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THEORY OF THE HYDROMAGNETIC DYNAMO

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The generation equations are generalized to the nonstationary case and are solved for a simple model of a hydromagnetic dynamo in the form of an infinite plane liquid layer. A hypothesis is advanced on the possibility of self-stabilization of a tangential discontinuity in a highly conducting liquid as a consequence of the dynamo-effect. The generation equations are used to investigate the hydromagnetic dynamo of the earth.

1. INTRODUCTION

THIS paper is devoted to the "kinematic" theory of the hydromagnetic dynamo, which considers the generation of a magnetic field as a result of a given motion of a conducting liquid. This problem usually arises in the investigation of magnetic fields in astrophysics and in geophysics, but it also has a more general significance in magnetohydrodynamics. The kinematic theory of the dynamo is related, essentially, to the investigation of the properties of the induction equation, i.e., of that particular new equation which in magnetohydrodynamics is added to the usual system of hydrodynamic equations.

In the text below the following notation is used: z, s, φ are cylindrical coordinates (s is the distance from the z axis); r, ϑ, φ are spherical polar coordinates; $\mathbf{1}_i$ is the basis vector along the i-th coordinate axis; **v** is the velocity, **B** is the magnetic field, **A** and **a** are the vector potentials of the field and of the velocity $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$, $\mathbf{v} = \mathrm{curl}\,\mathbf{a}$; $D_m = c^2/4\pi\sigma$, where σ is the electrical conductivity; $R_m = Lv/D_m$ is the magnetic Reynolds member, L and v are the characteristic values of the dimensions and of the velocity of the system; the brackets $\langle\ldots\rangle$ denote averaging over φ, which picks out from any quantity its axially-symmetric part; primes denote quantities periodic in φ, which as a result of such averaging yield zero, for example, $\langle v'\rangle = 0$; $\partial_1/\partial\varphi$ denotes differentiation with respect to φ, in the course of which the unit vectors $\mathbf{1}_s$, $\mathbf{1}_\varphi$ are regarded as constants; the caret over a primed quantity denotes the periodic part of the integral with respect to φ of this quantity, for example $\partial_1\hat{\mathbf{v}}'/\partial\varphi = \mathbf{v}$, $\langle\hat{\mathbf{v}}'\rangle = 0$; the index p on a vector denotes its component in the meridian plane, for example, $\mathbf{v}_p = \mathbf{v} - \mathbf{1}_\varphi v_\varphi$.

The initial equation is the induction equation in the form

$$\partial\mathbf{A}/\partial t + \nabla\Phi - [\mathbf{vB}] = -D_m\,\mathrm{rot}\,\mathbf{B}, \qquad (1.1)^*$$

where $c^{-1}\Phi$ is the scalar potential, or in the form

$$\partial\mathbf{B}/\partial t - \mathrm{rot}\,[\mathbf{vB}] = -\mathrm{rot}\,(D_m\,\mathrm{rot}\,\mathbf{B}), \qquad (1.2)$$

$$\mathrm{div}\,\mathbf{B} = 0. \qquad (1.3)$$

The liquid is assumed to be incompressible and homogeneous

$$\mathrm{div}\,\mathbf{v} = 0. \qquad (1.4)$$

As in [1], we shall assume that the electric conductivity of the liquid is very large $R_m \gg 1$, while the motion differs little from an axially-symmetric one. In [1] it was assumed that the non-symmetric part of the velocity $\mathbf{v}'(z, s, \varphi, t)$ depends slowly on the time $\partial/\partial t \sim D_m L^{-2} \sim R_m^{-1} v L^{-1}$, and equations, called the generation equations, were obtained for the determination of the symmetric components of the magnetic field. If the velocity $\mathbf{v}' = \mathbf{v}'(z, s, \varphi - \varphi_0(t), t)$ is represented by a single wave traveling in the φ direction, then a rapid dependence on the time which appears only through $\varphi_0(t)$ can be eliminated by going over to a system of coordinates rotating with angular velocity $\dot{\varphi}_0$. However, if the nonsymmetric part of the velocity has the form of several waves travelling in the φ direction with different velocities, then it is impossible to find such a system of coordinates in which $\mathbf{v}'$ would be quasistationary. In the present paper the generation equations are generalized to this more complicated case. Moreover, a simple example is considered of motion in a plane liquid layer, when

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*rot = curl, $[\mathbf{vB}] = \mathbf{v}\times\mathbf{B}$.

the equations of generation can be easily integrated, and some applications of the theory are pointed out.

2. OSCILLATIONS AND WAVES

We assume that the nonsymmetric part of the velocity can be represented in the form of a sum of waves travelling in the φ direction

$$\mathbf{v}' = \sum_\alpha \mathbf{v}_\alpha', \quad \mathbf{v}_\alpha' = \sum_m \mathbf{v}_{\alpha m} e^{im(\varphi - \varphi_\alpha)}, \tag{2.1}$$

where $\mathbf{v}_{\alpha,-m} = \mathbf{v}_{\alpha m}^*$. In particular, (2.1) can contain quasistationary motions (if $\varphi_0 = 0$, then $\mathbf{v}_0'$ is quasistationary) and standing waves (if $\varphi_1 = -\varphi_2$, $\mathbf{v}_{1m} = \mathbf{v}_{2m}$, then $\mathbf{v}_1' + \mathbf{v}_2'$ gives a standing wave). We also assume that in addition to a slowly varying part the axially-symmetric velocity also contains a rapidly oscillating additional component to which we shall simply refer as oscillations. We denote it by $\tilde{\mathbf{v}}$ and write it in the form

$$\tilde{\mathbf{v}} = \sum_\mu \mathbf{v}_\mu e^{-i\varphi_\mu}, \quad \varphi_{-\mu} = -\varphi_\mu, \quad \mathbf{v}_{-\mu} = \mathbf{v}_\mu^*. \tag{2.2}$$

We assume that the amplitudes $\mathbf{v}_{\alpha m}$, $\mathbf{v}_\mu$ and the frequencies $\dot{\varphi}_\alpha$, $\dot{\varphi}_\mu$ have a slow time variation, and that for them $\partial/\partial t \sim D_m/L^2$, so that the phases of the waves and of the oscillations φ_α, φ_μ are almost linear functions of the time.

We shall in future refer to the argument t appearing in the phases of the waves and of the oscillations as the "fast" time, we shall denote it by t′ (of course, t′ = t), and we shall discuss the dependence on it separately. Correspondingly, we shall in future write the derivative with respect to the time in the form $\partial/\partial t + \partial/\partial t'$, where the derivative $\partial/\partial t$ is taken at constant values of the phases of the waves and of the oscillations, while $\partial/\partial t'$ operates on the "fast" time appearing in these phases. We introduce averaging over the "fast" time (over the phases), and we denote it by $\langle \ldots \rangle^t$. We denote by a tilde quantities which vanish as a result of this averaging. A caret over a quantity with a tilde will denote the variable part of the integral over "fast" time, for example, $\partial\hat{\tilde{\mathbf{v}}}/\partial t' = \tilde{\mathbf{v}}$, $\langle \hat{\tilde{\mathbf{v}}} \rangle^t = 0$. We denote the double average over φ and t′ by $\langle\langle \ldots \rangle\rangle$.

We represent the vector for the total velocity appearing in (1.1) and (1.2) in the form

$$\mathbf{v}_{\text{tot}} = \mathbf{1}_\varphi v + \mathbf{v}_p + \tilde{\mathbf{v}} + \mathbf{v}', \tag{2.3}$$

where for brevity we write $v = \langle\langle v_\varphi \rangle\rangle$, $\mathbf{v}_p = \langle\langle \mathbf{v}_p \rangle\rangle$. We assume that in order of magnitude the quantities $\tilde{v} \sim v' \approx vR_m^{-1/2}$, $v_p \sim vR_m^{-1}$.

