PARTICLES AND THE S-MATRIX IN QUANTUM FIELD THEORY

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It is shown that the matrix element $\langle 2 | A(x) | 2 \rangle$ of a neutral scalar field $A(x)$, taken between two-particle in-states, admits poles which are not one-particle poles. The residues of these poles are determined explicitly in terms of the scattering amplitude. Only the fundamental axioms of quantum field theory are employed in this derivation. Similar poles also occur in a series of other matrix elements. By making use of the existence of such poles a solution has been given for the following problem: for a neutral scalar field $A(x)$ defined in an arbitrary basis α by its matrix elements $\langle \alpha | A(x) | \alpha' \rangle$ a method has been found for obtaining: a) the mass spectrum of the particles, b) the ensemble of quantities $\langle \alpha | 0 \rangle$, $\langle \alpha | \kappa_1 \mathbf{k}_1 \rangle$, $\langle \alpha | \kappa_1 \kappa_2 \mathbf{k}_1 \mathbf{k}_2 \rangle, \ldots,$ representing the transformation matrix from the given basis to the in-basis, and c) the S-matrix. The method can be extended to particles with spin and also to local quantities of arbitrary physical character and arbitrary tensorial dimension. In doing this the vacuum state is not assumed to be cyclic with respect to the field.

1. The physical idea underlying the present work consists in making use in quantum field theory of the following relatively obvious fact: if a particle is the source of a field $A(x)$, then a condensation of that field will move along with the particle (for instance, the Coulomb field of a charged particle). In the language of quantum field theory this means that the one-particle matrix element $\langle 1 | A(x) | 1 \rangle$ does not vanish. At infinite times the field remains different from zero along the world line of the particle. But if a quantity becomes constant at infinite times, the corresponding Fourier transform must possess a pole, and the residue in the pole must contain information about the asymptotic direction of the world line, i.e. about the S-matrix.

In Sec. 2 such pole terms are obtained for the two-particle matrix element $\langle 2 | A(x) | 2 \rangle$. If each particle is a source of the field $A(x)$, it is obvious that if the field $A(x)$ is given everywhere in four-dimensional space, one can trace the behavior of all the particles, i.e. establish a complete picture of the scattering process. In quantum field theory this assertion corresponds to the fact that if $A(x)$ is determined by its matrix elements with respect to any basis one can derive from these the mass spectrum, the transformation matrix to the in-basis and the S-matrix. This problem is solved in Sec. 3. We emphasize the fact, that in distinction from the usual treatment, the particles are treated here not as field quanta, but as sources of the field. Thus, by means of the usual approach a given electromagnetic field and the asymptotic condition, yield information only about the photons. The method expounded in Sec. 3, on the other hand, gives the possibility to obtain from a given electromagnetic field information about the motion of all the charged particles (this, by the way, corresponds completely to the experimental situation. All elementary particle reactions are studied through the interaction of the static Coulomb field of the outcoming particles with matter!). Thus, the method allows to describe particles of different kinds by means of one single field.

Section 4 contains several concluding remarks. Only the fundamental axioms of quantum field theory are used throughout this work.

2. It has been noted previously[1,2] that the Heisenberg two-particle matrix element $\langle \mathbf{pk} | A(x) | \mathbf{p'k'} \rangle$ of a neutral scalar field $A(x)$ is related to the S-matrix not only on the mass shell (here $\mathbf{p}$, $\mathbf{p'}$ denote momenta of particles of mass M and $\mathbf{k}$, $\mathbf{k'}$ are momenta of particles of mass κ; a complete set of in-states (in-basis) is used). It is the purpose of the present paper to obtain this relation explicitly.

Our only starting assumptions are: a) the existence of an S-matrix description; b) the existence of nonvanishing one-particle matrix elements $\langle \mathbf{k} | A(x) | \mathbf{k'} \rangle$, $\langle \mathbf{p} | A(x) | \mathbf{p'} \rangle$. The mathematical conditions for the existence of an S-matrix description in quantum field theory are formulated in Sec. 3. The physical content of condition b) is that the particles should possess a static field $A(x)$.

