# FLUCTUATIONS OF SPECTRAL ENERGY QUANTITIES IN THERMAL RADIATION

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It is demonstrated that the fluctuation-dissipation theorem [1-3] can be employed to calculate the correlation characteristics of fluctuations of energy quantities in thermal radiation. As examples the correlation functions for energy fluctuations in equilibrium radiation and for the output signal of a receiver measuring the energy radiated from a half-space are considered.

## 1. INTRODUCTION

In the general phenomenological theory of thermal fluctuations the correlation theory based on the fluctuation-dissipation theorem (FDT) has undergone particular development in recent years [1-3]. This theorem which is a generalization of the well-known Nyquist formula to dissipative systems of arbitrary physical nature has been proved originally for an arbitrary number of discrete fluctuating quantities describing the deviations of a system from the condition of thermodynamic equilibrium. The correlation theory of thermal fluctuations in electrodynamics, i.e., as applied to a distributed system—the electromagnetic field, was at first developed independently of the FDT on the basis of the concept of external fluctuation fields and of a generalization of Nyquist's theorem [4]. Later the FDT was applied to the equations of electrodynamics-at first in its discrete form [3], and then in a form generalized to arbitrary continuous dissipative systems [5].

As is well known, the correlation theory of random fields is restricted to moments of the first and of the second order. With respect to the electromagnetic field this means that for media which are in a state of equilibrium (or in a state sufficiently close to the equilibrium state so that the lack of equilibrium does not yet begin to affect the characteristics of the external fields) it is possible to calculate averages of arbitrary bilinear or quadratic functions of the field intensities. The question arises naturally whether one can extend the results obtained by means of the FDT and relating to moments of the second order to fluctuations of energy quantities, i.e., to moments of the fourth order. If this is possible, then, in particular, for equilibrium radiation in vacuo one should obtain the Einstein formula [6]:

$$\Psi_U(0) \equiv \langle U^2\rangle - \langle U\rangle^2 = \frac{\hbar^2}{c^3\pi^2}\omega^4 V n_\omega(n_\omega+1)\Delta\omega, \quad (1)$$

where U is the energy of the field in a sufficiently large volume V, while $n_\omega$ is the average occupation number of an oscillator of characteristic frequency $\omega$ at a temperature T.

Thus, the problem consists of expressing the fourth-order moments in terms of the second-order moments.

## 2. EVALUATION OF THE FOURTH-ORDER MOMENTS

It is natural to assume that the intensities **E** and **B** of the equilibrium electromagnetic field in vacuo (or in a transparent medium) being the result of a superposition of fields from numerous independent microscopic sources are distributed normally in virtue of the central limit theorem. Then in the region $\hbar\omega \ll kT$ where the classical description of the field is well known to be valid, we have for the fourth order moment the expression (we denote by $F_i$ the components of **E** and **B**)

$$\langle F_iF_kF_lF_m\rangle = \langle F_iF_k\rangle\langle F_lF_m\rangle + \langle F_iF_l\rangle\langle F_kF_m\rangle + \langle F_iF_m\rangle\langle F_kF_l\rangle. \quad (2)$$

The same expansion also holds for external inductions (electric and magnetic) since the latter can be represented in the form of linear operators in terms of the field vector components $F_i$. Thus, the fourth-order moment of external inductions can be expressed in terms of the second-order moments given by the FDT:

$$\langle D_{i\omega}D_{k\omega'}\rangle = i\hbar\,\mathrm{cth}\frac{\hbar\omega}{2kT}(\varepsilon_{ki}^* - \varepsilon_{ik})\delta(\omega+\omega')\delta(\mathbf{r}-\mathbf{r}'),$$

$$\langle M_{i\omega}M_{k\omega'}\rangle = i\hbar\,\mathrm{cth}\frac{\hbar\omega}{2kT}(\mu_{ki}{}^* - \mu_{ik})\,\delta(\omega+\omega')\,\delta(\mathbf{r}-\mathbf{r}'),$$
$$\langle D_{i\omega}M_{k\omega'}\rangle = 0. \qquad (3)^*$$

Here $D_{i\omega}$ and $M_{i\omega}$ denote the Fourier components of the electric and the magnetic external inductions respectively, while $\epsilon_{ik}$ and $\mu_{ik}$ are the electric and the magnetic permittivities of the medium.

