terms of the integral exponential functions Ei:

$$\Phi_n(x, y) = (-\alpha x^{-1})^n \Phi_0(x, y) - \sum_{m=1}^n (-\alpha x^{-1})^{n-m+1} \Phi_m(0; y),$$

$$\Phi_0(x, y) = \alpha x^{-1} \{\ln|1 + \alpha^{-1}x| + e^{-2yL}[\mathrm{Ei}(-2\alpha L) - e^{2xL}\,\mathrm{Ei}(-2\alpha L - 2xL)]\};$$

$$\Phi_{m-1}(0; y) = m^{-1} - e^{-2yL} \int_1^\infty x^{-m-1} e^{-2\alpha x L}\, dx. \tag{38}$$

The quantities $X^{(l)}_{\mu\lambda}$ can also be expressed in terms of Φ_n:

$$X_{\mu\lambda}^{(l)}(y) = [2(\lambda y - \beta)]^{-1} e^{(\lambda y - \beta)L} \beta\xi(-\mu\beta) \times \sum_n \mu^n C_{nl}(-\mu\beta)[X_n(\beta; \lambda y) - X_n(\lambda y, \lambda y)],$$

$$X_n(p, k) = \frac{(2n-1)!!}{n!} \Big[\Phi_n(p, k) - \frac{n(n-1)}{2(2n-1)} \Phi_{n-2}(p, k) + \frac{n(n-1)(n-2)(n-3)}{2\cdot4\cdot(2n-1)(2n-3)} \Phi_{n-4}(p, k) - \ldots\Big],$$

$$C_{nl}(x) = \sum_L [C^{L0}_{n0l0}]^2 h_L(x). \tag{39}$$

Inasmuch as the $X^{(l)}_{\mu\,-\mu}(s_1)$ contained in (33) make a noticeable contribution only when $\alpha s_1^{-1} = \cos\vartheta \approx 0$, we can replace $X_n(\beta,\ s_1)$ and $X_n(\lambda s_1,\ s_1)$ by $X_n(\beta,\ \infty)$ and $X_n(\lambda s_1,\ \infty)$. In addition, in the case when $0 \le \vartheta \le \pi/2$ it is sufficient to retain X_{-11} and discard X_{1-1}, and vice versa in the case when $0 \ge \vartheta \ge \pi/2$.

The integral in (20) can also be evaluated:

$$\Gamma_l(iu) = \varkappa_l + \frac{\varepsilon}{2}\Big[\mathcal{L}_l(u)\frac{\alpha}{u}\ln\frac{u+\alpha}{u-\alpha} - \sum_{n=1}\sum_{\mu=\pm1} n^{-1}\mu^{n-1} \times \sum_{m=n}\Big(\frac{\alpha}{u}\Big)^{m-n+1} d_{lm}\Big] + \alpha\sum_{\mu=\pm1}\Big[(\mu u + \beta)^{-1} h_l(-\mu\beta)\xi(-\mu\beta) + \frac{\varepsilon}{2}(\mu u + x_0)^{-1}\mathcal{T}(\varepsilon\delta_0)\mathcal{L}_l(-\mu x_0)\Big]. \tag{40}$$

With the same accuracy, the quantity (34) can be written in the form

$$B_{l\mu}(u, w) = \alpha\mathcal{T}(\varepsilon\delta_0)\,b(-\mu x_0)\,[\Delta(x_0)]^{-2}$$
$$\times\,[\xi(-\mu x_0)]^{-1}X^{(l)}_{\mu\mu}(u, x_0)f(i\mu x_0 + iw). \qquad (41)$$

The sum over L in (39) is determined by the number of the h_L that differ from zero, while the sum over n in (39) is determined by the requirement $L + l \geq n \geq |L - l|$, with $n + L + l$ of necessity even. The sums over n in (36) and (40) are bounded as a result of the rapid decrease in d_{ln} with increasing n.