The invariant parametrization of the one-particle matrix elements has the form[2]:

$$\langle \mathbf{p} | A(x) | \mathbf{p}' \rangle = \frac{e^{-ix(p-p')} f_1(t)}{(2\pi)^3 (4EE')^{1/2}},$$

$$\langle \mathbf{k} | A(x) | \mathbf{k}' \rangle = \frac{e^{-ix(k-k')} f_2(t')}{(2\pi)^3 (4\omega\omega')^{1/2}};$$

$$E = (\mathbf{p}^2 + M^2)^{1/2}, \quad E' = (\mathbf{p}'^2 + M^2)^{1/2},$$
$$\omega = (\mathbf{k}^2 + \varkappa^2)^{1/2}, \quad \omega' = (\mathbf{k}'^2 + \varkappa^2)^{1/2},$$
$$t = -(p-p')^2 \equiv (E-E')^2 - (\mathbf{p}-\mathbf{p}')^2,$$
$$t' = -(k-k')^2, \quad xp \equiv \mathbf{x}\mathbf{p} - x_0 E, \tag{1}$$

where $f_1(t)$ and $f_2(t')$ are invariant formfactors. For simplicity we assume $f_1(0) \neq 0$, $f_2(0) \neq 0$.

The result we are looking for is obtained on the basis of an analysis of the limiting behavior as time goes to infinity of the operator $D(\mathbf{V}, x_0)$:

$$D(\mathbf{V}, x_0) = (1 - \mathbf{V}^2)^{1/2} \int d^3x A(\mathbf{x}, x_0 + \mathbf{V}\mathbf{x}), \tag{2}$$

where $\mathbf{V}$ is a vector parameter satisfying the condition $\mathbf{V}^2 < 1$. One can verify directly that the one-particle matrix elements (1) of the operator $D(\mathbf{V}, x_0)$ are time-independent:

$$\langle \mathbf{p} | D(\mathbf{V}, x_0) | \mathbf{p}' \rangle = -\frac{\delta(\mathbf{p}-\mathbf{p}') f_1(0)}{2 n_\mu p_\mu},$$

$$\langle \mathbf{k} | D(\mathbf{V}, x_0) | \mathbf{k}' \rangle = -\frac{\delta(\mathbf{k}-\mathbf{k}') f_2(0)}{2 n_\mu k_\mu}, \tag{3}$$

where n_μ is a timelike unit 4-vector ($n_\mu^2 = -1$) with components $\mathbf{n} = \mathbf{V}(1 - \mathbf{V}^2)^{-1/2}$, $n_0 = n_4/i = (1 - \mathbf{V}^2)^{-1/2}$. The quantity $D(\mathbf{V}, x_0)$ is closely related to the new physical quantities introduced in refs. [1,2], the so-called dynamic moments. The dynamic moments are the coefficients in a Taylor expansion of $(1 - \mathbf{V}^2)^{-1/2} D(\mathbf{V}, x_0)$ in powers of $\mathbf{V}$.

The two-particle matrix element can be parametrized in the form[2]

$$\langle \mathbf{pk} | A(x) | \mathbf{p}'\mathbf{k}' \rangle = \delta(\mathbf{k}-\mathbf{k}') \frac{e^{-ix(p-p')} f_1(t)}{(2\pi)^3 (4EE')^{1/2}}$$
$$+ \delta(\mathbf{p}-\mathbf{p}') \frac{e^{-ix(k-k')} f_2(t')}{(2\pi)^3 (4\omega\omega')^{1/2}} + \frac{e^{-ixK} F(s, s', t, t', u, u')}{(2\pi)^3 (16EE'\omega\omega')^{1/2}}. \tag{4}$$

with

$$s = -(p+k)^2, \quad s' = -(p'+k')^2, \quad u = -(p'-k)^2,$$
$$u' = -(p-k')^2, \quad K = p + k - p' - k',$$

where F is an invariant formfactor depending on six variables. According to Eqs. (2) and (4) the two-particle matrix element of the operator $D(\mathbf{V}, x_0)$ equals

$$\langle \mathbf{pk} | D(\mathbf{V}, x_0) | \mathbf{p}'\mathbf{k}' \rangle$$
$$= -\delta(\mathbf{p}-\mathbf{p}')\delta(\mathbf{k}-\mathbf{k}') \left\{ \frac{f_1(0)}{2n_\mu p_\mu} + \frac{f_2(0)}{2n_\mu k_\mu} \right\}$$
$$+ \frac{\delta(\mathbf{K} - \mathbf{V}K_0) e^{ix_0 K_0} F(s, \ldots, u') (1 - \mathbf{V}^2)^{1/2}}{(16EE'\omega\omega')^{1/2}}. \tag{5}$$