For the magnetic field we seek the "steady state" solution in the analogous form:

$$\mathbf{B}_{\text{tot}} = \mathbf{1}_\varphi B + \mathbf{B}_p + \tilde{\mathbf{B}} + \mathbf{B}', \qquad \mathbf{B}_p = \operatorname{rot} \mathbf{1}_\varphi A, \tag{2.4}$$

where we have used the notation $B = \langle\langle B_\varphi \rangle\rangle$, $A = \langle\langle A_\varphi \rangle\rangle$, $\mathbf{B}_p = \langle\langle \mathbf{B}_p \rangle\rangle$.

$$\mathbf{B}' = \sum_\beta \mathbf{B}_\beta', \quad \mathbf{B}_\beta' = \sum_n \mathbf{B}_{\beta n} e^{in(\varphi - \varphi_\beta)}, \quad \mathbf{B}_{\beta, -n} = \mathbf{B}_{\beta n}^*, \tag{2.5}$$

$$\tilde{\mathbf{B}} = \sum_\nu \mathbf{B}_\nu e^{-i\varphi_\nu}, \quad \varphi_{-\nu} = -\varphi_\nu, \quad \mathbf{B}_{-\nu} = \mathbf{B}_\nu^*. \tag{2.6}$$

Substituting (2.3) and (2.4) into the φ-components of (1.1) and (1.2), and carrying out a double averaging, we obtain the equations for A, B:

$$\partial A / \partial t + s^{-1}\mathbf{v}_p \nabla (sA) = D_m \Delta_1 A + \mathscr{E}_\varphi, \tag{2.7a}$$

$$\partial B / \partial t + s\mathbf{v}_p \nabla (s^{-1}B) = D_m \Delta_1 B + [\nabla\zeta\nabla(sA)]_\varphi + (\operatorname{rot} \mathscr{E})_\varphi, \tag{2.7b}$$

where $\Delta_1 = \nabla^2 - s^{-2}$, ζ is the local angular velocity:

$$\zeta = v / s, \tag{2.8}$$

$\mathscr{E}$ has the interpretation of an emf for axially-symmetric currents:

$$\mathscr{E} = \langle\langle [(\tilde{\mathbf{v}} + \mathbf{v}')(\tilde{\mathbf{B}} + \mathbf{B}')] \rangle\rangle = \langle [\tilde{\mathbf{v}}\tilde{\mathbf{B}}] \rangle^t + \langle\langle [\mathbf{v}'\mathbf{B}'] \rangle\rangle. \tag{2.9}$$

If we subtract from (1.2) the same equation, but doubly averaged, we obtain an equation for $\tilde{\mathbf{B}} + \mathbf{B}'$. Picking out the axially-symmetric terms, we obtain equations for $\mathbf{B}$ and $\mathbf{B}'$. The equation for the waves $\mathbf{B}_p'$ has the form

$$\frac{\partial \mathbf{B}_p'}{\partial t'} + \frac{v}{s}\frac{\partial_1 \mathbf{B}_p'}{\partial\varphi} = \frac{B}{s}\frac{\partial_1 \mathbf{v}_p'}{\partial\varphi} + \{\operatorname{rot}(\mathbf{G}' + \mathbf{K}')\}_p + \mathbf{L}_p', \tag{2.10}$$

$$\mathbf{G}' = [\mathbf{v}'\mathbf{B}'] - \langle [\mathbf{v}'\mathbf{B}'] \rangle, \tag{2.11}$$

$$\mathbf{K}' = [\tilde{\mathbf{v}}\mathbf{B}'] + [\mathbf{v}'\tilde{\mathbf{B}}], \tag{2.12}$$

$$\mathbf{L}_p' = \{\operatorname{rot}([\mathbf{v}'\mathbf{B}_p] - D_m \operatorname{rot} \mathbf{B}')\}_p + (\mathbf{B}_p'\nabla)\mathbf{v}_p - d\mathbf{B}_p'/dt, \tag{2.13}$$

$$d/dt = \partial/\partial t + (\mathbf{v}_p\nabla). \tag{2.14}$$

The component B_φ' can be easily expressed in terms of $\mathbf{B}_p'$ from $\operatorname{div} \mathbf{B}' = 0$:

$$B_\varphi' = -s \operatorname{div} \hat{\mathbf{B}}_p'. \tag{2.15}$$

The equations for the oscillations $\tilde{\mathbf{B}}_p$ and $\tilde{B}_\varphi$ have the form

$$\partial\tilde{\mathbf{B}}_p/\partial t' = \{\operatorname{rot}(\tilde{\mathscr{E}} + \tilde{\mathbf{G}} + [\tilde{\mathbf{v}}\mathbf{B}_p] - D_m \operatorname{rot}\tilde{\mathbf{B}})\}_p + (\tilde{\mathbf{B}}_p\nabla)\mathbf{v}_p - d\tilde{\mathbf{B}}_p/dt, \tag{2.16}$$

$$\partial\tilde{B}_\varphi/\partial t' = \{\operatorname{rot}(\tilde{\mathscr{E}} + \tilde{\mathbf{G}} + [\tilde{\mathbf{v}}\mathbf{B}_p] - D_m \operatorname{rot}\tilde{\mathbf{B}})\}_\varphi + s\tilde{\mathbf{B}}_p\nabla(s^{-1}v) - s\tilde{\mathbf{v}}_p\nabla(s^{-1}B) - sd(s^{-1}\tilde{B}_\varphi)/dt, \tag{2.17}$$

$$\tilde{\mathscr{E}} = \langle [\mathbf{v}'\mathbf{B}'] \rangle - \langle\langle [\mathbf{v}'\ \mathbf{B}'] \rangle\rangle, \tag{2.18}$$

$$\tilde{\mathbf{G}} = [\tilde{\mathbf{v}}\tilde{\mathbf{B}}] - \langle [\tilde{\mathbf{v}}\tilde{\mathbf{B}}] \rangle^t. \tag{2.19}$$

The term $\tilde{\mathscr{E}}$ gives the excitation of the oscillations of the field by the waves.

We seek the solution of the equations for the oscillations and for the waves in the form of a series in powers of $R_m^{-1/2}$.

In (2.16) and (2.17), the terms proportional to $\mathbf{v}_p$ are small, and we neglect them. We set

$$\tilde{\mathbf{B}}_p = \operatorname{rot} \mathbf{1}_\varphi \tilde{A}_\varphi = [\nabla (s\tilde{A}_\varphi) \mathbf{1}_\varphi] / s. \quad (2.20)$$

From (2.16) we obtain with an accuracy up to terms of the third order in $R_m^{-1/2}$ inclusively

$$\tilde{A}_\varphi = \hat{\hat{\mathscr{E}}}_\varphi + \hat{\tilde{G}}_\varphi^{(3)} + [\hat{\tilde{\mathbf{v}}}\mathbf{B}_p]_\varphi + D_m \Delta_1 \hat{\tilde{A}}_\varphi. \quad (2.21)$$

Here the principal term (the first) is of order $R_m^{-1}BL$. It must be substituted into the second term; this yields

$$\tilde{A}_\varphi^{(2)} = \hat{\tilde{\mathscr{E}}}_\varphi^{(2)}, \ \tilde{G}_\varphi^{(3)} = -s^{-1} \operatorname{div} (s\hat{\tilde{\mathscr{E}}}_\varphi^{(2)} \tilde{\mathbf{v}}_p - s \langle \hat{\tilde{\mathscr{E}}}_\varphi^{(2)} \tilde{\mathbf{v}}_p \rangle^t). \quad (2.22)$$

The solution of (2.17), up to terms of the second order inclusive, has the form

$$\tilde{B}_\varphi = -s \hat{\tilde{\mathbf{v}}}_p \nabla (s^{-1} B) + (\operatorname{rot} \hat{\tilde{\mathscr{E}}}_p)_\varphi + s \hat{\tilde{\mathbf{B}}}_p \nabla (s^{-1} v) + s \frac{\partial}{\partial x_i} \left(\hat{\tilde{V}}_{ij} \frac{\partial}{\partial x_j} \frac{B}{s} \right) + D_m \Delta_1 \hat{\tilde{B}}_\varphi \quad (2.23)$$

Here the principal term (the first one) is of order $R_m^{-1/2}$. It is substituted in place of $\tilde{B}_\varphi$ into the penultimate term, where we put

$$\tilde{V}_{ij} = \tilde{v}_{pi} \hat{\tilde{v}}_{pj} - \langle \tilde{v}_{pi} \hat{\tilde{v}}_{pj} \rangle^t.$$

The last terms in (2.21) and (2.23) can be significant only in thin boundary layers, but ordinarily they are by two orders of magnitude, i.e., R_m^{-1} times, smaller than the principal terms, and we omit them.

We rewrite equation (2.10) for the waves in the form

$$\frac{\partial \hat{\mathbf{B}}_p'}{\partial t'} + \frac{v}{s} \mathbf{B}_p' = \sum_\beta \frac{v_\beta}{s} \mathbf{B}'_{\beta p} = \frac{B}{s} \mathbf{v}_p' + \{\operatorname{rot} (\hat{\mathbf{G}}' + \hat{\mathbf{K}}')\}_p + \hat{\mathbf{L}}_p'. \quad (2.24)$$

Here we have taken into account that from (2.5) the relation

$$\partial \hat{\mathbf{B}}'/\partial t' = -\sum_\beta \dot{\varphi}_\beta \mathbf{B}_\beta' \quad (2.25)$$

follows, and we have used the notation

$$v_\beta = v - s\dot{\varphi}_\beta. \quad (2.26)$$

Substituting into (2.24) and (2.15) $\mathbf{B}' = \sum_n \mathbf{B}^{(n)}$ and equating terms of the same order we can obtain in turn $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ etc.