But in an arbitrary frequency region the expression $\langle F_iF_kF_lF_m\rangle$ in which the order of the field operators $F_i$ is significant must play the role of the fourth order moment. Therefore, relation (2) for $F_i$ (or some other relation replacing (2)) requires separate justification.

We consider the radiation field in vacuo in a volume bounded by perfectly conducting walls. In this case the field operators can be represented in the form

$$F_i = f_i + f_i^+, \quad f_i = \sum_\lambda q_{i\lambda}a_\lambda, \qquad (4)$$

where a and $a_\lambda^+$ are the photon annihilation and creation operators satisfying the commutation relations

$$[a_\lambda, a_{\lambda'}{}^+] = \delta_{\lambda\lambda'}, \qquad [a_\lambda, a_{\lambda'}] = 0. \qquad (5)$$

The operator $a_\lambda^+a_\lambda$ is diagonal in the wave functions with definite occupation numbers, and its eigenvalues are equal to the occupation numbers of the oscillator of wave vector $\kappa_\lambda$.

In order to find $\langle F_iF_kF_lF_m\rangle$ we must know the diagonal matrix element of the operator $F_iF_kF_lF_m$ for the state of energy $E_m$. For this element we have

$$(F_iF_kF_lF_m)_n = (f_if_kf_lf_m)_n + \ldots + (f_i^+f_k^+f_l^+f_m^+)_n. \qquad (6)$$

From the meaning of the operators $a_\lambda$ and $a_\lambda^+$ it is clear that this sum will involve only those matrix elements which contain two creation and annihilation operators. Let us consider such an element, for example $(f_i^+f_kf_l^+f_m)_n$. Utilizing (4) and (5) we obtain

$$(f_i^+f_kf_l^+f_m)_n = \sum_{\lambda\neq\lambda'}[q_{i\lambda}{}^*q_{k\lambda}q_{l\lambda'}{}^*q_{m\lambda'}(a_\lambda{}^+a_\lambda)_n(a_{\lambda'}{}^+a_{\lambda'})_n$$
$$+ q_{i\lambda}{}^*q_{m\lambda}q_{k\lambda'}q_{l\lambda'}{}^*(a_\lambda{}^+a_\lambda)_n(a_{\lambda'}a_{\lambda'}{}^+)_n]$$
$$+ \sum_\lambda q_{i\lambda}{}^*q_{k\lambda}q_{l\lambda}{}^*q_{m\lambda}(a_\lambda{}^+a_\lambda a_\lambda{}^+a_\lambda)_n. \qquad (7)$$

The result of averaging the last term in (7) over the Gibbs distribution will be

$$\langle a_\lambda{}^+a_\lambda a_\lambda{}^+a_\lambda\rangle = \langle n_\lambda{}^2\rangle.$$

Using the Gibbs distribution it can be easily shown that $\langle n_\lambda^2\rangle = 2\langle n_\lambda\rangle^2 + \langle n_\lambda\rangle$, and, therefore,

$$\langle a_\lambda{}^+a_\lambda a_\lambda{}^+a_\lambda\rangle = \langle n_\lambda\rangle^2 + \langle n_\lambda\rangle(\langle n_\lambda\rangle + 1)$$
$$= \langle a_\lambda{}^+a_\lambda\rangle\langle a_\lambda{}^+a_\lambda\rangle + \langle a_\lambda{}^+a_\lambda\rangle\langle a_\lambda a_\lambda{}^+\rangle. \qquad (8)$$

But the other factors in the first sum in (7) can be averaged over the Gibbs distribution independently, since they refer to different oscillators which do not interact with one another. Substituting (8) into (7) we therefore obtain

$$\langle f_i{}^+f_kf_l{}^+f_m\rangle = \langle f_i{}^+f_k\rangle\langle f_l{}^+f_m\rangle + \langle f_i{}^+f_m\rangle\langle f_kf_l{}^+\rangle + \langle f_i{}^+f_l{}^+\rangle\langle f_kf_m\rangle$$

(the last term is equal to zero and is added to preserve symmetry).