For two free particles this matrix element reduces obviously to

$$\langle \mathbf{pk} | D^{free}(\mathbf{V}, x_0) | \mathbf{pk} \rangle$$
$$= -\delta(\mathbf{p}-\mathbf{p}')\delta(\mathbf{k}-\mathbf{k}') \left\{ \frac{f_1(0)}{2n_\mu p_\mu} + \frac{f_2(0)}{2n_\mu k_\mu} \right\}. \tag{6}$$

From the assumed existence of the S-matrix it follows that for infinite time the particles are infinitely separated and free. Consequently the matrix element (5) should converge to (6) as $x_0 \to -\infty$ in the in-basis, and as $x_0 \to +\infty$ in the out-basis. Therefore the operator relation

$$D(\mathbf{V}, +\infty) = S^{-1} D(\mathbf{V}, -\infty) S \tag{7}$$

should hold. The relation (7) is equivalent to the system of corresponding equations for the dynamic moments[1,2].

In order that (7) be satisfied, the formfactor, F in Eq. (4) must contain terms with a pole $(K_0 - i\epsilon)^{-1}$, since time occurs only in the exponent in Eq. (4). The necessary appearance of such terms is obvious from the well-known identity

$$\lim \frac{e^{ix_0 K_0}}{K_0 - i\varepsilon} = \begin{cases} 2\pi i \delta(K_0), & x_0 \to +\infty \\ 0, & x_0 \to -\infty \end{cases}. \tag{8}$$

The four independent combinations of the invariant variables in the formfactor F: $s - s'$, $t - t'$, $u - u'$, $(-K_\mu^2)^{1/2}$, have first order zeros as $K_\mu \to 0$. Therefore the most general form of the pole denominator having the required properties is

$$\alpha(s - s') + \beta(t - t') + \gamma(u - u') + \delta\sqrt{-K_\mu^2} - i\varepsilon,$$

where $\alpha, \beta, \gamma, \delta$ are numerical coefficients. Thus the number of pole terms becomes infinite and we can make no definite statement about the residues in each of these poles.

One can avoid this difficulty by replacing the invariant variables s, s′, t, t′, u, and u′ by the variables s, t, w, w_1, w_2, and w_3, with

$$w = \frac{K_0}{|K_0|} \sqrt{-K_\mu^2}, \quad w_1 = \frac{s - s'}{w},$$
$$w_2 = \frac{t - t'}{w}, \quad w_3 = \frac{u - u'}{w}. \tag{9}$$

The fact that the new variables are invariant only for timelike vectors K_μ is inessential, since in our computations only timelike K_μ will occur, and the final result will be rewritten in terms of the original fully invariant variables.

Among the variables s, t, w, w_1, w_2, and w_3 only w vanishes as $K_\mu \to 0$, however the limiting values of w_1, w_2, and w_3 depend on the unit 4-vector n_μ. Indeed, the presence of the delta-function in the last term in (5) implies

$$w = K_0(1 - \mathbf{V}^2)^{1/2}, \tag{10}$$

$$K_\mu = n_\mu w, \tag{11}$$

$$w_1 = -(p_\mu + k_\mu + p_\mu' + k_\mu')n_\mu. \tag{12}$$

It follows from (12) that w_1 depends on n_μ as $K_\mu \to 0$. The same follows for w_2, w_3. One can see from (11), (12) that the limiting value of the formfactor depends on the path along which the 4-vector K_μ vanishes.

In the new variables there is only one pole denominator with the required properties, so that the formfactor can be written in the form

$$F = \frac{\varphi(s, t, w_1, w_2, w_3)}{w - i\varepsilon} + \Phi(s, t, w, w_1, w_2, w_3), \tag{13}$$

where $w\Phi \to 0$ as $w \to 0$. Substituting (13) into (5) and making use of (8), we obtain in the limit as $x_0 \to +\infty$

$$\langle \mathbf{pk} | D(\mathbf{V}, +\infty) | \mathbf{p'k'} \rangle = -\delta(\mathbf{p}-\mathbf{p'})\delta(\mathbf{k}-\mathbf{k'})\left\{\frac{f_1(0)}{2n_\mu p_\mu} + \frac{f_2(0)}{2n_\mu k_\mu}\right\} + \frac{2\pi i\delta^4(K)\varphi(s, t, w_1, w_2, w_3)}{(16EE'\omega\omega')^{1/2}}. \tag{14}$$

On the other hand, according to (7), the matrix element $\langle \mathbf{pk} | D(\mathbf{V}, +\infty) | \mathbf{p'k'} \rangle$ equals

$$\langle \mathbf{pk} | D(\mathbf{V}, +\infty) | \mathbf{p'k'} \rangle = -\int d^3p'' d^3k'' \langle \mathbf{pk} | S^{-1} | \mathbf{p''k''} \rangle \times \left\{\frac{f_1(0)}{2n_\mu p_\mu''} + \frac{f_2(0)}{2n_\mu k_\mu''}\right\} \langle \mathbf{p''k''} | S | \mathbf{p'k'} \rangle. \tag{15}$$

For simplicity it has been assumed here that only the elastic channel is open. In the presence of inelastic channels corresponding additional terms would appear in the right hand side of Eq. (15).