For $n = 1$ we obtain

$$\mathbf{B}_p^{(1)} = \sum_\alpha \frac{B}{v_\alpha} \mathbf{v}_{\alpha p}' = B \sum_\alpha \mathbf{u}_{\alpha p}, \quad (2.27)$$

$$\mathbf{u}_{\alpha p} = \mathbf{v}_{\alpha p}' / v_\alpha. \quad (2.28)$$

Using (2.15) and $v_\varphi' = -s \operatorname{div} \hat{\mathbf{v}}_p'$, we find $B_\varphi^{(1)}$ and obtain

$$\mathbf{B}^{(1)} = \sum_\alpha \mathbf{B}_\alpha^{(1)} = \sum_\alpha \{(B/v_\alpha)\mathbf{v}_\alpha' - \mathbf{1}_\varphi (s\hat{v}_{\alpha p}' \nabla (B/v_\alpha))\}. \quad (2.29)$$

The first approximation $\mathbf{B}^{(1)}$ depends linearly on $\mathbf{v}'$, and, therefore (2.29) contains a sum of the waves of the field $\mathbf{B}_\alpha^{(1)}$, each of which is expressed in terms of the corresponding wave of velocity $\mathbf{v}_\alpha'$ quite independently of the other waves, just as in [1]. For $n = 2$, nonlinear effects already play a role. The quantities $\mathbf{G}^{(2)}$ and $\mathbf{K}^{(2)}$ contain combination frequencies $m\dot{\varphi}_\alpha + m'\dot{\varphi}_{\alpha'}$, $m\dot{\varphi}_\alpha + \dot{\varphi}_\mu$, and, therefore, the corresponding terms in $\mathbf{B}^{(2)}$ have the same frequencies.

It is convenient to introduce the notation

$$v_{\alpha m \alpha' m'} = v - s(m\dot{\varphi}_\alpha + m'\dot{\varphi}_{\alpha'})(m + m')^{-1}, \quad (2.30)$$

$$v_{\alpha m \mu} = v - s(m\dot{\varphi}_\alpha + \dot{\varphi}_\mu) m^{-1}. \quad (2.31)$$

The expression for the curl in cylindrical coordinates yields

$$(\operatorname{rot} \hat{\mathbf{G}}')_p = s^{-1} [\mathbf{1}_\varphi (\mathbf{G}_p' - \nabla s\hat{G}_\varphi')]. \quad (2.32)$$

Taking into account (2.32), (2.11), (2.12), (2.29), and (2.23), we obtain $\mathbf{B}_p^{(2)}$ in the form

$$\mathbf{B}_p^{(2)} = \sum_{\substack{\alpha m \alpha' m' \\ m+m' \neq 0}} v_{\alpha m \alpha' m'}^{-1} [\mathbf{1}_\varphi \{\mathbf{G}_{\alpha m \alpha' m'}^{(2)} - (im + im')^{-1} \nabla (s G_{\alpha m \alpha' m'}^{(2)})_\varphi\}] \exp (i\psi_{\alpha m \alpha' m'}) + \sum_{\alpha m \mu} v_{\alpha m \mu}^{-1} [\mathbf{1}_\varphi \{\mathbf{K}_{\alpha m \mu}^{(2)} - (im)^{-1} \nabla (s K_{\alpha m \mu}^{(2)})_\varphi\}] \exp (i\psi_{\alpha m \mu}), \quad (2.33)$$

where we have used the notation

$$\mathbf{G}_{\alpha m \alpha' m'}^{(2)} = (B/v_\alpha)[\mathbf{v}_{\alpha' m'} \mathbf{v}_{\alpha m}] - [\mathbf{v}_{\alpha' m'} \mathbf{1}_\varphi](im)^{-1} s \mathbf{v}_{\alpha m} \nabla (B/v_\alpha), \quad (2.34)$$

$$\psi_{\alpha m \alpha' m'} = (m + m')\varphi - (m\varphi_\alpha + m'\varphi_{\alpha'}), \quad (2.35)$$

$$\mathbf{K}_{\alpha m \mu}^{(2)} = (B/v_\alpha)[\mathbf{v}_\mu \mathbf{v}_{\alpha m}] - [\mathbf{v}_\mu \mathbf{1}_\varphi](s/im) \mathbf{v}_{\alpha m} \nabla (B/v_\alpha) - [\mathbf{v}_{\alpha m} \mathbf{1}_\varphi](-i\dot{\varphi}_\mu)^{-1} s \mathbf{v}_\mu \nabla (B/s), \quad (2.36)$$

$$\psi_{\alpha m \mu} = m\varphi - (m\varphi_\alpha + \varphi_\mu). \quad (2.37)$$

The expressions obtained for $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ are valid if the quantities v_α, $v_{\alpha m \alpha m'}$, $v_{\alpha m \mu}$ differ from zero, and we assume that this is the case.

At the outer surface of the conducting liquid, where it is bounded either by vacuum or by an insulator, both the φ-component of the axially-symmetric field $B = 0$, $\tilde{B}_\varphi = 0$ and the component of velocity normal to the boundary $v_n = \tilde{v}_n = v_n' = 0$ vanish. As a result of this, as can be verified by means of the expressions given above, at the boundary $\tilde{B}^{(1)} = 0$, $\tilde{B}^{(2)} = 0$, $\tilde{\mathbf{B}}^{(1)} = 0$, $\tilde{\mathbf{B}}^{(2)} = 0$.

The n-component of all these terms in the neighborhood of the boundary is proportional to ξ^2, while the ϑ- and φ-components are proportional to ξ, where ξ is the distance from the boundary (it is assumed that at the boundary $B \sim \xi$, $\tilde{v}_n \sim v'_n \sim \xi$, while $\tilde{v}_\vartheta$ and v'_ϑ are finite). Thus, the oscillations and the waves of the field of order $R_m^{-1/2}$ and R_m^{-1} do not pass to the outside from the liquid, as in [1].

Oscillations of the field of order $R_m^{-3/2}$ can pass to the outside, since (2.21) contains at the boundary the term $\sim R_m^{-3/2}$ which is equal to

$$\tilde{A}_\varphi^{(3)} = [\hat{\tilde{\mathbf{v}}}_p \mathbf{B}_p]_\varphi = s^{-1}\hat{\tilde{\mathbf{v}}}_p \nabla s A. \qquad (2.38)$$

The oscillations of the field outside are determined by the equation $\Delta\tilde{A}_{\varphi\text{out}} = 0$ with the boundary condition $\tilde{A}_{\varphi\text{out}} = \tilde{A}_\varphi^{(3)}$ in accordance with (2.38). The normal derivative $\partial\tilde{A}_\varphi/\partial n$ in this case turns out to be discontinuous, but in fact it varies continuously in the boundary layer of thickness of the order of $\delta \sim (D_m/\dot{\varphi}_\mu)^{1/2}$, analogous to the layers investigated in [1].

The third approximation for the waves is very awkward, but we are interested only in the normal component $B_n^{(3)}$ at the boundary, which, as shown in [1], determines the wave field which penetrates outside. Taking into account the relations given above, one can verify that at the boundary curl $\mathbf{G}^{(3)} = 0$, curl $\mathbf{K}^{(3)} = 0$, so that in the right-hand side of the equation for the waves only the term $\mathbf{L}_p^{(3)}$ linear in $\mathbf{v}'$ differs from zero. Thus, the wave field penetrating outside is of order $R_m^{-3/2}$, and $B_n^{(3)}$ can be expressed in the form of mutually independent waves found in [1]:

$$B_n^{(3)} = \sum_\alpha \left\{ - s\hat{\mathbf{u}}_{\alpha p} \nabla B_n + \left(sB_n + \frac{2D_m}{s^{-1}v_\alpha}\frac{\partial B}{\partial n}\right)\frac{\partial \hat{u}_{\alpha n}}{\partial n}\right\}. \qquad (2.39)$$

3. THE GENERATION EQUATIONS

In order to obtain the generation equations we have to evaluate $\mathscr{E}$ and substitute it in (2.7). The quantity $\mathscr{E}_p$, which appears in (2.7b), is of the second order of smallness. Up to this accuracy it can be easily obtained by substituting (2.29) and the first term of (2.23) into (2.9). This gives

$$\mathscr{E}_p = \frac{1}{2}\langle[\tilde{\mathbf{v}}\hat{\tilde{\mathbf{v}}}]_\varphi\rangle^t s\nabla\frac{B}{s} + \sum_\alpha w_\alpha v_\alpha^2 \nabla\frac{B}{v_\alpha}, \qquad (3.1)$$

$$w_\alpha = (s/2v_\alpha^2)\langle[\mathbf{v}_\alpha' \hat{\mathbf{v}}_\alpha']_\varphi\rangle = (s/2)\langle[\mathbf{u}_{\alpha p}\hat{\mathbf{u}}_{\alpha p}]_\varphi\rangle. \qquad (3.2)$$

