THERMAL CONDUCTIVITY OF THE INTERMEDIATE STATE OF SUPERCONDUCTORS. II

A. F. ANDREEV

Institute of Physics Problems, Academy of Sciences, U.S.S.R.

Submitted to JETP editor June 18, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) **47**, 2222-2228 (December, 1964)

We have calculated the coefficient of thermal conductivity of the lamellar and filamentary structure of the intermediate state, assuming heat transfer along the interfaces between the phases, for the case where the electronic mean free path is appreciably longer than the characteristic dimensions a of the normal regions. We assume an arbitrary dispersion law for the electrons. The coefficient of thermal conductivity of the lamellar structure is independent of the electronic mean free path l, while that of the filamentary structure decreases with increasing l as $\ln(l/a)/l$.

If a superconductor of finite dimensions is placed in an external magnetic field H and H is increased, there arrives a moment when the field at some places on the surface of the sample reaches the critical value while H is still less than H_c. Under such conditions the superconductor goes over into the intermediate state[1], i.e., its volume is split into a large number of alternating layers of the normal and superconducting phases. The dimensions of these layers can be expressed in terms of the intensity of the external field and the coefficient of the surface tension at the boundary between the phases (cf. [2]). Near the boundaries of the region of the intermediate state the thermodynamically most advantageous structure is a "filamentary" one.[3] The normal regions are then filaments with a cylindrical form arranged along the magnetic field.

A number of experimental investigations[4,5] established that the thermal conductivity (when heat is transferred across the layers) of the intermediate state is appreciably less than the thermal conductivity of the purely superconducting state, especially that of the normal phase. We showed in an earlier paper[6] that this is connected with the reflection of electronic thermal excitations from the interface boundaries. A specific peculiarity of this reflection is the unusual connection between the momenta and the velocities of the incident and the reflected excitations. In fact, the momentum **p** is practically unchanged during the reflection while the distance (in energy) from the Fermi surface $\xi(\mathbf{p}) = E(\mathbf{p}) - E_F$ and at the same time the velocity $\mathbf{v} = \partial\epsilon/\partial\mathbf{p}$ ($\epsilon = |\xi|$) changes sign. If the excitation incident from the normal phase is a "hole" ($\xi < 0$), the reflected one is an "electron" ($\xi > 0$) and the other way round. The probability for reflection when $|\xi| < \Delta$, where Δ is the energy gap in the superconducting phase, is equal to unity.

The aim of the present paper is to show that the unusual character of the reflection of excitations leads to a distinctive peculiarity of the heat conductivity of superconductors in the intermediate state when the heat transfer takes place along the interfaces between the phases.

1. HEAT CONDUCTIVITY OF A LAYER STRUCTURE

Let the temperature of the system be low compared to the critical temperature. The main contribution to the heat conductivity will then be given by the electrons in the normal regions. The thermal conductivity of the lattice and of the electrons in the superconducting regions can be neglected. We choose a system of coordinates such that the z-axis is normal to the interface boundary and that the z = 0 plane lies in the middle of one of the normal layers. We denote by a_n the thickness of that layer.

We shall start from the kinetic equation for the distribution function $n(\mathbf{r}, \mathbf{p})$ of the excitations ("electrons" and "holes"):

$$\mathbf{v}\,\partial n / \partial \mathbf{r} = -(n - \bar{n}) / \tau, \quad (1)$$

where τ is the electron relaxation time caused by the scattering by impurities, lattice point defects, etc., while the bar indicates averaging over an equal-energy surface. Instead of n we introduce a new function to be found, χ:

$$n = n_0[T(x_i)] + \chi\,\partial n_0 / \partial T, \quad (2)$$

where $n_0(T) = \{e^{\epsilon/T} + 1\}^{-1}$ is the equilibrium distribution function, and $T(x_i)$ a linear function of x_i; the indices i and k take on the values x and y.[1)]

We assume that the mean free path of the electrons, l, is appreciably larger than the thickness, a_n, of the normal layers, but considerably smaller than the Larmor radius of the electrons in a magnetic field equal to H_c. The last condition enables us to neglect in the kinetic equation the magnetic field present in the normal layers.