The S-matrix is related to the invariant scattering amplitude g(s, t) by means of the usual relation

$$\langle \mathbf{pk} | S | \mathbf{p'k'} \rangle = \delta(\mathbf{p}-\mathbf{p'})\delta(\mathbf{k}-\mathbf{k'}) - 2\pi i\delta^4(K)\frac{g(s, t)}{(16EE'\omega\omega')^{1/2}}. \tag{16}$$

The final step of our calculation consists in substituting (15) and (16) into (14) which yields the final result in the form

$$\frac{\varphi(s, t, w_1, w_2, w_3)}{w - i\varepsilon} = g(s, t)\left\{\frac{f_1(0)}{2pK - i\varepsilon} + \frac{f_2(0)}{2kK - i\varepsilon}\right\} - g^*(s, t)\left\{\frac{f_1(0)}{2p'K - i\varepsilon} + \frac{f_2(0)}{2k'K - i\varepsilon}\right\} + 2\pi i\int \frac{d^3p''}{2E''}\frac{d^3k''}{2\omega''}\delta^4(p - k - p'' - k'') \times g^*(s, t'')g(s, t''')\left\{\frac{f_1(0)}{2p''K - i\varepsilon} + \frac{f_2(0)}{2k''K - i\varepsilon}\right\}, \tag{17}$$

where $t'' = -(p - p'')^2$, $t''' = -(p' - p''')^2$. It follows from the form of the right hand side of Eq. (18) that the pole term is invariant for spacelike as well as for timelike K_μ. We emphasize the facts that the new poles are not one-particle poles and that the S-matrix in (17) is related to the formfactor off the mass-shell.

It is interesting to compute the other singularities of the matrix element $\langle \mathbf{pk} | A(x) | \mathbf{p'k'} \rangle$, which are connected with the derivatives of the one-particle formfactors at the origin. This can be done by means of the same technique as used in the present paper, but using instead of (2) operators connected with dynamical moments of higher orders[2]. For instance, in order to study the singularities which are connected with the first order derivatives at the origin of the one-particle formfactors, one should use the operator $D^2(\mathbf{V}, x_0)$ defined by

$$D^2(\mathbf{V}, x_0) = \left\{1 - x_0\frac{\partial}{\partial x_0} + \frac{1}{2}x_0^2\left(\frac{\partial}{\partial x_0}\right)^2\right\} \times \int d^3x x_i^2 A(\mathbf{x}, x_0 + \mathbf{V}\mathbf{x})(1 - \mathbf{V}^2)^{1/2}. \tag{18}$$

3. Let the operator of some local quantity A(x) be given in an arbitrary basis α in terms of its matrix elements $\langle \alpha | A(x) | \alpha' \rangle$. How can one obtain from here a particle description? By particle description we understand here, obtaining: a) the mass spectrum and the spin spectrum of the particles, b) the transformation matrix from the given representation to the in-basis, and c) the scattering matrix. For simplicity we restrict ourselves to spinless particles, but admit the existence of several kinds of particles with different masses. Also for simplicity, we consider the local quantity A(x) a scalar neutral field.

It is a consequence of relativistic invariance that in the initial basis the transformations of the inhomogeneous Lorentz group are well defined, i.e., the operators P_μ of 4-momentum and $M_{\mu\nu}$ of angular 4-momentum are given and satisfy the standard commutation relations.

In addition to the general quantum and relativistic conditions we impose, as in the preceding section, a condition of completeness, consisting in the requirement that the one-particle matrix elements $\langle 1 | A(x) | 1 \rangle$ taken between states of the same particle should not vanish for all particles in the in-basis. Roughly speaking the sense of this condition is that one cannot detect a particle by measuring a field which that particle does not possess.