For the evaluation of (3.1) we utilize formulas for integration by parts of the type

$$\langle F_1' \hat{F}_2'\rangle = -\langle \hat{F}_1' F_2'\rangle, \quad \langle \tilde{F}_1 \hat{\tilde{F}}_2\rangle^t = -\langle \hat{\tilde{F}}_1 \tilde{F}_2\rangle^t. \qquad (3.3)$$

The double averaging of terms quadratic in $\mathbf{v}'$, is carried out very simply, since in this case the individual waves are averaged independently of one another. Waves with different velocities have a phase shift rapidly varying with time, and as a result of averaging with respect to time this leads to a vanishing result. Indeed, the quantity $\langle\langle \exp\{i[(m + m')\varphi - (m\varphi_\alpha + m'\varphi_{\alpha'})]\}\rangle\rangle$ differs from zero only for $m + m' = 0$, $m\varphi_\alpha + m'\varphi_{\alpha'} = 0$, and consequently for $\varphi_\alpha = \varphi_{\alpha'}$.

Equation (2.7b), after (3.1) has been substituted into it, can be brought, as in [1], to the form corresponding to axial symmetry, if in place of A, $\mathbf{v}_p$ we introduce the "effective" quantities A_e and $\mathbf{v}_{ep}$:

$$A_e = A + WB, \qquad (3.4)$$

$$\mathbf{v}_{ep} = \mathbf{v}_p + \mathbf{v}_{\text{osc}} + \mathbf{v}_{\text{wave}}, \qquad (3.5)$$

where

$$W = \sum_\alpha w_\alpha, \qquad (3.6)$$

$$\mathbf{v}_{\text{osc}} = {}^1\!/_2 \operatorname{rot}(\mathbf{1}_\varphi \langle[\tilde{\mathbf{v}}\hat{\tilde{\mathbf{v}}}]_\varphi\rangle^t), \quad \mathbf{v}_{\text{wave}} = \operatorname{rot}\Big(\mathbf{1}_\varphi \sum_\alpha v_\alpha w_\alpha\Big). \qquad (3.7)$$

Evaluation of $\mathscr{E}_\varphi$ is considerably more complicated, since this is a quantity of the fourth order, and in $\langle\langle \mathbf{v}' \times \mathbf{B}'\rangle\rangle$ one would have to substitute $\mathbf{B}'$ with an accuracy up to terms of the third order. However, we proceed differently. For the evaluation of $\langle\langle \mathbf{v}' \times \mathbf{B}'_\varphi\rangle\rangle$ we take the vector product of (2.24) with $\mathbf{B}'_p$, and take its double average. As a result of this the left-hand side will vanish, while on the right-hand side we can carry out the following transformation:

$$\langle\langle [s\mathbf{B}_p'(\operatorname{rot}\mathbf{G}')_p]_\varphi\rangle\rangle = \langle\langle \mathbf{B}'\mathbf{G}' - B_\varphi' G_\varphi' - \mathbf{B}_p'\nabla s\hat{G}_\varphi'\rangle\rangle$$
$$= \langle\langle \mathbf{B}'\mathbf{G}'\rangle\rangle + \operatorname{div}\langle\langle sG_\varphi'\hat{\mathbf{B}}_p'\rangle\rangle \qquad (3.8)$$

and similarly for the term with $\mathbf{K}'$. As a result of this we obtain

$$B\langle\langle[\mathbf{v}'\mathbf{B}']_\varphi\rangle\rangle = \operatorname{div}\langle\langle s[\mathbf{v}_p'\mathbf{B}_p']_\varphi \hat{\mathbf{B}}_p'\rangle\rangle - \langle\tilde{\mathscr{E}}\hat{\tilde{\mathbf{B}}}\rangle^t$$
$$+ \operatorname{div}\langle\langle sK_\varphi'\hat{\mathbf{B}}_p'\rangle\rangle + \langle\langle[s\mathbf{B}_p'\hat{\mathbf{L}}_p']_\varphi\rangle\rangle. \qquad (3.9)$$

Into the last term above it is sufficient to substitute for $\mathbf{B}_p'$ the first approximation (2.29). Into the other terms we must substitute $\mathbf{B}'$ in the second approximation, but even this can be avoided by means of subterfuge. We first state a number of identities which will be useful for subsequent transformations:

$$[\mathbf{ab}]_\varphi \mathbf{c}_p + [\mathbf{ca}]_\varphi \mathbf{b}_p + [\mathbf{bc}]_\varphi \mathbf{a}_p \equiv 0, \qquad (3.10)$$

$$\left[\frac{\partial\hat{\mathbf{F}}'}{\partial t'}\mathbf{F}'\right]_\varphi \equiv \frac{1}{2}\frac{\partial}{\partial t'}[\hat{\mathbf{F}}'\mathbf{F}']_\varphi + \frac{1}{2}\frac{\partial_1}{\partial\varphi}\left[\frac{\partial\hat{\mathbf{F}}'}{\partial t'}\hat{\mathbf{F}}'\right]_\varphi, \qquad (3.11)$$

$$\left[\frac{\partial\hat{\mathbf{F}}'}{\partial t'}\mathbf{F}'\right]_\varphi \hat{\mathbf{F}}_p' \equiv \frac{1}{3}\frac{\partial}{\partial t'}([\hat{\mathbf{F}}'\mathbf{F}']_\varphi \hat{\mathbf{F}}_p')$$
$$+ \frac{1}{3}\frac{\partial_1}{\partial\varphi}\left(\left[\frac{\partial\hat{\mathbf{F}}'}{\partial t'}\hat{\mathbf{F}}'\right]_\varphi \hat{\mathbf{F}}_p'\right), \qquad (3.12)$$

$$\left[\frac{\partial \hat{\mathbf{F}}'}{\partial t'}\mathbf{F}'\right]_\varphi \hat{F}_i'\hat{F}_j' \equiv \frac{1}{4}\frac{\partial}{\partial t'}([\hat{\mathbf{F}}'\mathbf{F}']_\varphi \hat{F}_i'\hat{F}_j') + \frac{1}{4}\frac{\partial_1}{\partial\varphi}\left(\left[\frac{\partial \hat{\mathbf{F}}'}{\partial t'}\hat{\mathbf{F}}'\right]_\varphi \hat{F}_i'\hat{F}_j'\right), \quad (3.13)$$

$$[\mathbf{A}\mathbf{F}']_\varphi \hat{F}_i'\hat{F}_j' \equiv \frac{1}{3}\frac{\partial_1}{\partial\varphi}([\mathbf{A}\hat{\mathbf{F}}']_\varphi \hat{F}_i'\hat{F}_j') + [\hat{\mathbf{F}}'\mathbf{F}']_\varphi(\hat{F}_i'A_j + A_i\hat{F}_j'). \quad (3.14)$$

Here **a**, **b**, and **c** are arbitrary vectors, **A** is an axially symmetric vector, and **F′** is a vector variable with respect to φ such that $\langle \mathbf{F}' \rangle = 0$. All the identities contain only components of vectors in the meridian plane, and the indices i, j in (3.13) and (3.14) take on only two values, for example, z and s. Identity (3.10) can be easily obtained by means of the well-known formula for the double vector product, (3.11) is obvious, while the others are obtained with the aid of these two. From (3.11)—(3.13) it follows that the left-hand sides vanish as a result of double averaging.