3. We present the explicit form of h_L, d_{ln} in (19) for the important case when

$$\varkappa(\vartheta) \equiv \sigma(\vartheta)\sigma_s^{-1} = 1 + \varkappa_1 P_1(\cos\vartheta) + \varkappa_2 P_2(\cos\vartheta).$$

In this case

$$h_0 = 1, \quad h_1 = z\varkappa_1(1-\varepsilon),$$
$$h_2 = -{}^1/_2\varkappa_2[1 - z^2(1-\varepsilon)(3-\varepsilon\varkappa_1)],$$
$$\Delta = 1 + \varepsilon z^2\{\varkappa_1(1-\varepsilon) - 0.75\varkappa_2[1 - z^2(1-\varepsilon)(3-\varepsilon\varkappa_1)]\},$$
$$b = \sum_{l=0}^{2} h_l, \qquad d_{ln} = \sum_k \zeta_{lk}\gamma_{n-k}, \qquad z = \alpha s_1^{-1} = \cos\vartheta,$$
$$\zeta_{00} = 1 - 0.5\varkappa_2,\ \zeta_{01} = \varkappa_1(1-\varepsilon),\ \zeta_{1,\,n+1} = \varkappa_1(1-\varepsilon)\zeta_{0n},$$
$$\zeta_{02} = 0.5\varkappa_2(1-\varepsilon)(3-\varepsilon\varkappa_1),$$
$$\zeta_{2,\,n} = 0.5\varkappa_2[(1-\varepsilon)(3-\varepsilon\varkappa_1)\zeta_{0,\,n-2} - \zeta_{0n}],$$
$$\gamma_0 = 1,\ \gamma_2 = -2\varepsilon[\varkappa_1(1-\varepsilon) - 0.75\varkappa_2],$$
$$\gamma_4 = -1.5\varepsilon(1-\varepsilon)(3-\varepsilon\varkappa_1)\varkappa_2. \qquad (42)$$

The remaining ζ_{lk} and γ_{n-k} are small and can be discarded.

4. If $2(\alpha - \beta)L \geq 1$, $\epsilon < 0.7$, and $0 < \vartheta < \pi/2$, further simplification of (33) is possible:

$$K_l(i\alpha, is) \approx \frac{e^{(\alpha-\beta)L}}{2(\alpha-\beta)}\left[\frac{\xi_{l1}\,e^{(s-\beta)L}}{s-\beta} - \varepsilon\varkappa_l X_{-11}{}^{(l)}(s)\right], \quad (43)$$

and when $\alpha s^{-1} = \cos\vartheta \approx 1$, we can neglect $\epsilon\kappa_l X_{-11}$.

5. We now consider the case a) with $1 - \epsilon \ll 1$ and $2(\alpha - \beta)L \gtrsim 1$. Discarding in (25) the terms smaller than $0.1(\epsilon/2)^2$, we obtain

$$K_l(i\alpha, is) = \mathfrak{M}_{l\sigma}(\alpha)\rho_{l\sigma}(\alpha, |s|)$$
$$+ D_{l_0}\sum_{\mu n}\xi_{n\mu}\frac{e^{(\mu\alpha-\beta)L}}{\mu\alpha-\beta}H_{l\mu}{}^{(n)}(\beta, s),$$

$$\mathfrak{M}_{l\sigma}(s) = \varkappa_l + \sum_\mu \frac{\xi_{l\mu}}{\beta-\mu s} + \frac{1}{2}\alpha\int_\alpha^\infty \frac{dx}{x}\frac{\eta_{l-\sigma}(x)}{|s|+x}, \qquad (44)$$

where $\sigma = 1$ or -1 respectively for $\sec\vartheta > 0$ or < 0. In this case, accurate to terms $\sim 1 - \epsilon$, we obtain

$$\beta \approx \alpha[(1-\varepsilon)\varepsilon^{-1}(3-\varepsilon\varkappa_1)]^{1/2},$$
$$h_1 = \varkappa_1[(1-\varepsilon)\varepsilon(3-\varepsilon\varkappa_1)]^{1/2},$$
$$h_0 = 1,\ h_2 = -0.5\varkappa_2(1-\varepsilon) \qquad (45)$$

etc.

The quantities $\xi_{0\mu}$ and $D^{-1}_{l_0}$ are proportional to $(1-\epsilon)^{-1/2}$. They enter in (44) only in a combination of $\xi_{0\mu}$ and D_{l_0} which, in the limit as $\epsilon = 1$, becomes equal to

$$\lim \xi_{l\mu}D_{l_0} \equiv \alpha h_l(0)\Lambda_{l_0}$$
$$= {}^1/_2\alpha h_l(0)\,\{[1 + 2\omega_{l_0}(\alpha L t_{01}(0) - t_{02}(0))]T_0(0)$$
$$\times[1 - 2\varkappa_1(3(3-\varkappa_1))^{-1/2}\omega_{l_0}T_1(0)]\}^{-1}. \qquad (46)$$