Substituting (2) into Eq. (1) we get

$$v_z\partial\chi / \partial z + \chi / \tau + v_i \partial T / \partial x_i = 0. \quad (3)$$

We dropped in Eq. (3) a term containing $\bar{\chi}$; this is legitimate, since in the case considered by us the region of velocities v where $\chi \gg \bar{\chi}$ is the important one, as will be shown in the following. Let us put

$$\chi = -\tau v_i \partial T / \partial x_i + \nu. \quad (4)$$

From (3) we can then write

$$\partial\nu / \partial z + \nu / v_z\tau = 0. \quad (5)$$

The general solution of Eq. (5) has the form

$$\nu = C(\mathbf{n}, \xi) \exp\left(-\frac{z}{v_z\tau}\right), \qquad \mathbf{n} = \frac{\partial E}{\partial \mathbf{p}} \Big/ \left|\frac{\partial E}{\partial \mathbf{p}}\right|. \quad (6)$$

The arbitrary function $C(\mathbf{n}, \xi)$ must be determined from the boundary conditions at $z = \pm a_n/2$ which depend on the character of the reflection of the excitations from the boundaries. Since the temperature $T \ll \Delta \approx T_c$ the main role is played by excitations for which $\epsilon \ll \Delta$. As was shown above, in that case total reflection of the excitations occurs; the vector $\mathbf{n}$ remains then unchanged and ξ changes to $-\xi$. The boundary conditions have thus the following form:

$$n(\mathbf{n}, \xi) = n(\mathbf{n}, -\xi) \text{ when } z = \pm a_n / 2 \quad (7)$$

or, using (2) and (3)

$$\nu(n, \xi) - \nu(\mathbf{n}, -\xi) = 2\tau v_i \partial T / \partial x_i \text{ when } z = \pm a_n / 2. \quad (8)$$

Substituting (6) into the conditions (8) we get

$$C(\mathbf{n}, \xi) \exp(-a_n / 2v_z\tau) - C(\mathbf{n}, -\xi) \exp(a_n / 2v_z\tau) = 2\tau v_i \partial T / \partial x_i,$$

$$C(\mathbf{n}, \xi) \exp(a_n / 2v_z\tau) - C(\mathbf{n}, -\xi) \exp(-a_n / 2v_z\tau) = 2\tau v_i \partial T / \partial x_i, \quad (9)$$

[1)]We note that the quantity $T(x_i)$ is not a local temperature, as the second term in (2) introduces, as can be seen from (11), a non-vanishing contribution to the energy density. However, since χ is independent of x_i, $\partial T/\partial x_i$ is the same as the derivative of the local temperature with respect to x_i.

whence we find

$$C(\mathbf{n}, \xi) = \tau v_i \frac{\partial T}{\partial x_i}\left[\operatorname{ch}\left(\frac{a_n}{2v_z\tau}\right)\right]^{-1}. \quad (10)^*$$

Thus,

$$\chi(\mathbf{n}, \xi) = -\tau v_i \frac{\partial T}{\partial x_i}\left[1 - \frac{\exp(-z/v_z\tau)}{\operatorname{ch}(a_n/2v_z\tau)}\right]. \quad (11)$$