We take the vector product of (2.24) with $\mathbf{B}_p'$, obtain the φ-component of this product, multiply it by $\hat{\mathbf{B}}_p'$ and take a double average. As a result of this the left-hand side will vanish in accordance with (3.12) and we obtain

$$\langle\!\langle[\mathbf{v}_p'\mathbf{B}_p']_\varphi \hat{\mathbf{B}}_p'\rangle\!\rangle = sB^{-1}\langle\!\langle[\mathbf{B}_p' \operatorname{rot}(\hat{\mathbf{G}}'+\hat{\mathbf{K}}')]_\varphi \hat{\mathbf{B}}_p'\rangle\!\rangle + sB^{-1}\langle\!\langle[\mathbf{B}_p'\hat{\mathbf{L}}_p']_\varphi \hat{\mathbf{B}}_p'\rangle\!\rangle. \quad (3.15)$$

Substituting (3.15) into (3.9) we obtain from (2.9) the following exact formal expression for $\mathscr{E}_\varphi$:

$$\mathscr{E}_\varphi = \mathscr{E}_\varphi{}^I + \mathscr{E}_\varphi{}^{II} + sB^{-1}\langle\!\langle[\mathbf{B}_p'\hat{\mathbf{L}}_p']_\varphi\rangle\!\rangle + B^{-1}\operatorname{div}\langle\!\langle s^2B^{-1}[\mathbf{B}_p'\hat{\mathbf{L}}_p']_\varphi\hat{\mathbf{B}}_p'\rangle\!\rangle, \quad (3.16)$$

$$\mathscr{E}_\varphi{}^I = \langle[\tilde{\mathbf{v}}_p\tilde{\mathbf{B}}_p]_\varphi\rangle^t - B^{-1}\langle\tilde{\mathscr{E}}\,\tilde{\mathbf{B}}\rangle^t + B^{-1}\operatorname{div}\langle\!\langle sK_\varphi'\hat{\mathbf{B}}_p'\rangle\!\rangle + B^{-1}\operatorname{div}\langle\!\langle s^2B^{-1}[\mathbf{B}_p'\operatorname{rot}\hat{\mathbf{K}}']_\varphi\hat{\mathbf{B}}_p'\rangle\!\rangle, \quad (3.16a)$$

$$\mathscr{E}_\varphi{}^{II} = B^{-1}\operatorname{div}\langle\!\langle s^2B^{-1}[\mathbf{B}_p'\operatorname{rot}\hat{\mathbf{G}}']_\varphi\hat{\mathbf{B}}_p'\rangle\!\rangle. \quad (3.16b)$$

The last term in (3.16) is small and can be neglected, while in the evaluation of $\langle\!\langle \mathbf{B}_p'\hat{\mathbf{L}}_{p\varphi}' \rangle\!\rangle$ and $\mathscr{E}_\varphi^{II}$ it is sufficient to utilize for the waves the first approximation (2.29).

The quantity $\tilde{\mathscr{E}}_\varphi$ must be known up to terms of the third order inclusively. In order to evaluate it we take the vector product of (2.24) with $\mathbf{B}_p'$, average the result over φ, take a double average and subtract. Utilizing the indentity (3.11) and the transformation (3.8) we obtain with the required accuracy

$$\tilde{\mathscr{E}}_\varphi = -(s/2B)\,\partial\langle[\mathbf{B}'\hat{\mathbf{B}}']_\varphi\rangle/\partial t' - B^{-1}\operatorname{div}(\tilde{\mathscr{E}}_\varphi^{(2)}\tilde{\mathbf{B}}^{(1)} - \langle\tilde{\mathscr{E}}_\varphi^{(2)}\tilde{\mathbf{B}}^{(1)}\rangle^t) + B^{-1}\operatorname{div}[\langle s(G_\varphi' + K_\varphi')\hat{\mathbf{B}}_p'\rangle - \langle\!\langle s(G_\varphi' + K_\varphi')\hat{\mathbf{B}}_p'\rangle\!\rangle]. \quad (3.17)$$

With an accuracy up to terms of the second order we obtain from this

$$\hat{\tilde{\mathscr{E}}}_\varphi^{(2)} = -B\sum_{\alpha\neq\beta}\tilde{w}_{\alpha\beta}, \quad \tilde{w}_{\alpha\beta} = -\frac{s}{2}\langle[\mathbf{u}_{\alpha p}\hat{\mathbf{u}}_{\beta p}]_\varphi\rangle. \quad (3.18)$$

After substitution of (2.21), (2.22) and (3.17) into (3.16a) all the third-order terms contained in $\mathscr{E}_\varphi^I$ cancel each other. Thus, $\mathscr{E}_\varphi$ contains only fourth-order terms. In evaluating them it is sufficient to utilize the first approximation for the waves $\mathbf{B}_p' = \mathbf{B}_p^{(1)} = B\sum_\alpha \mathbf{u}_{\alpha p}$, and in the course of this calculation we can utilize the relations

$$\frac{\partial\hat{\mathbf{B}}_p'}{\partial t'} + \frac{v}{s}\mathbf{B}_p' = \frac{B}{s}\mathbf{v}_p' \qquad [\mathbf{v}'\mathbf{B}']_\varphi = \frac{s}{B}\left[\frac{\partial\mathbf{B}'}{\partial t'}\mathbf{B}'\right]_\varphi. \quad (3.19)$$

Further calculations are fairly awkward, and we shall, therefore, not reproduce them in detail, but only outline them. It is simplest of all to calculate $\langle\!\langle \mathbf{B}_p'\hat{\mathbf{L}}_p' \rangle\!\rangle$, since after the substitution $\mathbf{B}_p' = B\sum_\alpha \mathbf{u}_{\alpha p}$ we obtain here an expression quadratic in $\mathbf{u}_{\alpha p}$ in which each wave is averaged independently of the others, so that we can utilize the results of the preceding paper [1]. After relatively uncomplicated transformations and after utilization of equation (2.7b) we obtain

$$sB^{-1}\langle\!\langle[\mathbf{B}_p'\hat{\mathbf{L}}_p']_\varphi\rangle\!\rangle = D_mB\sum_\alpha \Gamma(\mathbf{u}_{\alpha p}) - s^{-1}d(sWB)/dt + D_m\Delta_1(WB) - s^{-1}\mathbf{v}_{\text{wave}}\nabla sA - W(\operatorname{rot}\mathcal{E}_p)_\varphi, \quad (3.20)$$

where $\Gamma(\mathbf{u})$ is given by expression (3.21) from reference [1].

The part of $\mathscr{E}_\varphi$ associated with the oscillations (3.16a) after substitution of (3.17) and cancellation of third-order terms can be represented as a sum of three groups of terms: terms independent of $\tilde{\mathbf{v}}$, terms linear in $\tilde{\mathbf{v}}$, and terms quadratic in $\tilde{\mathbf{v}}$. The terms linear in $\tilde{\mathbf{v}}$ cancel one another. This can be shown by utilizing integration by parts (3.3) and identities (3.12), (3.14) averaged over φ. The majority of terms quadratic in $\tilde{\mathbf{v}}$ also cancel, while the others can be reduced to a simple form. By also transforming the terms not containing $\tilde{\mathbf{v}}$ with the aid of (3.18), (3.19) we obtain:

$$\mathscr{E}_\varphi{}^I = -s^{-1}\mathbf{v}_{\text{osc}}\nabla(sA) - B^{-1}\operatorname{div}(WB\mathbf{v}_{\text{osc}}) - B^{-1}\operatorname{div}\{\langle\hat{\tilde{\mathscr{E}}}_\varphi[\mathbf{1}_\varphi\tilde{\mathscr{E}}_p]\rangle^t + \langle\!\langle sB^{-1}(\frac{\partial\hat{\mathbf{B}}_p'}{\partial t'}\nabla(s\tilde{\mathscr{E}}_\varphi))\hat{\mathbf{B}}_p'\rangle\!\rangle\}. \quad (3.21)$$

Expression (3.16b) after lengthy transformations with repeated utilization of integration by parts (3.3) and of identities (3.10), (3.11), and also (3.13) and (3.19) is reduced to

$$\mathscr{E}_\varphi{}^{II} = B^{-1}\operatorname{div}\left\{-BW\left[\mathbf{1}_\varphi \sum_\alpha w_\alpha v_\alpha{}^2\nabla(B/v_\alpha)\right] + \langle\hat{\tilde{\mathscr{E}}}_\varphi[\mathbf{1}_\varphi\tilde{\mathscr{E}}_p]\rangle^t\right.$$

$$+ \langle\!\langle sB^{-1} \left(\frac{\partial \hat{\mathbf{B}}_p'}{\partial t} \nabla s \hat{\hat{\mathscr{E}}}_\varphi\right) \hat{\mathbf{B}}_p'$$

$$+ WB\left[\mathbf{1}_\varphi \nabla \left(\left[\frac{\partial \hat{\mathbf{B}}_p'}{\partial t} \hat{\mathbf{B}}_p'\right]_\varphi (s^2/2B)\right)\right]\rangle\!\rangle\Big\}. \qquad (3.22)$$

The calculations are concluded by the joining together of (3.20), (3.21) and (3.22), substitution into (2.7a) and a certain rearrangement of terms. Finally, all these lengthy calculations lead to an amazingly simple result, viz., the generation equations for A_e, B after introducing the effective meridian velocity $v_{ep} = v_p + v_{osc} + v_{wave}$ are reduced in the nonstationary case to the same form as the equations for the stationary case obtained previously [1] with only Γ being replaced by $\sum_\alpha \Gamma(\mathbf{u}_{\alpha p})$.

$$\partial A_e / \partial t + s^{-1}\mathbf{v}_{ep} \nabla (sA_e) = D_m \Delta_1 A_e + D_m \Gamma B, \qquad (3.23a)$$

$$\partial B/\partial t + s\mathbf{v}_{ep} \nabla (s^{-1}B) = D_m \Delta_1 B + [\nabla \zeta \nabla s A_e]_\varphi, \quad (3.23b)$$

$$\Gamma = \sum_\alpha \Gamma(\mathbf{u}_{\alpha_p}),$$

$$\Gamma(\mathbf{u}_p) = s^{-1} \langle [\mathbf{u}_p (\hat{\mathbf{u}}_p + \partial_1 \mathbf{u}_p / \partial \varphi)]_\varphi \rangle + 2 \langle (\nabla_p r u_r) \nabla_p \hat{u}_z \rangle. \qquad (3.24)$$

Here $ru_r = zu_z + su_s$, $\nabla_p = \mathbf{1}_z \partial/\partial z + \mathbf{1}_s \partial/\partial s$. The quantities A_e and $\mathbf{v}_{ep}$ are given by expressions (3.4)—(3.7).