The solution of the problem is based on the use of the new physical quantities introduced in [1,2], the so called dynamic moments $D_{i_1 \ldots i_n}(x_0)$:

$$D_{i_1 \ldots i_n}(x_0) = \frac{1}{n!} \int d^3x x_{i_1} \ldots x_{i_n} A(x), \qquad (19)$$

where $i_1 \ldots i_n = 1, 2, 3$. The fundamental property of these dynamic moments is that at time infinity they go over into constants of the motion which are additive with respect to the particles. As has been shown in [1,2], if one projects the operator $D_{i_1 \ldots i_n}(x_0)$ onto states with a given fixed number of particles in the infinite past, then in the limit $x_0 \to -\infty$ it will have the form

$$D_{i_1 \ldots i_n}(-\infty) = \sum_N v_{i_1}^{(N)} \ldots v_{i_n}^{(N)} \frac{f_N(0)}{2\omega_N}. \qquad (20)$$

In Eq. (20) $v_i^{(N)}$ and ω_N are the operators of the velocity and energy of the N-th particle, respectively, $f_N(0)$ is a constant characterizing the distribution of the field $A(x)$ within the N-th particle (the one-particle formfactor at the origin). All operators $v_i^{(N)}$, ω_N commute with each other and also with the total energy and momentum.

In order to obtain a particle description it is natural to start from the vacuum state $\langle \alpha | 0 \rangle$, in which the energy vanishes, and, according to (20) all dynamic moments vanish

$$\langle \alpha | P_0 | \alpha' \rangle \langle \alpha' | 0 \rangle = 0, \qquad (21)$$

$$\langle \alpha | D_{i_1 \ldots i_n}(x_0) | \alpha' \rangle \langle \alpha' | 0 \rangle = 0. \qquad (22)$$

Since the energy is positive definite, condition (21) itself suffices to determine uniquely the vacuum state $\langle \alpha | 0 \rangle$.

The one particle states $\langle \alpha | 1 \rangle$ must, first of all, be orthogonal to the vacuum state

$$\langle 0 | \alpha \rangle \langle \alpha | 1 \rangle = 0. \qquad (23)$$

Further, Eq. (20) and the stability of the state $\langle \alpha | 1 \rangle$ imply that the one-particle states must satisfy the condition

$$\langle \alpha | D(x_0) D_{ij}(x_0) - D_i(x_0) D_j(x_0) | \alpha' \rangle \langle \alpha' | 1 \rangle = 0. \qquad (24)$$

Conditions (23) and (24) determine the one-particle states completely. One can separate particles of different masses κ by taking into consideration the eigenvalues and eigenfunctions of the operator P_μ^2 in the states $\langle \alpha | 1 \rangle$:

$$\langle \alpha | P_\mu^2 | \alpha' \rangle \langle \alpha' | \varkappa_1 \rangle = -\varkappa^2 \langle \alpha | \varkappa_1 \rangle. \qquad (25)$$

One-particle states belonging to the same mass are classified by the eigenvalues $\mathbf{k}$ of the total 3-momentum $\mathbf{P}$:

$$\langle \alpha | \mathbf{P} | \alpha' \rangle \langle \alpha' | \varkappa_1 \mathbf{k}_1 \rangle = \mathbf{k} \langle \alpha | \varkappa_1 \mathbf{k}_1 \rangle. \qquad (26)$$

The state vectors (26) may still contain a two-fold degeneracy due to the existence of particles and antiparticles with identical masses. However such states will always differ in the values of some conserved charges (if the contrary is true, as for instance for K_0 mesons, the mass degeneracy is lifted). Therefore the one-particle matrix element

$$\langle \varkappa_1 \mathbf{k}_1 | A(x) | \varkappa_1 \mathbf{k}_1' \rangle = \langle \varkappa_1 \mathbf{k}_1 | \alpha \rangle \langle \alpha | A(x) | \alpha' \rangle \langle \alpha' | \varkappa_1 \mathbf{k}_1' \rangle \qquad (27)$$

must vanish between states of a particle and its antiparticle. This permits us to complete the classification of one particle states.

