AN EXAMPLE OF STATIONARY UNBOUNDED CUMULATION

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An example of stationary unbounded cumulation of energy has been found. This occurs in the case of a converging conical electromagnetic shock wave. Just as in the case of a cylindrically converging field wave, it is found that the amplitude of the wave reflected from the axis remains infinite at a finite distance from the axis. The problems of stationary conic and unstationary cylindrical waves reduce to the same equations.

THERE are known examples of phenomena which are accompanied by cumulation of energy, that is, an unbounded increase in energy density per unit volume at a point (spherical cumulation) or on a line (cylindrical cumulation). In all these cases the motion is essentially nonstationary.

It might appear that a transition from a cylindrical converging wave into a conical wave should lead to stationary unbounded cumulation at the vertex of the cone, but this does not occur because of the following physical reason: as the wave approaches the axis its amplitude increases and consequently its velocity grows, too. As a result, a section of the wave normal to the axis is formed on the axis and moves with finite velocity, that is, carries a finite pressure. This impossibility of conical flows was proved analytically in[1].

By considering an acoustic wave with constant velocity, we can construct formally for a converging conical wave a solution that results in stationary unbounded cumulation, but this would be contradictory and physically meaningless, for then the particles would move near the axis faster than the wave passing through them and would cross the axis, something incompatible with the continuity assumed in the problem.

The foregoing difficulties do not arise if a conical field shock wave is considered, with a velocity that remains constant and equal to the velocity of light. In this case, unbounded cumulation is realized and is furthermore stationary.

As far as we know, this is the first example of a phenomenon of this type. Let us examine it in greater detail.

Assume a magnetic field H_0 parallel to the axis inside a cylindrical cavity in an ideal conductor. Assume further that a conical shock wave emerges from the conductor to the surface of the cavity, as a result of which the surface suddenly acquires a velocity in the direction of the field, and the place of emergence of the shock wave moves along the generatrix with superluminal velocity D. Then a conical converging electromagnetic shock wave of compression will arrive at the axis. The scheme of the phenomenon is shown in the figure.

The nonvanishing field components are H_x, H_r, and E_φ. Bearing in mind that they depend only on r and x + Dt, we obtain Maxwell's equations in the form

$$\frac{\partial}{\partial x}(H_r - ME_\varphi) = \frac{\partial H_x}{\partial r}, \quad \frac{\partial H_r}{\partial r} + \frac{H_r}{r} + \frac{\partial H_x}{\partial x} = 0,$$
$$\frac{\partial}{\partial x}(E_\varphi - MH_r) = 0, \quad \frac{\partial E_\varphi}{\partial r} + \frac{E_\varphi}{r} = M\frac{\partial H_x}{\partial x},$$

where $M = D/c = 1/\sin\theta$.

Eliminating E_φ, we obtain

$$(M^2-1)\frac{\partial^2 H_r}{\partial x^2} - \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(rH_r)}{\partial r}\right] = 0,$$
$$(M^2-1)\frac{\partial^2 H_x}{\partial x^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_x}{\partial r}\right) = 0.$$

H_r and H_x are connected by the equations

$$(M^2-1)\frac{\partial H_r}{\partial x} = \frac{\partial H_x}{\partial r}, \quad \frac{\partial H_r}{\partial r} + \frac{H_r}{r} = -\frac{\partial H_x}{\partial x}.$$

The electric field can be expressed in terms of the magnetic field

$$E_\varphi = MH_r + \text{const}/r.$$

These equations of the problem turn out to coincide fully with the equations that describe the nonstationary cylindrical field wave considered in [2], the only difference being that the roles of H, E, and t are now assumed by

$$H_x, \quad H_r\sqrt{M^2-1} \quad \text{and} \quad x/c\sqrt{M^2-1}.$$

Thus, the stationary problem for a conical wave has been reduced to a nonstationary problem for a cylindrical wave.

We are interested in a self-similar solution describing the vicinity of the vertex of the cone. A self-similar solution of the resultant equations was obtained and analyzed in [2]. We can use the solution and rewrite it in the notation of the new problem. As a result we get

$$H_r = H_{0r}\sqrt{R/r}h_r(\tau),$$
$$H_x = H_0 + H_{0x}\sqrt{R/r}h_x(\tau),$$
$$\tau = x/r\sqrt{M^2-1},$$

where H_{0r} and H_{0x} are the radial and axial components of the amplitude of the magnetic field on the radius R.

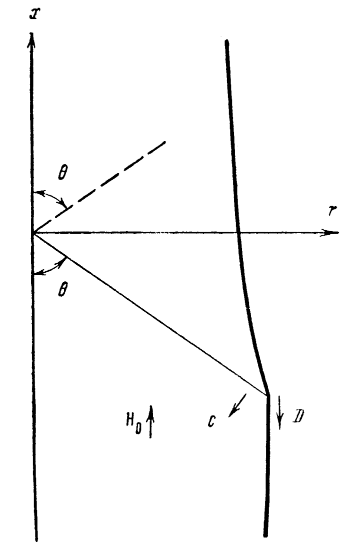
Thus, on approaching the axis the amplitude of the wave increases without limit like 1/r, and unbounded cumulation actually takes place at the vertex of the cone. H_{0r} and H_{0x} are connected by the relation

$$\sqrt{M^2-1}\frac{H_{0r}}{H_{0x}} = -1,$$

i.e., the amplitude of the magnetic field on the front of the wave is parallel to the generatrix of the cone.

Without writing out the cumbersome formulas for $h_r(\tau)$ and $h_x(\tau)$, which coincide with the expressions for $h(\tau)$ and $e(\tau)$ in [2], we mention only the remarkable property that they diverge as $\tau \to 1$, that is, on the front of the reflected wave.

This means that for a conical wave, as well as for a cylindrical one, the unbounded amplitude is conserved in the reflected wave also at a constant distance from the axis. We have previously considered this effect to be peculiar only to a cylindrical wave, for which this interesting property



was confirmed by Zel'dovich[3], who regarded a cylindrical acoustical wave as a superposition of plane waves.

The described solution pertains to the case of ideal symmetry and zero front width. Apparently even a slight violation of these conditions destroys the effect of unbounded cumulation.

[1] H. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience, N. Y., 1948.

[2] E. I. Zababakhin and M. N. Nechaev, JETP **33**, 442 (1957), Soviet Phys. JETP **6**, 345 (1958).

[3] Ya. B. Zel'dovich, JETP **33**, 700 (1957), Soviet Phys. JETP **6**, 537 (1958).

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