At the boundary of the liquid $w_\alpha = 0$, $B = 0$, and, therefore, we have there $A_e = A$, $\nabla A_e = \nabla A$ and the boundary conditions for A_e, $\mathbf{v}_{ep}$ are the same as those for A, $\mathbf{v}_p$.

According to (3.24) waves with different velocities, and also the different harmonics in the Fourier expansion of a single wave $\mathbf{v}'_{\alpha p}$ with respect to φ, give independent contributions to the generation coefficient Γ. For generation it is necessary that in a certain wave there should be present a Fourier harmonic in which v'_p simultaneously contains terms with amplitudes of $\cos m\varphi$ and of $\sin m\varphi$ which are not proportional to one another. Sometimes this requirement can turn out to be definitely nontrivial. For example, let $\mathbf{v}'$ represent characteristic oscillations determined by a linear equation with coefficients independent of φ, which is quite natural for a system which differs little from axial symmetry. If these coefficients are real, then the characteristic functions are proportional to $\cos m\varphi$ or to $\sin m\varphi$, and generation is possible only in the presence of degeneracy, when more than one characteristic function corresponds to a certain frequency.

Concentrated generation. The previous paper by the author [1] investigates the quasistationary case of concentrated generation at surfaces of velocity discontinuities and at resonance surfaces $\zeta = 0$, and expressions are obtained for coefficients of concentrated generation Γ^S. In the presence of several waves the same expressions can be utilized. Indeed, in the derivation of Γ^S at a surface of discontinuity in [1], averages are taken of expressions quadratic in $\mathbf{v}'$ in which the averages can be carried out independently for each wave, so that the Γ^S_α are additive. If the resonance surfaces $\zeta_\alpha = 0$ do not coincide for different waves, then at each surface only one wave participates in the resonance, and Γ^S_α are calculated in accordance with [1]. However, in the presence of several waves there also appears the possibility of combined resonance generation, for example, at surfaces $v_{\alpha m \alpha' m'} = 0$, i.e., $m\zeta_\alpha + m'\zeta_{\alpha'} = 0$, where the denominator of (2.33) vanishes. The distribution of the field in layers near such surfaces and the coefficient Γ^S can be obtained in the same way as was done in [1] for resonance generation at the surfaces $\zeta = 0$.

We note that in the present paper the generation equations (3.23) and (3.24) are obtained as a result of long formal calculations. One can naturally suppose that for these comparatively simple equations there must exist a simple and more transparent derivation. It would be very desirable to find it, since this would enable us to better understand and to graphically represent the mechanism of generation.

A certain graphical visualization of the generation mechanism is given in the paper by Parker [2] who investigated the mechanism of generation of a meridian field from an azimuthal one in the case when there exists in the liquid a large number of randomly oriented vortices. On the basis of qualitative considerations Parker has shown that in the equation for $\partial A/\partial t$ a term arises proportional to B, and he has written equations analogous to (3.23), but without a specific expression for Γ and with the quantities A and $\mathbf{v}_p$ in place of A_e and $\mathbf{v}_{ep}$.

4. A PLANE MODEL OF A HYDROMAGNETIC DYNAMO

The generation equations obtained for the case of a small deviation from axial symmetry determine the quantities A, B, which do not depend on φ. Translational symmetry can be regarded as a limiting case of axial symmetry for $s \to \infty$, and, therefore, there is no need to repeat the derivation of the equations for a small deviation from translational symmetry. We introduce in place of the coordinates z, s, φ the coordinates z_{new}

$= s_0\varphi$, $x = z$, $y = s - s_0$, and we go over in the generation equations (3.23) to the limit $s_0 \to \infty$. As a result we obtain

$$\partial A_e/\partial t + \mathbf{v}_{ep}\nabla A_e = D_m\nabla^2 A_e + D_m\Gamma B,$$
$$\partial B/\partial t + \mathbf{v}_{ep}\nabla B = D_m\nabla^2 B + [\nabla v \nabla A_e]_z,$$
$$A_e = A + \sum_\alpha w_\alpha B, \quad \mathbf{v}_{ep} = \mathbf{v}_p + \mathbf{v}_{osc} + \mathbf{v}_{wave},$$
$$w_\alpha = {}^1/_2 \langle[\mathbf{u}_{\alpha p}\hat{\mathbf{u}}_{\alpha p}]_z\rangle, \quad \mathbf{u}_{\alpha p} = \mathbf{v}_{\alpha p}'/v_\alpha,$$
$$\mathbf{v}_{osc} = {}^1/_2 \operatorname{rot}(\mathbf{1}_z \langle[\tilde{\mathbf{v}}\hat{\tilde{\mathbf{v}}}]_z\rangle^t; \quad \mathbf{v}_{wave} = \operatorname{rot}\Big(\mathbf{1}_z \sum_\alpha w_\alpha v_\alpha\Big),$$
$$\Gamma = \sum_\alpha \Gamma(\mathbf{u}_{\alpha p}), \Gamma(\mathbf{u}_p) = \langle[\mathbf{u}_p \partial \mathbf{u}_p/\partial z]_z\rangle - 2\langle(\nabla_p u_x)(\nabla_p \hat{u}_y)\rangle. \tag{4.1}$$

All the notation here is similar to that used in the case of axial symmetry. Instead of the meridian planes (subscript p) we now have planes perpendicular to the z-axis, in place of the φ coordinate we now have the (new) z coordinate. The velocity and the magnetic field are now given in the form

$$\mathbf{1}_z v(x, y, t) + \mathbf{v}_p(x, y, t) + \tilde{\mathbf{v}} + \mathbf{v}'.$$

$$\mathbf{1}_z B(x, y, t) + \mathbf{B}_p(x, y, t) + \tilde{\mathbf{B}} + \mathbf{B}', \ \mathbf{B}_p = \operatorname{rot} \mathbf{1}_z A(x, y, t).$$

Primes denote quantities periodic in z ("waves"), while a tilde denotes quantities which are independent of z and which oscillate rapidly in time ("oscillations"), for example,

$$\mathbf{v}' = \sum_\alpha \mathbf{v}_\alpha', \quad \mathbf{v}_\alpha' = \sum_k \mathbf{v}_{\alpha k}(x, y, t)\, e^{ik(z - z_\alpha)},$$
$$\mathbf{v}_{\alpha, -k} = \mathbf{v}_{\alpha k}^*, \quad \tilde{\mathbf{v}} = \sum_\mu \mathbf{v}_\mu(x, y, t)\, e^{-i\varphi_\mu}; \quad \mathbf{v}_{-\mu} = \mathbf{v}_\mu^*. \tag{4.2}$$

In (4.1) we have used the notation $v_\alpha = v - dz_\alpha/dt$. The angle brackets $\langle \ldots \rangle$ and $\langle \ldots \rangle^t$ respectively denote averaging over z and over the "fast time" t′ which appears in z_α and φ_μ, for example $\langle \mathbf{v}' \rangle = 0$, $\langle \tilde{\mathbf{v}} \rangle^t = 0$. Quantities marked with a caret represent the periodic part of the integral with respect to z or with respect to t′, for example, $\widehat{\cos kz} = k^{-1}\sin kz$, $\partial\hat{\mathbf{v}}'/\partial z = \mathbf{v}'$, $\langle \hat{\mathbf{v}}' \rangle = 0$, and likewise $\partial\hat{\tilde{\mathbf{v}}}/\partial t' = \tilde{\mathbf{v}}$, $\langle \hat{\tilde{\mathbf{v}}} \rangle^t = 0$. Expressions for $\tilde{\mathbf{B}}$ and $\mathbf{B}'$ can be easily obtained from the corresponding formulas (2.21), (2.23), (2.29), (2.33), and (2.39) for small deviations from axial symmetry.