The two-particle states are orthogonal to the one-particle states and to the vacuum

$$\langle 0 | \alpha \rangle \langle \alpha | 2 \rangle = 0, \qquad \langle 1 | \alpha \rangle \langle \alpha | 2 \rangle = 0. \qquad (28)$$

In addition, for two-particle states in the limit as $x_0 \to -\infty$ all three-dimensional tensors are coplanar, being situated in the plane defined by the vectors $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ used in Eq. (20). Therefore two particle amplitudes must, for instance, satisfy the condition

$$\langle \alpha | \varepsilon_{ijk} D_i(-\infty) D_{jl}(-\infty) D_{kmq}(-\infty) | \alpha' \rangle \langle \alpha' | 2 \rangle = 0. \qquad (29)$$

The conditions (28) and (29) determine the two particle states. Similar conditions can be written down also for states with a higher number of particles. Note that in distinction from Eqs. (22) and (24), the relation (29) and similar relations for a larger number of particles, are valid only for $x_0 \to -\infty$. Therefore in solving Eqs. (28) and (29) for $\langle \alpha | 2 \rangle$ not only two particle states, but also three- and in general many-particle states may result. The problem of separating such "false" states will be discussed at the end of this section.

After separating the false states, the particle kinematic variables of the state vector $\langle \alpha | 2 \rangle$ can be expressed in terms of the total momentum, energy and also in terms of the asymptotic values of the dynamic moments. Indeed, the operators P_0, $\mathbf{P}$, $D(-\infty)$, $D_i(-\infty)$, $D_{ij}(-\infty)$, projected onto the two-particle states have the form

$$P_0 = \omega_1 + \omega_2, \qquad \mathbf{P} = \mathbf{k}_1 + \mathbf{k}_2; \qquad (30)$$

$$D(-\infty)=\frac{f_1(0)}{2\omega_1}+\frac{f_2(0)}{2\omega_2'},$$

$$D_i(-\infty)=\frac{f_1(0)v_i^{(1)}}{2\omega_1}+\frac{f_1(0)v_i^{(2)}}{2\omega_2},$$

$$D_{ij}(-\infty)=\frac{f_1(0)v_i^{(1)}v_j^{(1)}}{2\omega_1}+\frac{f_2(0)v_i^{(2)}v_j^{(2)}}{2\omega_2}, \quad (31)$$

where $f_1(0)$ and $f_2(0)$ are constants (formfactors at the origin), and

$$\omega_1=(\mathbf{k}_1^2+\varkappa_1^2)^{1/2}, \quad \omega_2=(\mathbf{k}_2^2+\varkappa_2^2)^{1/2},$$

$$\mathbf{k}_1=\omega_1\mathbf{v}^{(1)}, \quad \mathbf{k}_2=\omega_2\mathbf{v}^{(2)}.$$

The equations (30) and (31) are more than sufficient for expressing the operators $\mathbf{k}_1$, $\mathbf{k}_2$, κ_1, and κ_2 and the constants $f_1(0)$ and $f_2(0)$ in terms of the dynamic moments. The respective computations are elementary since the operators P_0, $\mathbf{P}$, $D(-\infty)$, $D_i(-\infty)$, $D_{ij}(-\infty)$, $\mathbf{k}_1$, $\mathbf{k}_2$, κ_1, and κ_2 all commute with each other. Writing down for the state vector $\langle\alpha\,|\,2\rangle$ a system of equations analogous to (25) and (26), but for the eigenvalues and eigenvectors of the operators κ_1, κ_2, $\mathbf{k}_1$, $\mathbf{k}_2$, we obobtain the eigenstates $\langle\alpha\,|\,\kappa_1\kappa_2\mathbf{k}_1\mathbf{k}_2\rangle$ connecting the initial basis with the two-particle states in the in-basis. A possible mass degeneracy due to the existence of antiparticles is lifted by the same method as described above for one-particle states.

One can obtain the three-particle states $\langle\alpha\,|\,\kappa_1\kappa_2\kappa_3\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$ in a similar manner, and also for states with a higher number of particles. The totality of eigenstates $\langle\alpha\,|\,0\rangle$, $\langle\alpha\,|\,\kappa_1\mathbf{k}_1\rangle$, $\langle\alpha\,|\,\kappa_1\kappa_2\mathbf{k}_1\mathbf{k}_2\rangle$ represents just the matrix of the unitary transformation from the initial basis to the in-basis. Replacing in Eq. (29) and in other similar equations the limiting process $x_0 \to -\infty$ by $x_0 \to +\infty$, we obtain the transformation from the initial basis to the out-basis, and thus also the S-matrix.

In order to complete the solution of the problem which was formulated at the beginning of this section, we must still learn how to recognize the many-particle states (mentioned above, in the text following Eq. (29)) which might sometimes satisfy the conditions imposed on states with a smaller number of particles.