One sees easily that in a state described by the distribution function (2) the electrical current density $\mathbf{j}$ vanishes. Indeed,

$$\mathbf{j} = 2e\int v\mathbf{n}\frac{\partial n_0}{\partial T}\chi\frac{d^3\mathbf{p}}{(2\pi\hbar)^3} = \frac{2e}{(2\pi\hbar)^3}\int_{-\infty}^{+\infty}\frac{\partial n_0}{\partial T}d\xi\int dS\mathbf{n}\chi$$

$$= \frac{2e}{(2\pi\hbar)^3}\left[\int_0^{\infty}\frac{\partial n_0}{\partial T}d\xi\int dS\mathbf{n}\chi + \int_{-\infty}^{0}\frac{\partial n_0}{\partial T}d\xi\int dS\mathbf{n}\chi\right]$$

$$= \frac{2e}{(2\pi\hbar)^3}\int_0^{\infty}\frac{\partial n_0}{\partial T}d\xi\int dS\mathbf{n}[\chi(\mathbf{n}, \xi) + \chi(\mathbf{n}, -\xi)], \quad (12)$$

where e is the electronic charge and dS an element of the Fermi surface. From (11) we have

$$\chi(\mathbf{n}, \xi) + \chi(\mathbf{n}, -\xi) = -2\tau v_i \frac{\partial T}{\partial x_i}\frac{\operatorname{sh}(z/v_z\tau)}{\operatorname{ch}(a_n/2v_z\tau)}. \quad (13)^{**}$$

As the Fermi surface has a center of symmetry, the last integral in (12) vanishes, since the integrand is multiplied by -1 when we change from $\mathbf{n}$ to $-\mathbf{n}$.

Let us evaluate the energy current density

$$q_i = 2\int \varepsilon v_i n \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}. \quad (14)$$

Substituting here (2), we get

$$q_i = \frac{2}{(2\pi\hbar)^3}\int_{-\infty}^{+\infty}\xi\frac{\partial n_0}{\partial T}d\xi\int dSn_i\chi$$

$$= \frac{2}{(2\pi\hbar)^3}\int_0^{\infty}\xi\frac{\partial n_0}{\partial T}d\xi\int dSn_i[\chi(\mathbf{n}, \xi) - \chi(\mathbf{n}, -\xi)]. \quad (15)$$

Using (11) we find

$$\chi(\mathbf{n}, \xi) - \chi(\mathbf{n}, -\xi) = -2\tau v_k \frac{\partial T}{\partial x_k}\left[1 - \frac{\operatorname{ch}(z/v_z\tau)}{\operatorname{ch}(a_n/2v_z\tau)}\right]. \quad (16)$$

Noting that

$$\int_0^{\infty}\xi\frac{\partial n_0}{\partial T}d\xi = \frac{\partial}{\partial T}\int_0^{\infty}\frac{\xi d\xi}{e^{\xi/T} + 1} = \frac{\pi^2 T}{6}, \quad (17)$$

we have

$$q_i = -\frac{T}{12\pi\hbar^3}\frac{\partial T}{\partial x_k}\int dSn_in_kl\left[1 - \frac{\operatorname{ch}(z/n_zl)}{\operatorname{ch}(a_n/2n_zl)}\right], \quad (18)$$

*ch = cosh.

**sh = sinh.

where $l = v\tau$ is the electron mean free path.

We shall be interested in the heat current density averaged over the volume of the superconductor, $\bar{q}_i$, which is equal to

$$\bar{q}_i = \frac{1}{a}\int_{-a_n/2}^{a_n/2} q_i(z)\,dz, \tag{19}$$

where $a = a_s + a_n$ is the period of the structure of the intermediate state with a_s the thickness of the superconducting layers.

Substituting (18) into (19), we get

$$\bar{q}_i = -\varkappa_{ik}\partial T / \partial x_k, \tag{20}$$

where

$$\varkappa_{ik} = \frac{T}{12\pi\hbar^3}\frac{a_n}{a}\int dS n_i n_k l\left[1 - \frac{2n_z l}{a_n}\operatorname{th}\frac{a_n}{2n_z l}\right]. \tag{21*}$$

In the case considered by us ($l \gg a_n$) we must perform in Eq. (21) the limiting process $l \to \infty$. Let us consider the expression

$$\lim_{l\to\infty} l\left[1 - \frac{2n_z l}{a_n}\operatorname{th}\frac{a_n}{2n_z l}\right].$$