In a similar manner one can obtain expressions for the remaining terms of (6), and this yields the result

$$\langle F_iF_kF_lF_m\rangle = \langle F_iF_k\rangle\langle F_lF_m\rangle$$
$$+ \langle F_iF_l\rangle\langle F_kF_m\rangle + \langle F_iF_m\rangle\langle F_kF_l\rangle. \qquad (9)$$

We now use the expansion (9) to calculate the central moments of the second order of physical quantities W bilinear in the field components. We consider quantities W described by hermitian operators and expressed linearly in terms of symmetrized products of the field operators:

$$W = {}^1/_2\sum a_{ik}(F_iF_k + F_kF_i),$$

where $a_{ik}$ are real coefficients. The central moment of the second order of W defined by

$$\psi_W = \mathrm{Re}\langle WW'\rangle - \langle W\rangle\langle W'\rangle,$$

is expressed linearly in terms of the quantities

$$\langle F_iF_kF_lF_m\rangle_0 \equiv {}^1/_4\mathrm{Re}\langle(F_iF_k + F_kF_i)(F_lF_m + F_mF_l)\rangle$$
$$-\langle F_iF_k\rangle_+\langle F_lF_m\rangle_+,$$

where

$$\langle F_iF_k\rangle_+ = {}^1/_2\langle F_iF_k + F_kF_i\rangle, \qquad (10)$$
$$\langle F_iF_k\rangle_- = {}^1/_2\langle F_iF_k - F_kF_i\rangle. \qquad (10a)$$

Using (9) we obtain

$$\langle F_iF_kF_lF_m\rangle_0 = \langle F_iF_l\rangle_+\langle F_kF_m\rangle_+ + \langle F_iF_l\rangle_-\langle F_kF_m\rangle_-$$
$$+ \langle F_kF_l\rangle_+\langle F_iF_m\rangle_+ + \langle F_kF_l\rangle_-\langle F_iF_m\rangle_-. \qquad (11)$$

The correlation functions for the Fourier components of external inductions (3) given by the FDT refer to the evaluation of the quantities (10), while in (11) the quantities (10a) also appear. But by slightly altering the derivation of the FDT one can easily obtain the correlation functions for the external inductions which also correspond to the correlation functions (10a):

*cth = coth.

$$\langle D_{i\omega}D_{k\omega'}\rangle = i\hbar(\varepsilon_{ki}^* - \varepsilon_{ik})\delta(\omega+\omega')\delta(\mathbf{r}-\mathbf{r}'),$$
$$\langle M_{i\omega}M_{k\omega'}\rangle = i\hbar(\mu_{ki}^* - \mu_{ik})\delta(\omega+\omega')\delta(\mathbf{r}-\mathbf{r}'),$$
$$\langle D_{i\omega}M_{k\omega'}\rangle = 0. \tag{3a}$$

Starting with (11), (3) and (3a) we must define the central moment of the fourth order of the Fourier components of the external electric induction in the following manner:

$$\langle D_{i\omega}D_{k\omega'}D_{l\omega_1}D_{m\omega_1'}\rangle$$
$$= -\hbar^2[\operatorname{cth}(\hbar\omega/2kT)\operatorname{cth}(\hbar\omega'/2kT)+1]$$
$$\times[(\varepsilon_{il}(\omega)-\varepsilon_{li}^*(\omega))(\varepsilon_{km}(\omega')$$
$$-\varepsilon_{mk}^*(\omega'))\delta(\omega+\omega_1)\delta(\omega'+\omega_1')$$
$$\times\delta(\mathbf{r}-\mathbf{r}_1)\delta(\mathbf{r}'-\mathbf{r}_1')+(\varepsilon_{im}(\omega)-\varepsilon_{mi}^*(\omega))(\varepsilon_{kl}(\omega')$$
$$-\varepsilon_{lk}^*(\omega'))\delta(\omega+\omega_1')\delta(\omega'+\omega_1)\delta(\mathbf{r}-\mathbf{r}_1')\delta(\mathbf{r}'-\mathbf{r}_1)]. \tag{12}$$

One can easily write down analogous formulas also for the fourth-order moments containing both the electric and the magnetic external induction, but we shall not do this, but shall assume in what follows that there are no magnetic losses.