In analogous fashion, the calculation of $\rho_{l\mu}$ and $H^{(n)}_{l\mu}$ for $\epsilon = 1$ yields

$$\mathfrak{M}_{l\mu}\rho_{l\mu}(\alpha, s) = \mathfrak{M}_{l\mu}f(i\mu s - i\alpha)$$
$$- \Lambda_{l_0}\sum_{\lambda nk}\mathfrak{M}_{n\mu}(-\lambda\mu)^k\,T_n(0)h_k(0)H^{(k)}_{l-\lambda}(0; s),$$

$$sH_{0\mu}{}^{(0)}(0;\ s) = \operatorname{sh} sL\{1 + 2\omega_{l_0}[\alpha L t_{01}(0) - t_{02}(0)]\}$$
$$- 2\mu\omega_{l_0}T_0(0)\left[\alpha L\operatorname{ch} sL - \frac{\alpha}{s}\operatorname{sh} sL\right], \qquad (47)^*$$

where $\operatorname{sh}_1(0)H^{(1)}_{l\mu}$ is obtained by multiplying by

$$2\mu^{l+1}\varkappa_1[3(3-\varkappa_1)]^{-1/2}\omega_{l_0}T_{1-l}(0)$$

the first term in (47) with $l = 1$ and the second with $l = 0$; $\operatorname{sh}^{(0)}_1$ is obtained by multiplying the second term in the curly brackets by

$$-2\varkappa_1\omega_{l_0}[3(3-\varkappa_1)]^{-1/2}T_0(0)\operatorname{sh} sL,$$

ω_{l_0} is a number the value of which for $l \leq l_0$ is equal to

$$\omega_2 = (3-\varkappa_1)^3\left\{9\left(1-\frac{\varkappa_2}{4}\right)^2\left[2-\varkappa_1-\frac{2}{9}(3-\varkappa_1)\right.\right.$$
$$\left.\left.\times\left(\varkappa_1-\frac{3}{2}\varkappa_2\left(1-\frac{\varkappa_2}{8}\right)\right)\left(1-\frac{\varkappa_2}{4}\right)^{-2}\right]\right\}^{-1}; \qquad (48)$$

$$T_l(s) = \alpha^2\int_\alpha^\infty dx\,x^{-2}(x+s)^{-1}e^{-xL}P_l(\alpha/x)\operatorname{sh}(x+s)L; \quad (49)$$

$t_{l1}(s)$ is obtained by replacing $\sinh(x+s)L$ in (49) by $\cosh(x+s)L$, and $t_{l2}(s)$ is obtained by replacing $(x+s)^{-1}$ by $\alpha(x+s)^{-2}$. The quantity

$$\gamma_l \equiv \sum_\mu(\mu s - \beta)^{-1}\xi_{l\mu}$$

in (44) makes a contribution in the case of $\epsilon = 1$ only if $l = 0$ or 1:

$$\gamma_0 = 2\omega_{l_0}\alpha s^{-1}[\varkappa_1(3-\varkappa_1)^{-1} + \alpha s^{-1}],$$
$$\gamma_1 = 2\omega_{l_0}\alpha s^{-1}\varkappa_1(3-\varkappa_1)^{-1}. \qquad (50)$$

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* ch = cosh, sh = sinh

6. In the case when $2(\alpha - \beta)L \ll 1$, and (44) is not applicable, we obtain for large-angle scattering

$$K_l(i\alpha, is) = \mathfrak{M}_{l\sigma}(\alpha) f(is - i\alpha). \quad (51)$$

Single scattering through small angles occurs when $2(\alpha - \beta)L \ll 1$.

From (25) and (27) follows an expression for single scattering, which differs from (51) only in that $\mathfrak{M}_{l\sigma}$ is replaced by κ_l.