We consider a model dynamo in the form of a plane layer of fluid choosing the velocities in such a way as to enable us to obtain a simple analytic solution of the generation equation (4.1). This gives a simple proof of the possibility of a stationary and oscillatory dynamo and illustrates certain properties of the solution and of the spectrum of characteristic frequencies. A similar model, but without taking boundary conditions into account, was discussed by Parker [2].

We assume that the conducting fluid occupies a plane layer infinite in the y, z directions, and extending from −L to +L along the x-axis. We assume that in the chosen coordinate system the velocity is stationary, and we take for it the following expression satisfying the equation div v = 0 and the boundary condition $v_x = 0$ for $x = \pm L$:

$$v = v(x), \quad \mathbf{v}_p = -[\nabla(wv)\mathbf{1}_z] + \mathbf{1}_y u_0(x).$$

$$v_x' = v u_{sx}(x)\sin kz, \quad u_{sx}(L) = u_{sx}(-L) = 0,$$
$$v_y' = v u_{cy}(x)\cos kz, \quad v_z' = k^{-1}(dv u_{sx}/dx)\cos kz. \tag{4.3}$$

At the same time we have

$$w = (2k)^{-1}u_{sx}u_{cy}, \quad \mathbf{v}_{ep} = \mathbf{1}_y u_0,$$
$$[\nabla v \nabla A_e]_z = (dv/dx)(\partial A_e/\partial y),$$
$$\Gamma = \Gamma(x) = -k u_{sx}u_{cy} - k^{-1}(du_{sx}/dx)(du_{cy}/dx). \tag{4.4}$$

As a result of (4.4) the coefficients in the generation equations do not depend on t, y and, therefore, we can seek a solution in the form

$$A_e = A(x)\,e^{i(qy - \omega t)}, \quad B = B(x)\,e^{i(qy - \omega t)}. \tag{4.5}$$

We assume further that $k = \pi/L$,

$$v = v_0\frac{2x}{L}, \quad u_0 = \text{const}, \quad u_{sx} = -u_{cx} = \frac{v_1}{v_0\pi^{1/2}}\sin\frac{\pi x}{L}. \tag{4.6}$$

Then $\Gamma = v_1^2/L$, and for A(x), B(x) we obtain equations with constant coefficients. Choosing L for the unit of length and L^2/D_m for the unit of time, B_1 for the unit of the magnetic field, and the quantity $A_1 = B_1 L(v_1/v_0)(D_m/Lv_0)^{1/2}$ for the unit of the vector potential, we write these equations in dimensionless variables:

$$(-i\omega + iq\alpha)A = A'' - q^2A + \beta B, \tag{4.7a}$$
$$(-i\omega + iq\alpha)B = B'' - q^2B + \beta 2iqA, \tag{4.7b}$$

where primes denote differentiation with respect to x, and where we have introduced

$$\alpha = \frac{Lu_0}{D_m} = \frac{u_0}{v_0}R_m, \quad \beta = \frac{v_1}{v_0}\left(\frac{Lv_0}{D_m}\right)^{1/2} = \frac{v_1}{v_0}R_m^{1/2}. \tag{4.8}$$

Outside the fluid where no current can flow we have

$$A'' - q^2A = 0, \quad B = 0. \tag{4.9}$$

Taking into account the fact that at the boundary of the fluid $w = 0$, so that there $A_e = A$, $A_e' = A'$, and solving (4.9) subject to the condition $A = 0$ at infinity, we obtain the boundary conditions for $x = \pm 1$

$$A_e' \pm |q|A_e = 0, \quad B = 0. \tag{4.10}$$

Equations (4.7) with the boundary conditions

(4.10) have solutions of two types—those symmetric and those antisymmetric with respect to x. These solutions can be obtained in an elementary fashion in terms of trigonometric functions, while the corresponding condition that these equations have a solution yields the "dispersion equation" for the characteristic frequencies. The equations are not selfconjugate, and, therefore, the frequencies are complex

$$\omega = \omega(\alpha, \beta, q) = \omega(0, \beta, q) + \alpha q.$$

Omitting the calculations we shall reproduce, for example, the expression for the frequencies of the symmetric modes for $q \gg 1$:

$$\omega(0, \beta, q) = -\beta\sqrt{q} - i(x_n^2 + q^2 - \beta\sqrt{q}), \qquad (4.11)$$

where $x_n = (2n + 1)\pi/2$, $n = 0, 1, 2, \ldots$ By a suitable choice of $\beta = \beta_n(q)$ we can make the frequency real, i.e., obtain waves of constant amplitude. In such a case from (4.11) one obtains

$$\omega_n = -(x_n^2 + q^2). \qquad (4.12)$$

Solutions of (4.11) also exist, but when β is replaced by $-\beta$ they are always damped. Numerical calculations show that for $n \geq 1$ expression (4.12) qualitatively correctly represents the shape of the curves $\omega_n(q)$ for all values of q, and not only for $q \gg 1$. For $n = 0$ when $q \ll 1$ one obtains in place of (4.12) $\omega_0 = -(21/8)^{1/2} q^{1/2}$. The value of the parameter $\beta = \beta_n(q)$ for which a transition from damping to build-up occurs passes through a minimum at a certain value of q and increases with the value of n. The smallest value is obtained for $n = 0$ for $q \approx 0.5$ and is equal to $\beta_{0\,\min} = 2.3$.

These results show that for $\alpha = 0$ one cannot obtain a stationary solution by suitable choice of β. By a suitable choice of the parameter α one can easily obtain a stationary solution from the purely oscillatory solutions found above. Indeed, it can be seen from (4.7) that $\omega(\alpha, \beta, q) = \omega(0, \beta, q) + \alpha q$, and, therefore, for $\beta = \beta_n(q)$, $\alpha = \alpha_n(q) = -\omega_n/q$ one obtains $\omega = 0$. In other words, a wave being propagated in the fluid with the velocity $V_{phase} = \omega/q$ can be brought to rest if the velocity $-V_{phase}$ is imparted to the whole fluid.

Outside the fluid the magnetic field consists of the basic field

$$\mathbf{B}_p(x, y) = \mathbf{1}_x \frac{\partial A}{\partial y} - \mathbf{1}_y \frac{\partial A}{\partial x}$$

and of the small correction

$$B^{(3)} \sim BR_m^{-3/2} \sim B_p R_m^{-1/2},$$

which can be found by using the analog of formula (2.39) for the plane model. In the present case $u_p(L) = 0$, $v(L) = 2v_0$, and, therefore,

$$B_{x^{(3)}}(L, y, z) = \{(B_x + D_m v_0^{-1} \partial B / \partial x) du_{sx} / dx\}_{x=L} (-k^{-1} \cos kz).$$

It is possible without changing the value of Γ to add to the stationary velocities (4.3) arbitrarily many "oscillations" $\tilde{\mathbf{v}}$ and "waves" $\mathbf{v}'_\alpha$ (4.2) as long as these waves do not contain for a given velocity simultaneously the Fourier components cos kz and sin kz. By a suitable choice of v_y one can achieve the former value $v_{ey} = u_0$. As a result of this calculation A_e and B retain their former values, and outside the fluid only the magnetic field of order $B_p R_m^{-1/2}$ will change, and in particular, new nonstationary terms will be added to the former stationary value $\mathbf{B}^{(3)}$: waves $\mathbf{B}'_\alpha$, propagated along the z-axis and axially-symmetric oscillations $\tilde{\mathbf{B}}$.

The hydromagnetic dynamo at a tangential "discontinuity." We briefly consider a system which is similar to the preceding one, but which has outside the layer $-L < x < L$ [characterized by the velocities (4.3) and (4.6)] another conducting medium moving with the velocities $\pm 2v_0 \mathbf{1}_z$ for $x \gtrless L$. For the velocities chosen above concentrated generation does not occur, so that Eqs. (4.7) are valid, but the boundary conditions are changed. Instead of (4.10) we have for $x = \pm 1$

$$A_e' \pm \varkappa A_e = 0, \qquad B' \pm \varkappa B = 0, \qquad (4.13)$$

where $\kappa = (q^2 - i\omega)^{1/2}$. The solution can again be obtained in an elementary manner and we shall not give it here. The frequency spectrum is analogous to the spectrum in the preceding problem, for example, for the symmetric modes for $q \gg 1$ we obtain instead of (4.11)

$$\omega(0, \beta, q) = -(x_n^2 + \beta\sqrt{q}) - i(q^2 - \beta\sqrt{q}). \qquad (4.14)$$

From this for real frequencies we again obtain expression (4.12).