The significance of expression (12) consists of the fact that it leads to the correct result in the calculation of the central moments (11) (and consequently also of $\Psi_W$) by means of the classical field equations. By neglecting in (11) the product of the antihermitian parts of the operators compared to the product of their hermitian parts, we introduce into (12) an error whose magnitude is proportional to $\hbar^2$. In the limiting transition to classical statistics ($\hbar \to 0$) this quantity related to the zero-point fluctuations of the field disappears, and then we obtain formula (2) for the fourth-order moments.

In conclusion we note that in an arbitrary frequency region expansions analogous to (9) are valid for moments of any even order. In this sense one can say that the multidimensional normal distribution is formally valid for the operators $F_i$.

## 3. ENERGY FLUCTUATIONS IN EQUILIBRIUM RADIATION

We now apply the results obtained above to the calculation of the correlation function for the fluctuations of electromagnetic energy in equilibrium radiation. The spectral density of the fluctuations can then be found as the Fourier transform of the correlation function.

We consider radiation in an unbounded homogeneous medium with no magnetic losses. We are interested in the case of a transparent medium or a vacuum, but we shall first of all solve the problem in the presence of absorption and will go over later to the limit $\operatorname{Im}\epsilon = 0$. As Landau and Lifshitz [3] have shown such a procedure leads to correct results for the second order moments. Moreover, we shall assume that the electromagnetic waves enter the volume under consideration through perfect reactive filters with a frequency characteristic $\Phi(\omega)$.

We write down the space-time expansion of the field intensities in terms of Fourier integrals:*

$$\mathbf{E}(\mathbf{r},t) = \int \mathbf{a}\Phi(\omega)\exp[i(\mathbf{kr}+\omega t)]\,d\omega d\mathbf{k},$$
$$\mathbf{H}(\mathbf{r},t) = \int \frac{[\mathbf{ka}]}{\varkappa}\Phi(\omega)\exp[i(\mathbf{kr}+\omega t)]\,d\omega d\mathbf{k},$$

$\kappa = \omega/c$. The amplitude $\mathbf{a} = \mathbf{a}(\omega, \mathbf{k})$ is expressed in the following manner in terms of the Fourier amplitude $\mathbf{g}$ of the external electric induction [3]:

$$a_i = [\varkappa^2\sqrt{\varepsilon}g_i - k_i(\mathbf{kg})]/\varepsilon\Delta,$$
$$\mu = 1, \quad \Delta = k^2 - \varkappa^2\sqrt{\varepsilon}. \tag{13}$$

Since in future we shall set $|\epsilon| \approx 1$ the formula for the electromagnetic energy in the volume V can be directly written in the form

$$U = \frac{1}{8\pi}\int_V (E_i^2 + H_i^2)\,d\mathbf{r}.$$

The time correlation function U is

$$\Psi_U(\tau) = \frac{1}{(8\pi)^2}\int T_{ik}T'_{i'k'}\langle g_i g_k' g_{1i'} g_{1k'}'\rangle$$
$$\times\Phi(\omega)\Phi(\omega')\Phi(\omega_1)\Phi(\omega_1')\exp\{i[(\omega+\omega')t$$
$$+(\omega_1+\omega_1')t'+(\mathbf{k}+\mathbf{k}')\mathbf{r}+(\mathbf{k}_1+\mathbf{k}_1')\mathbf{r}']\}\,d\omega d\omega'$$
$$\times d\omega_1 d\omega_1' d\mathbf{k} d\mathbf{k}' d\mathbf{k}_1 d\mathbf{k}_1' d\mathbf{r} d\mathbf{r}',$$

where, for example, $\mathbf{g}_1' = \mathbf{g}(\omega_1', \mathbf{k}_1)$, while $T_{ik}$ and $T'_{ik}$ are given by the expressions