4. INTENSITY IN THE CASE OF COULOMB SCATTERING

1. We now consider the case when the scattering from the force center has a sharp maximum at small angles. For example, for the screened Coulomb field of the atom we have

$$\varkappa(\mathbf{n}_1\mathbf{n}_0) = A[1 + \eta - \cos\vartheta]^{-2} = (4\pi)^{-1} \sum_l \varkappa_l P_l(\cos\vartheta),$$

$$4\pi A = \eta(\eta + 2)(\eta + 1)^{-1}; \quad \varkappa_l \equiv (2l + 1) b_l;$$

$$b_l = -4\pi A Q_l'(1 + \eta); \quad (52)$$

$Q_l'(1 + \eta)$ is the derivative of the Legendre polynomial of the second kind; η is a screening parameter whose magnitude is usually very small:

$$\eta = a_1 Z^{2/3}[a_2 + a_3 Z^2], \quad a_1 = 8.25 \cdot 10^{-3},$$

$$a_2 = 1.13 T^{-1}(T + 2)^{-1},$$

$$a_3 = 2 \cdot 10^{-4}(T + 1)^2 T^{-1}(T + 2)^{-1}; \quad (53)$$

T is the kinetic energy of the particle in electron $m_e c^2$ units.

We begin the solution of (12) with a determination of (11), and consider normal incidence of a particle beam on the surface of the layer. We make use of the sharpness of (52) and take $(\alpha + ip \cdot \mathbf{n})^{-1}$ outside the integral sign at the point $\mathbf{n} = \mathbf{n}_0$. In this approximation we can solve (11) exactly:

$$\Gamma(\mathbf{n}_1\mathbf{n}_0;\ p) = \frac{1}{4\pi} \sum_l \frac{(2l + 1) b_l(\alpha + ip)}{\alpha + ip - \alpha\varepsilon b_l} P_l(\cos\vartheta). \quad (54)$$

Substituting (54) in (12), we can easily verify that when $\epsilon^2(1 + \eta - \cos\vartheta)\alpha L \ll 1$ the integral term in (12) can be omitted. Then

$$K(\mathbf{n}_1\mathbf{n}_0;\ is, i\alpha) = (4\pi)^{-1} \sum_l (2l + 1) b_l e^{\alpha\varepsilon b_l L}$$
$$\times f[i\alpha(\sec\vartheta - 1 + \varepsilon b_l)] P_l(\cos\vartheta). \quad (55)$$

Multiple scattering of charged particles is usually described by the Goudsmit and Saunderson formula[6], which is valid for small angles in purely elastic scattering, and is obtained by solving (8) in which $\sec\vartheta$ is replaced by unity. This formula follows also from (1), if we substitute in it (55) and put $\epsilon = 1$ and $\sec\vartheta = 1$. In addition, it is necessary to add the transmitted beam $I_0[\exp(-2\alpha L)](\cos\vartheta - 1)$, since it is not included in (1). Our approximation, which is connected with taking $(\alpha + ip\mathbf{n} \cdot \mathbf{n}_0)^{-1}$ outside the integral sign in (12), is valid if $\epsilon(1 + \eta - \cos\vartheta)\alpha L \ll 1$ up to thicknesses $L \sim (\alpha\eta)^{-1}$. At large thicknesses this approximation is not valid, since the particle distribution comes close to isotropic, and K and Γ are no longer sharp functions. With increasing angle, the degree of multiplicity of the scattering, taken into account by formula (55), decreases. The interval of angles and thicknesses for which (55) is valid is much wider than for the Goudsmit-Saunderson formula. In addition, (55) takes into account inelastic collisions (absorption). Further refinement of (55) and increase of the intervals of ϑ and of L can be attained by successive approximations, choosing (54) as the initial equation.

2. The expansion (55) converges slowly, since terms with large l play an important role. However, if we confine ourselves to small angles $\vartheta \sim \eta^{1/2}$, then the summation over l can be replaced by integration. Using the asymptotic expression $P_l(\cos\vartheta) \to J_0[(l + 1/2)\vartheta]$, which is valid for large l and small ϑ, and the analogous expression

$$Q_l'(1 + \eta) \to (\eta + 1)[(2l + 1)\eta(2 + \eta)]^{-1}$$
$$\times \exp[-{}^1/_2\eta(l + {}^1/_2)^2],$$

and then substituting in (55) and integrating with respect to l, we obtain

$$K(\mathbf{n}_1\mathbf{n}_0;\ is, i\alpha) = (2\pi)^{-1} \int_0^\infty dx \exp[-{}^1/_2\eta x^2 + \alpha\varepsilon e^{-\eta x^2/2}]$$
$$\times J_0(x\vartheta) f[is - i\alpha + i\alpha\varepsilon \exp(-{}^1/_2\eta x^2)]. \quad (56)$$

This formula can be obtained by other means. To this end it is necessary to replace (52) by

$$\varkappa(\mathbf{n}_1\mathbf{n}_0) \to [\pi\eta(2 + \eta)]^{-1} e^{-\vartheta^2/2\eta}, \quad (57)$$

Substitute (57) in (11), take $(\alpha + ip\mathbf{n} \cdot \mathbf{n}_0)^{-1}$ outside the integral sign at $\mathbf{n} = \mathbf{n}_0$, and solve the obtained equation for Γ. After substituting the solutions in (12) we obtain (56).