If $n_z \neq 0$, the limit written down here vanishes. If, however, $n_z = 0$, it is infinite. This enables us to write

$$\lim_{l\to\infty} l\left[1 - \frac{2n_z l}{a_n}\operatorname{th}\frac{a_n}{2n_z l}\right] = a_n A\delta(n_z). \tag{22}$$

To determine the constant A we integrate both sides of Eq. (22) over n_z. We get

$$A = \int_0^\infty \frac{dt}{t^3}(t - \operatorname{th} t) = 7\zeta(3)/\pi^2, \tag{23}$$

where $\zeta(x)$ is Riemann's zeta-function.

Let us now substitute (22) into (21):

$$\varkappa_{ik} = \frac{7\zeta(3)}{12(\pi\hbar)^3} T\frac{a_n^2}{a}\int\frac{d\cos\theta\, d\varphi}{K(\theta,\varphi)} n_i n_k \delta(\cos\theta), \tag{24}$$

where $K(\theta, \varphi)$ is the Gaussian curvature of the Fermi surface; θ and φ are angles determining the relative position of **n** and the z-axis. Integrating over $\cos\theta$ we get finally

$$\varkappa_{ik} = \frac{7\zeta(3)}{6\pi^2\hbar^3} T\frac{a_n^2}{a} B_{ik}, \tag{25}$$

$$B_{ik} = \int_0^{2\pi}\frac{d\varphi}{2\pi}\frac{n_i n_k}{K(\varphi)}, \qquad K(\varphi) \equiv K\left(\frac{\pi}{2}, \varphi\right). \tag{26}$$

We note that the coefficient of thermal conductivity in the conditions considered is independent of the mean free path. Since $K(\varphi) \sim p_F^{-2}$, $\kappa \sim p_F^2 T a_n^2/\hbar^3 a$, which is by a factor $l a/a_n^2 \gg 1$ less than the thermal conductivity of the pure, normal phase.

It is clear from Eq. (24) that the main contribution to the thermal conductivity is given by excitations moving parallel to the interface boundaries between the phases. It is therefore not accidental that the thermal conductivity coefficient κ_{ik} is determined by the same integral, B_{ik}, over the Fermi surface as the surface impedance of a metal under the conditions of the anomalous skin-effect (see [7]).

*th = tanh.

2. THERMAL CONDUCTIVITY OF A FILAMENTARY STRUCTURE

Let us consider the case of sufficiently low temperatures when the main contribution to the thermal conductivity is given by the electrons of the normal regions. We shall also assume that the electronic mean free path is much longer than the diameter of the normal filaments.

We choose a cylindrical system of coordinates (r, θ, z) where the z-axis is along the axis of the filament while r is the distance from the axis. We shall look for the distribution function of the excitations $n(\mathbf{n}, \xi, \mathbf{r})$ in the form

$$n = n_0[T(z)] + \chi\partial n_0 / \partial T. \tag{27}$$

Substituting (27) into Eq. (1), we get

$$v_x\frac{\partial\chi}{\partial x} + v_y\frac{\partial\chi}{\partial y} + \frac{\chi}{\tau} = -v_z\frac{\partial T}{\partial z}. \tag{28}$$

The general solution of this equation has the form (see [8])

$$\chi = -v_z\tau\frac{\partial T}{\partial z}\left\{1 - f(rv_\theta, v_r^2 + v_\theta^2)\exp\left(-\frac{1}{\tau}\frac{rv_r}{v_r^2+v_\theta^2}\right)\right\}, \tag{29}$$

where $f(\alpha, \beta)$ is an arbitrary function determined by the boundary conditions at $r = a$ (a: radius of the filament), according to which

$$\chi(\mathbf{n}, \xi) = \chi(\mathbf{n}, -\xi) \text{ for } r = a. \tag{30}$$

Using (29) and (30) and the fact that it turns out that f is an even function of its first argument, we get

$$f(av_\theta, v_r^2 + v_\theta^2) = \left\{c\left[\frac{1}{\tau}\frac{av_r}{v_r^2+v_\theta^2}\right]\right\}^{-1}. \tag{31}$$