$$T_{ik} = T_{ik}(\omega,\omega',\mathbf{k},\mathbf{k}') = \frac{1}{\Delta\Delta'}\Big[\delta_{ik}\varkappa^2\varkappa'^2\Big(1+\frac{\mathbf{kk}'}{\varkappa\varkappa'}\Big) - k_i k_k \varkappa'^2$$
$$- k_i'k_k'\varkappa^2 + k_i k_k'(\mathbf{kk}') - \varkappa\varkappa' k_i' k_k\Big],$$
$$T_{ik}' = T_{ik}(\omega_1, \omega_1', \mathbf{k}_1, \mathbf{k}_1').$$

In accordance with (12) the expression for the central moment of the Fourier amplitude of the external induction is

$$\langle g_i g_k' g_{1i'} g_{1k'}'\rangle = \Big(\frac{\hbar}{4\pi^3}\Big)^2\Big[\operatorname{cth}\Big(\frac{\hbar\omega}{2kT}\Big)\operatorname{cth}\Big(\frac{\hbar\omega'}{2kT}\Big)+1\Big]$$
$$\times \operatorname{Im}\varepsilon\operatorname{Im}\varepsilon'\,[\delta_{ii'}\delta_{kk'}\delta(\omega+\omega_1)\delta(\omega'+\omega_1')$$
$$\times\delta(\mathbf{k}+\mathbf{k}_1)\delta(\mathbf{k}'+\mathbf{k}_1') + \delta_{ik'}\delta_{i'k}\delta(\omega+\omega_1')\delta(\omega'+\omega_1)$$
$$\times\delta(\mathbf{k}+\mathbf{k}_1')\delta(\mathbf{k}'+\mathbf{k}_1)], \quad \varepsilon_{ik} = \varepsilon\delta_{ik}. \tag{15}$$

*$[\mathbf{k}\,\mathbf{a}] = \mathbf{k}\times\mathbf{a}$.

Substituting this expression into (14) and integrating over $\omega_1$, $\omega_1'$, $\mathbf{k}_1$ and $\mathbf{k}_1'$, we obtain the expression

$$\Psi_U(i)=\frac{2}{(8\pi)^2}\left(\frac{\hbar}{4\pi^3}\right)^2\int \operatorname{Im}\varepsilon \operatorname{Im}\varepsilon'\left(\operatorname{cth}\frac{\hbar\omega}{2kT}\operatorname{cth}\frac{\hbar\omega'}{2kT}+1\right)$$
$$\times|T_{ik}|^2|\Phi(\omega)|^2|\Phi(\omega')|^2$$
$$\times\exp\{i[(\omega+\omega')\tau+(\mathbf{k}+\mathbf{k}')\rho]\}\,d\omega d\omega' d\mathbf{k} d\mathbf{k}' d\mathbf{r} d\mathbf{r}',\quad(16)$$

where $\rho = \mathbf{r} - \mathbf{r}'$, $\tau = t - t'$.

We can go over to the limit $\operatorname{Im}\epsilon = 0$ in which we are interested not only in the final result, but also in the integrand. In order to do this we must set

$$\lim_{\operatorname{Im}\varepsilon\to 0}\frac{\operatorname{Im}\varepsilon}{|\Delta|^2}=\frac{\pi}{\varkappa^2}\operatorname{sign}\varkappa\cdot\delta(k^2-\varkappa^2).\quad(17)$$

Since in this section we want to obtain only formula (1) we must assume that the dimension $l$ of the volume V is considerably greater than the maximum wavelength of the waves transmitted by the filter. Therefore in (16) we can go over from integrating over $\mathbf{r}$, $\mathbf{r}'$ to integrating over $\mathbf{r}$ and $\rho$ and to extend the limits of integration with respect to $\rho$ to infinity. This yields

$$\int_{-\infty}^{\infty} e^{i(\mathbf{k}+\mathbf{k}')\rho}\,d\mathbf{r}\,d\mathbf{r}'=V(2\pi)^3\delta(\mathbf{k}+\mathbf{k}').\quad(18)$$