The functional form of $v(x)$ assumed in the second problem of this section schematically describes the variation of the velocity at a tangential discontinuity. It is well known that a tangential discontinuity is unstable—small perturbations within it grow exponentially. What is the subsequent fate of a tangential discontinuity in a highly conducting fluid? One might suppose that perturbations of the velocity which play the role of a nonsymmetric component of $\mathbf{v}'$ after reaching a certain value $\sim vR_m^{-1/2}$ give rise to self-excitation of the magnetic field. The magnetic field then grows and its energy increases until the opposing forces of the Maxwell stresses of the magnetic field stop the growth of the perturbations of the velocity. Autostabilization of a tangential discontinuity in a highly conducting fluid can occur in this manner. The magnitude of the field at the discontinuity in the autostabilized state can be estimated by equating the inertial force $\sim \rho v v'/L$ to the magnetic force $\sim BB'/4\pi L$, and taking into account that $v' \sim v R_m^{-1/2}$ $B' \sim B R_m^{-1/2}$, this yields $B^2/4\pi \sim \rho v^2$. Autostabilized tangential discontinuities can, probably, play a significant role in astrophysics.

5. KINEMATIC MODELS OF THE EARTH'S DYNAMO

A most interesting example of a hydromagnetic dynamo is provided by the dynamo functioning in the liquid core of the earth which gives rise to the Earth's magnetic field. The expansion of the magnetic potential of the earth in spherical harmonics has the form (R_0 is the earth's radius)

$$V = R_0 \sum_{mn} \left(\frac{R_0}{r}\right)^{n+1} g_n{}^m P_n{}^m(\cos\vartheta)\cos m(\varphi - \varphi_n{}^m).$$

Here the principal role is played by the axial dipole g_1^0. The field undergoes secular variations in which the nondipolar components drift (predominantly towards the West) with different velocities $\dot{\varphi}_n^m$ of the order of 10^{-10} rad sec^{-1}, the axial dipole has a small oscillating part $\tilde{g}_1^0$, etc. The field of the plane model discussed above for $\omega = 0$ is quite similar to the above picture (in that picture the y-axis is the analog of latitude, while the z-axis is the analog of longitude). Paleomagnetic investigations indicate that the main dipole has reversed its polarity in the past. An analogy with the plane model shows that this phenomenon is quite natural, for if some perturbation should change the ratio of the velocities required for a stationary condition (an analog of this is the variation of the parameters α and β), then the stationary dynamo will be converted into an oscillatory one, and the field will reverse its sign with a frequency $\sim L^2/D_m$, until a stationary regime is reestablished as a result of an internal rearrangement of the system.

In [3] the present author discusses with the aid of the generation equations some kinematic models of the earth's dynamo. The simultaneous determination of the magnetic field and of the motion taking place in that field is quite complicated and requires a number of hypotheses on the details of the structure of the earth's core, and, therefore, a kinematic discussion represents a natural first stage in the solution of the problem. In a kinematic discussion the choice of the velocities is partially suggested by the form of the generation equation and by the expression for Γ, and partially by the symmetry of the problem and by data from observations of the earth's magnetic field, but to a large extent this choice is arbitrary.

The dynamics of the core is essentially related to the rotation of the earth, and, therefore, one can assume that the symmetry of **A**, **B** is determined by the angular velocity vector $\mathbf{\Omega}$. Taking into account the fact that **A** is a polar vector, while **B** is an axial vector we obtain

$$\mathbf{A} \propto [\mathbf{\Omega r}], \qquad \mathbf{B} \propto (\mathbf{\Omega r})[\mathbf{\Omega r}],$$

i.e.,

$$A(-z, s) = A(z, s), \qquad B(-z, s) = -B(z, s).$$

In accordance with the generation equations we have

$$\zeta(-z, s) = \zeta(z, s), \qquad a_e(-z, s) = -a_e(z, s),$$

where $\mathbf{v}_{ep} = \mathrm{curl}\, \mathbf{1}_\varphi a_e$, $\Gamma(-z, s) = -\Gamma(z, s)$, with one of the components of $\mathbf{v}'$ being symmetric, while the other one is antisymmetric. The axis of the earth's dipole is inclined, i.e., in addition to g_1^0 there also exists a transverse dipole g_1^1 which yields field components proportional to $\cos\varphi$ and $\sin\varphi$. It is natural to identify them with the field $\mathbf{B}^{(3)}$, which in accordance with (2.39) is related to the velocities $\mathbf{v}' \propto \sin\varphi, \cos\varphi$. Thus, the inclination of the dipole axis is closely related to the mechanism of generation, and is not accidental. Moreover we have

$$v_r'(-z, s) = -v_r'(z, s), \quad v_\vartheta'(-z, s) = v_\vartheta'(z, s).$$

In [3] we have assumed for the liquid core

$$\zeta = v_0 R_1^{-1}\zeta(x), \quad a_e = \alpha v_0 R_1 R_m^{-1} 3\mu\mu_s F(x),$$

$\mathbf{v}'$ is chosen in such a manner that

$$\Gamma = \beta^2 R_1^{-1} R_m^{-1} 3\mu\mu_s^2 G(x),$$

where v_0 is the unit of azimuthal velocity, R_1 is the external radius of the liquid core, $R_m = R_1 v_0/D_m$, $x = r/R_1$, $\mu = \cos\vartheta$, $\mu_s = \sin\vartheta$; $\zeta(x)$, $F(x)$, $g(x)$ are dimensionless functions ~ 1. The function $\zeta(x)$ grows monotonically with decreasing x and undergoes a discontinuity at the boundary of the internal core $r = R_2 = 0.4R_1$. The magnitude of the discontinuity is determined from the condition of equilibrium of the internal core

$$\int_{r \leqslant R_2} [\mathbf{jB}]_\varphi \, dV = 0.$$

The observed value of g_3^0 is very small. Calculations based on different models have shown that in order for this to occur $G(x)$ must change sign. By prescribing $G(x, \lambda)$ with one parameter λ, it can be so chosen as to obtain the correct value of $g_3^0/g_1^0 \approx -4 \times 10^{-2}$.

The stationary solution of the generation equations was obtained in [3] numerically by means of an electronic computer. The parameter β turned out to be quite large, for different models $\beta \approx 30$. It turns out that in contrast to the plane model the parameter α can be chosen arbitrarily within certain limits, and in order of magnitude $a_e \sim D_m$. The unit of velocity v_0 can be estimated from the observed value of the westward drift of the secular variations which is equal to 0.3 deg/year [4], and this gives $v_0 \approx 0.2$ cm/sec. The value of the field B_p is known from observations, but in order to determine the azimuthal field $B \sim R_m B_p$ we must know R_m. It can be estimated by three independent methods. Extrapolation of experimental data on conductivity yields $\sigma \approx 0.6 \times 10^{16}$ sec^{-1}, while for the model under investigation we obtain $R_m = 0.6 \times 10^4$, $[(B^2)_{av}]^{1/2} = 740$ G, $B_{max} = 2 \times 10^3$ G. The second method of making an estimate makes use of the inclination of the dipole axis: $g_1^1/g_1^0 = \kappa\beta R_m^{-1/2}$, where $\kappa \sim 1$. Agreement with the first estimate is obtained for $\kappa = 0.5$. The third method is a dynamic one. Equating the order of magnitude of the magnetic and the Coriolis forces in the core $B^2/4\pi R_1 \sim \rho 2\Omega v_0$, we obtain $B \sim 1.2 \times 10^3$G.

The model discussed above gives a Joule dissipation in the core of $Q_J = 3.8 \times 10^{19}$ erg/sec. This exceeds by a factor of 45 the value obtained by Bullard and Gellman in their kinematic model [5], and represents approximately $1/8$ of the total flux of heat observed at the earth's surface. Such a large value of Q_J shows that convection in the earth's core is not thermal. The efficiency of a heat engine is lower than the Carnot efficiency, and, therefore, in the earth's core where $\Delta T/T \sim 10^{-1}$ the liberation of heat would be several dozen times greater than Q_J, and this is improbable. A possible mechanism of convection is indicated in [6], it is associated with the floating upwards of the excess of the light component—silicon formed as the internal core is crystallized. Estimates show that the rate of liberation of gravitational energy equal to $2Q_J$ corresponds to the formation of the internal core during a time of the order of 10^9 years.

The theory of the hydromagnetic dynamo explains in a natural manner not only the existence of the basic geomagnetic field, but also such of its properties as the inclination of the axis of the geomagnetic dipole, the small value of the g_3^0 harmonic, the westward drift and the secular variations, the oscillations in the magnitude of the dipole and the reversal of the polarity of the dipole in the past.

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