This last equation can be satisfied for all values of v_r and v_θ only provided

$$f(rv_\theta, v_r^2 + v_\theta^2) = \left\{\operatorname{ch}\left(\frac{1}{\tau}\frac{[(v_r^2+v_\theta^2)a^2 - r^2v_\theta^2]^{1/2}}{v_r^2+v_\theta^2}\right)\right\}^{-1}. \tag{32}$$

Using (29) and (32) we find

$$\chi = -v_z\tau\frac{\partial T}{\partial z}\left\{1-\exp\left(-\frac{1}{\tau}\frac{rv_r}{v_r^2+v_\theta^2}\right)\right.$$
$$\left.\times\left[\operatorname{ch}\left(\frac{1}{\tau}\frac{[(v_r^2+v_\theta^2)a^2-r^2v_\theta^2]^{1/2}}{v_r^2+v_\theta^2}\right)\right]^{-1}\right\}. \quad (33)$$

As in the preceding section, the thermal current density can be evaluated from the formula

$$q = \frac{2}{(2\pi\hbar)^3}\frac{\pi^2 T}{6}\int dSn_z[\chi(\mathbf{n},\xi)-\chi(\mathbf{n},-\xi)], \quad (34)$$

where $\xi > 0$. For the difference $\chi(-\xi)-\chi(\xi)$ we have from (33)

$$\chi(\mathbf{n},\xi)-\chi(\mathbf{n},-\xi) = -2ln_z\frac{\partial T}{\partial z}$$
$$\times\left\{1-\operatorname{ch}\left(\frac{1}{l}\frac{rn_r}{n_r^2+n_\theta^2}\right)\right.$$
$$\left.\times\left[\operatorname{ch}\left(\frac{1}{l}\frac{[(n_r^2+n_\theta^2)a^2-r^2n_\theta^2]^{1/2}}{n_r^2+n_\theta^2}\right)\right]^{-1}\right\}. \quad (35)$$

Substituting (35) into (34) and denoting the angles determining the position of the vector **n** relative to the z-axis by ψ and φ, we get after a few simple transformations:

$$q = -\frac{T}{12\pi\hbar^3}\frac{\partial T}{\partial z}\int_0^{2\pi}d\varphi\int_0^{\pi}\frac{\cos^2\psi\sin\psi\,d\psi}{K(\psi,\varphi)}l$$
$$\times\left\{1-\operatorname{ch}\left(\frac{1}{l}\frac{r\cos(\varphi-\theta)}{\sin\psi}\right)\right.$$
$$\left.\times\left[\operatorname{ch}\left(\frac{1}{l}\frac{[a^2-r^2\sin^2(\varphi-\theta)]^{1/2}}{\sin\psi}\right)\right]^{-1}\right\}. \quad (36)$$

To calculate the total heat current through the filament we must integrate Eq. (36) over the cross-section of the filament. Introducing instead of r, θ the variables $x = r\cos(\varphi-\theta)$, $y = r\sin(\varphi-\theta)$, we get

$$Q = \int_0^a r\,dr\int_0^{2\pi}q\,d\theta = -\frac{T}{12\pi\hbar^3}\frac{\partial T}{\partial z}\int_0^{2\pi}d\varphi\int_0^{\pi}\frac{\sin\psi\cos^2\psi\,d\psi}{K(\psi,\varphi)}$$
$$\times\int_{-a}^{a}dy\int_{-(a^2-y^2)^{1/2}}^{(a^2-y^2)^{1/2}}dx\,l\left\{1-\operatorname{ch}\left(\frac{x}{l\sin\psi}\right)\Big/\operatorname{ch}\left(\frac{(a^2-y^2)^{1/2}}{l\sin\psi}\right)\right\} \quad (37)$$