However, we must remember that this limiting transition has meaning only in the case if further integration over $\omega$ is performed over sufficiently slowly varying functions of $\omega$. The $\delta$-function appearing in (18) is in fact smeared out in $\mathbf{k}$-space over the volume $4\pi k^2\delta k \approx 1/l^3$ (or, which is the same thing, over the frequency interval $\delta\omega \approx c^3/4\pi l^3\omega^2$). Therefore, in order for (18) to be applicable we must require that in the integrand of (16) the functions $\Phi(\omega)$ and $\exp[i(\omega+\omega')\tau]$ should vary sufficiently slowly: $\tau \ll 1/\delta\omega$ and $\Delta\omega \gg \delta\omega$. The last requirement coincides with the condition for the validity of formula (1): the number $\Delta Z = l^3\omega^2\Delta\omega/\pi^2c^3$ of characteristic oscillations in the interval $\Delta\omega$ must be large.

Assuming that the condition $\tau \ll 1/\delta\omega$ is also satisfied we shall carry out in (16), taking (17) and (18) into account, the integration over $\mathbf{k}$, $\mathbf{k}'$ and $\omega'$ and we shall then go over to an integral over positive frequencies. We obtain

$$\Psi_U(\tau)=\frac{\hbar^2 V}{4c^3\pi^2}\int_0^\infty\left(\operatorname{cth}^2\frac{\hbar\omega}{2kT}-1\right)\omega^4|\Phi(\omega)|^4 d\omega.$$

If we now assume that $|\Phi(\omega)|$ is equal to unity in the interval $\Delta\omega$ in the neighborhood of the frequency $\omega$, and vanishes outside this interval, then

$$\Psi_U(\tau)=\Psi_U(0)=\frac{\hbar^2 V\omega^4}{4c^3\pi^2}\left(\operatorname{cth}^2\frac{\hbar\omega}{2kT}-1\right)\Delta\omega,\quad(19)$$

which, in view of the fact that $\coth(\hbar\omega/2kT) = 2(n_\omega + 1/2)$, coincides with (1).

The fact that the energy correlation function is independent of $\tau$ is, of course, a consequence of the assumptions made in the derivation of (19). In actual fact the width $\delta\tau$ of the function $\Psi_U(\tau)$ is determined by the smaller one of the intervals $\Delta\omega$ and $\delta\omega$, and in particular by: $\delta\tau \approx 1/\delta\omega$ for $\delta\omega \lesssim \Delta\omega$ and $\delta\tau \approx 1/\Delta\omega$ for $\delta\omega \gtrsim \Delta\omega$. We note that in the last case $\Psi_U(0) \sim (\Delta\omega)^2$.

## 4. FLUCTUATIONS OF THE ENERGY FLUX IN EQUILIBRIUM RADIATION

We now consider the signal fluctuations at the output of an idealized energy receiver which we shall imagine in the form of a plane area $\sigma$ which absorbs all the radiation incident on it. The output signal of such a receiver is given by

$$i(t)=\int_\sigma \mathbf{S}(\mathbf{r},t)\,d\mathbf{r},\quad(20)$$

where $\mathbf{S}(\mathbf{r}, t)$ is the Poynting vector, while the integral is extended over all the waves incident on the absorbing side of the area. In order to take into account the finite time constant of such a device we shall assume that the signal $i(t)$ is passed through a low frequency filter of frequency characteristic $\Phi_0(\omega)$; we shall denote the output signal of such a filter by $I(t)$. Moreover, it is assumed, as before, that the radiation incident on the receiver is passed through a filter of frequency characteristic $\Phi(\omega)$.[1)]

We take the receiver to be situated in the (1, 2) plane. In order to pick out the waves propagated in the positive direction of the 3 axis we must in the expansions of the fields into Fourier integrals integrate over $k_1$ and $k_2$ from $-\infty$ to $\infty$, and over $k_3$ from $-\infty$ to 0 for $\omega > 0$ and from 0 to $\infty$ for $\omega < 0$. Taking into account the effect of the high frequency and the low frequency filters we obtain the following expression for the output signal of the complete arrangement:[2)]

[1)]Fluctuations at the output of the receiver can also arise due to thermal noise in the receiver itself. We do not take this source of fluctuations into account by assuming that the temperature of the receiver is sufficiently low, while the power of the radiation incident on the receiver is great.