The method of separating the "false" states is the same for all state vectors with the exception of the vacuum state and the one-particle states, where it is simpler to make use of the stability condition, according to which Eqs. (22) and (24) hold for arbitrary values of x_0. In order not to complicate the treatment we carry through the elimination of "false" states for the one-particle vector $\langle\alpha\,|\,\kappa_1\mathbf{k}_1\rangle$ (which for simplicity will be denoted as $\langle\alpha\,|\,\kappa\mathbf{k}\rangle$) without using the stability condition. This means we will not assume that the vector satisfies Eq. (24), but only the weaker condition

$$\langle\alpha|D(-\infty)D_{ij}(-\infty)-D_i(-\infty)$$
$$\times D_j(-\infty)\,|\alpha'\rangle\langle\alpha'|\varkappa\mathbf{k}\rangle=0. \quad (32)$$

In general, Eq. (32) can be satisfied besides by one-particle states, also by: a) the two-particle state in which the 3-momenta of the two particles are parallel; b) the two-particle state in which the two particles are in a relative S-state of motion and c) the two-particle state in which the 4-momenta of the particles are parallel. In order to test the state $\langle\alpha\,|\,\kappa\mathbf{k}\rangle$ for "one-particleness", one has to construct the one-particle matrix element $\langle\kappa\mathbf{k}\,|\,A(x)\,|\,\kappa\mathbf{k}'\rangle$ in (27). An invariant parametrization has the form (1):

$$\langle\varkappa\mathbf{k}\,|A(x)\,|\varkappa\mathbf{k}'\rangle=\frac{e^{-ix(k-k')}f(t)}{(2\pi)^3(4\omega\omega')^{1/2}} \quad (33)$$

where $t=-(k-k')^2$; $f(t)$ is an invariant formfactor. The parametrization (33) is valid both for an individual particle and for an invariantly separated ensemble of states of the system of colliding particles. In case a) the relativistic parametrization (33) is simply not valid. In cases b) and c) relativistic invariance is not violated, but the formfactor $f(t)$ will have singularities of a definite type for $t = 0$. And only for the true one-particle states will the formfactor $f(t)$ be regular in the origin.

To illustrate this point we construct the matrix element (33) for "false" states of types b) and c), obtained from the two-particle state. Assuming for simplicity that both particles are interacting, the parametrization of the matrix element

$$\langle\varkappa_1\varkappa_2\mathbf{k}_1\mathbf{k}_2|A(x)\,|\varkappa_1\varkappa_2\mathbf{k}_1'\mathbf{k}_2'\rangle$$

has the form

$$\langle\varkappa_1\varkappa_2\mathbf{k}_1\mathbf{k}_2|A(x)\,|\varkappa_1\varkappa_2\mathbf{k}_1'\mathbf{k}_2'\rangle=\delta(\mathbf{k}_2-\mathbf{k}_2')\frac{e^{-ix(k_1-k_1')}f_1(t_1)}{(2\pi)^3(4\omega_1\omega')^{1/2}}$$
$$+\,\delta(\mathbf{k}_1-\mathbf{k}_1')\,\frac{e^{-ix(k_2-k_2')}f_2(t_2)}{(2\pi)^3(4\omega_2\omega_2')^{1/2}}. \quad (34)$$

In order to obtain the case b) one must multiply the matrix element (34) by

$$C\delta^4(k_1+k_2-k)\,\delta^4(k_1'+k_2'-k')\,(64\omega_1\omega_2\omega_1'\omega_2'\omega\omega')^{-1/2},$$

where C is a normalization factor, and also integrate over $d^3k_1d^3k_1'd^3k_2d^3k_2'$. As a result one indeed obtains the expression (33), but the "false one-particle formfactor" f(t) has the form

$$f(t)=\frac{\pi C\{f_1(t)\theta(t_{01}-t)+f_2(t)\theta(t_{02}-t)\}}{2\sqrt{-t}\,(4\varkappa^2-t)^{1/2}}, \qquad (35)$$

where $\theta(t)$ is the step-function, and

$$t_{01}=\varkappa_2^{-2}\{\varkappa^2-(\varkappa_1-\varkappa_2)^2\}\{\varkappa^2-(\varkappa_1-\varkappa_2)^2\},$$

$$t_{02}=\varkappa_2^2\varkappa_1^{-2}t_{01}.$$

The normalization constant C in (35) may depend on the masses but not on t. It is obvious from (35) that for a "false" state of type b) the formfactor f(t) in (33) has a singularity of the type $t^{-1/2}$ in the origin.