or, integrating over x, we have

$$Q = -\frac{T}{3\pi\hbar^3}\frac{\partial T}{\partial z}\int_0^a(a^2-y^2)^{1/2}dy\int_0^{2\pi}d\varphi$$
$$\times\int_0^1\frac{t(1-t^2)^{1/2}dt}{\widetilde{K}(t,\varphi)}l\left[1-\frac{lt}{\alpha}\operatorname{th}\frac{\alpha}{lt}\right], \quad (38)$$

where $t = \sin\psi$, $\alpha = (a^2-y^2)^{1/2}$

$$\frac{1}{\widetilde{K}(t,\varphi)} = \frac{1}{K(\arcsin t,\varphi)}+\frac{1}{K(\pi-\arcsin t,\varphi)}. \quad (39)$$

The region $\alpha/l \ll t \ll 1$ gives the main contribution in the integral over t in (38). In this region we can expand $\tanh(\alpha/lt)$ in a power series and put $\widetilde{K}(t,\varphi) = \widetilde{K}(0) \equiv \widetilde{K}(0,\varphi)$, $l = l_0 \equiv l(0)$. We get

$$\int_0^1\frac{t(1-t^2)^{1/2}dt}{\widetilde{K}(t,\varphi)}l\left[1-\frac{lt}{\alpha}\operatorname{th}\frac{\alpha}{lt}\right] = \frac{\alpha^2}{3l_0\widetilde{K}(0)}\int_{t_1}^{t_2}\frac{dt}{t}, \quad (40)$$

where $t_1 \sim \alpha/l$, $t_2 \sim 1$. Finally, performing in (40) and (38) elementary integrations we get (up to terms of logarithmic order of magnitude)

$$Q = -\frac{\pi T}{24\hbar^3}\frac{a^4\ln(l/a)}{\widetilde{K}(0)l_0}\frac{\partial T}{\partial z}. \quad (41)$$

If there are on the Fermi surface more than two points in which $|n_z| = 1$, we must have instead of $1/\widetilde{K}(0)l_0$ the sum $\Sigma(1/Kl)$ taken over those points.

We get easily for the effective thermal conductivity for a filamentary structure the following expression from (41)

$$\varkappa^{\text{eff}} = \eta\frac{T}{24\hbar^3}\frac{a^2\ln(l/a)}{\widetilde{K}(0)l_0}, \quad (42)$$

where η is the concentration of the normal phase. We note that the coefficient of thermal conductivity decreases with increasing electronic mean free path like $\ln(l/\alpha)/l$ while the main contribution to the thermal conductivity comes from excitations moving at small angles to the axis of the filament.

I express my gratitude to A. A. Abrikosov and I. M. Khalatnikov for valuable discussions of this paper.

[1] L. D. Landau, JETP **7**, 371 (1937), Phys. Zs. Sowjetun. **11**, 129 (1937), Collected Papers, Pergamon and Gordon and Breach, 1965, p. 217.

[2] E. M. Lifshitz and Yu. V. Sharvin, DAN SSSR **79**, 783 (1951).

[3] E. R. Andrew, Proc. Roy. Soc. (London) **A194**, 98 (1948).

[4] K. Mendelssohn and J. L. Olsen, Proc. Phys. Soc. (London) **A63**, 2 (1950); Phys. Rev. **80**, 859 (1950).

[5] N. V. Zavaritskii, JETP **38**, 1673 (1960), Soviet Phys. JETP **11**, 1207 (1960).

[6] A. F. Andreev, JETP **46**, 1823 (1964), Soviet Phys. JETP **19**, 1228 (1964).

[7] M. I. Kaganov and M. Ya. Azbel', DAN SSSR **102**, 49 (1955).

[8] R. B. Dingle, Proc. Roy. Soc. (London) **A201**, 545 (1950).

Translated by D. ter Haar
320