From (56) we can obtain a rapidly converging series in powers of $2\epsilon\alpha L$:

$$K(\mathbf{n}_1\mathbf{n}_0;\ is, i\alpha) = (2\pi\eta)^{-1} L \exp[-\alpha(\sec\vartheta - 1)L]$$
$$\times \sum_{mn} \frac{(2\alpha\varepsilon L)^n [2\alpha L(\sec\vartheta - 1)]^m}{(n + 1)!\, m!\, (n + m + 1)} \exp\left[-\frac{\vartheta^2}{2\eta(n + 1)}\right]. \quad (58)$$

On the other hand, if $\epsilon\alpha L > 1$, then we obtain from (56)

$$K(\mathbf{n}_1\mathbf{n}_0; is, i\alpha) = [4\pi\eta\varepsilon\alpha^2 L(2\varepsilon + \vartheta^2)]^{-1}$$
$$\times\{1 - (2\varepsilon\alpha L)^{-1}[1 - \theta^2 + {}^1/{}_2\theta^4$$
$$+ {}^1/{}_2(1 - \theta^2)\vartheta^2(2\varepsilon + \vartheta^2)^{-1}\}$$
$$\times \exp[\alpha L(2\varepsilon + \sec\vartheta - 1) - \theta^2],$$
$$\theta^2 \equiv (4\eta\varepsilon\alpha L)^{-1}\vartheta^2. \tag{59}$$

Substituting (59) in (1) we obtain for $\epsilon = 1$ and $\alpha L \to \infty$ a Gaussian angular distribution which coincides with the well known results[3].

5. PARTICLE DISTRIBUTION INSIDE THE LAYER

In many cases it is of interest to determine the number of particles at some arbitrary point inside the layer. To this end, it is necessary to find the mean square of the wave function $\langle|\psi(z)|^2\rangle$ at the point z. A similar problem was solved earlier for the case of a spherical body and an isotropic cross section[1]. The entire calculation procedure remains in force also in the case of a plane layer, with the exception of the fact that in place of (7) we must take (5). As a result we obtain

$$\langle|\Psi(z)|^2\rangle = e^{-\alpha z} + \pi^{-1}\alpha\varepsilon\int d\Omega_{\mathbf{n}}\int_{-\infty}^{+\infty} dp e^{ipz}(\alpha + ip\mathbf{n}\mathbf{n}_0)^{-1}$$
$$\times K(\mathbf{n}\mathbf{n}_0;\ p, i\alpha)\exp[-(\alpha + ip)L]. \tag{60}$$

CONCLUSION

The theory of multiple scattering contains as dimensionless parameters ϵ, αL, and βL, where ϵ determines the inelasticity of the scattering, αL the mean multiplicity of scattering with account of absorption, and βL the ratio of the thickness of the layer to the mean path covered prior to absorption. The solution of the transport equation contains terms which are due to poles of the integral kernel, to its branch points, as well as to crossing terms that depend on both. We evaluated the pole terms exactly and considered various approximations in ϵ, αL, and βL for terms that depend on the branch points.

In the case of Coulomb scattering, an additional parameter appears, connected with the sharp anisotropy of the transverse cross section. This leads to an increase in the role of the pole terms and makes it possible to obtain an expression which is more accurate than the well-known expression of Goudsmit and Saunderson.

It is seen from the resultant formulas that with increasing thickness of the layer the relative contribution of the particles scattered backwards increases, and the difference between the cases of anisotropic and isotropic scattering from an individual force center becomes smoother and smoother, especially as $\epsilon \to 1$. This can be readily seen from (27), since the diffusion (pole) terms, which make the main contribution when αL is large, decrease with increasing l and ϵ.

The approximations used to solve the transport equation do not lead to errors larger than several per cent. In many cases this accuracy is of the same order as the accuracy of the equation itself, in the derivation of which it was assumed that recoil can be neglected, that the Doppler shift is insignificant, that the scattered particles have a random distribution, etc.

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73