In the case b) one must multiply the matrix element (34) by

$$B\delta^4(k_1+k_2-k)\delta^4(k_1'-k_2'-k')\delta^4(k_1-\alpha k_2)\times\delta^4(k_1'-\alpha k_2')(64\omega_1\omega_2\omega_1'\omega_2'\omega\omega')^{-1/2},$$

where B is a normalization factor, and after that one has to integrate over $d\alpha d\beta d^3k_1d^3k_2d^3k_1'd^3k_2'$. As a result one obtains for the matrix element (33) the coordinate-independent expression

$$\langle\varkappa\mathbf{k}|A(x)|\varkappa\mathbf{k}'\rangle=\delta(\mathbf{k}-\mathbf{k}')(\varkappa_1+\varkappa_2)^{-2}\{\varkappa_1^2f_1(0)+\varkappa_2^2f_2(0)\}. \qquad (36)$$

The normalization factor is $B = 4(\kappa_1+\kappa_2)\kappa_1^2\kappa_2^2(2\pi)^3$. One can consider the expression (36) as a particular case of (33) with a formfactor f(t) having a delta-type singularity in the origin. One can obtain it from (35) by means of the limiting process $\kappa\to\kappa_1+\kappa_2$.

The physical significance of the coordinate-independence of (36) consists in the fact that if the relative 4-momentum of the two particles is fixed, then, in the center of inertia system, matter is not concentrated around any centers but is evenly distributed throughout space. Thus we have shown that "false" states can always be separated either using relativistic invariance, or by means of considering the singularities of one-particle formfactors in the origin. Thus the problem posed at the beginning of this article is completely solved.

4. We now discuss the result we have obtained. The described method of obtaining the particle description from the field description contains the conditions of existence of particles and of the S-matrix within the framework of quantum field theory. The particles and the S-matrix exist within quantum field theory only in the case when the system of equations determining the amplitudes $\langle\alpha|0\rangle$, $\langle\alpha|1\rangle$, ... has solutions with one-particle formfactors which are not singular in the origin.

The conditions (21)—(31) admit of various generalizations and modifications. First of all, they can be generalized to the case when particles with spin are present. The conditions (21)—(31) remain valid in this case, but are insufficient for a complete classification of states. In addition it becomes necessary to make use of the helicity operator[3] $\Gamma_\sigma = -\frac{1}{2}i\epsilon_{\sigma\mu\nu\lambda}M_{\mu\nu}P_\lambda$, and, for more than one particle, of the dynamic moments of the first order[2].

Instead of the scalar field A(x) one can use a neutral local operator of arbitrary tensorial dimension for obtaining a particle operator; such operators are, for instance, the electric current $j_\mu(x)$, the energy-momentum tensor $T_{\mu\nu}(x)$ etc. The method does not need any change. For spinor fields, and also for charged fields $\Psi(x)$, the one-particle matrix elements $\langle 1|\Psi(x)|1\rangle$ vanish, and hence also the dynamic moments. In this case one can use for constructing conditions of the type (22), (24), and (29) a quantity $C(\mathbf{x}, x_0)$ defined as

$$C(\mathbf{x},x_0)=\int d^3y\,d^3z\,F(\mathbf{x}-\mathbf{y},\ \mathbf{x}-\mathbf{z})\Psi(\mathbf{y}\ x_0)\Psi^+(\mathbf{z},x_0). \qquad (37)$$

where F is a sufficiently smooth and rapidly decreasing function of its arguments. The non-invariance and nonlocality of the quantity $C(\mathbf{x}, x_0)$ do not play any role. Only the method of separation of "false" states suffers slight modifications. Finally, if the formfactor vanishes at the origin for a particle, F(0) = 0, then by making use of higher dynamic moments[2] one may go over to the derivatives at the origin f′(0) of that formfactor.

Thus we reach the conclusion that the spectrum of masses and spins as well as the S-matrix can be obtained from any local quantity, defined in an arbitrary basis. This is particularly true of the energy-momentum tensor $T_{\mu\nu}(x)$, the exclusive role of which is determined by the fact that its existence is a direct consequence of purely geometric properties of space-time.[1] We also note that for $T_{\mu\nu}(x)$ the completeness condition is automatically fulfilled, since all particles possess inertial properties.

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[2] Yu. M. Shirokov, JETP **46**, 583 (1964), Soviet Phys. JETP **19**, 397 (1964).

[3] Yu. M. Shirokov, JETP **33**, 1196 (1957), Soviet Phys. JETP **6**, 919 (1958).

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