[2)]Strictly speaking, here we should also take into account the "zero-point" radiation field of the light filter placed before the receiver. However, a special analysis shows that in the case $\Delta\omega \gg \Delta\Omega$ this need not be done. Restricting ourselves here for the sake of simplicity to the expression for the current in the form (21) we shall later give without derivation only the final expression for the opposite case $\Delta\omega \ll \Delta\Omega$.

$$I(t) = \frac{}{4\pi} S_{ik} g_i g_k' \Phi(\omega) \Phi(\omega') \Phi_0(\omega + \omega') \exp\{i[(\omega + \omega')t + (\mathbf{k} + \mathbf{k}')\mathbf{r}]\} \, d\omega \, d\omega' \, d\mathbf{k} \, d\mathbf{k}' \, d\mathbf{r}, \tag{21}$$

$$S_{ik} = \frac{1}{\varkappa' \Delta \Delta'} \{\varkappa^2 \varkappa'^2 (k_3' \delta_{ik} - k_i' \delta_{k3}) - \varkappa'^2 k_i (k_3' k_k - \delta_{k3} (\mathbf{k}\mathbf{k}'))\}, \qquad |\varepsilon| \approx 1.$$

The time correlation function of the output signal is given by the expression

$$\Psi_I(\tau) = \left(\frac{c}{4\pi}\right)^2 \int S_{ik} S'_{i'k'} \langle g_i g_k' g_{1i'} g_{1k'}' \rangle \times \Phi(\omega)\Phi(\omega')\Phi(\omega_1)\Phi(\omega_1')\Phi_0(\omega + \omega')\Phi_0(\omega_1 + \omega_1') \times \exp\{i[(\omega + \omega')t + (\omega_1 + \omega_1')t' + (\mathbf{k} + \mathbf{k}')\mathbf{r} + (\mathbf{k}_1 + \mathbf{k}_1')\mathbf{r}']\} d\omega d\omega' d\omega_1 d\omega_1' d\mathbf{k} d\mathbf{k}' d\mathbf{k}_1 d\mathbf{k}_1' d\mathbf{r} d\mathbf{r}'.$$

Substitution of (15) yields

$$\Psi_I(\tau) = \left(\frac{\hbar}{4\pi^3}\right)^2 \left(\frac{c}{4\pi}\right)^2 \int \left(\operatorname{cth}\frac{\hbar\omega}{2kT}\operatorname{cth}\frac{\hbar\omega'}{2kT} + 1\right) \operatorname{Im}\varepsilon \operatorname{Im}\varepsilon' \times (|S_{ik}|^2 + S_{ik}R_{ik}) |\Phi(\omega)|^2 |\Phi(\omega')|^2 |\Phi_0(\omega + \omega')|^2 \times \exp\{i[(\omega + \omega')\tau + (\mathbf{k}_0 + \mathbf{k}_0')\rho]\} \, d\omega d\omega' d\mathbf{k} d\mathbf{k}' d\mathbf{r} d\mathbf{r}', \tag{22}$$

$\mathbf{k}_0 = (k_1, k_2)$; at the same time we also have $R_{ik} = R_{ik}(\omega, \omega', \mathbf{k}, \mathbf{k}') = S_{ki}(-\omega', -\omega, -\mathbf{k}', -\mathbf{k})$.

Transition to the limit $\operatorname{Im} \epsilon = 0$ yields

$$|S_{ik}|^2 \operatorname{Im}\varepsilon \operatorname{Im}\varepsilon' = \pi^2 \{\varkappa^2\varkappa'^2 - [(\mathbf{k}_0\mathbf{k}_0')^2 - (k_3k_3')^2]\} \times \delta(k^2 - \varkappa^2)\delta(k'^2 - \varkappa'^2) \operatorname{sign}\varkappa \operatorname{sign}\varkappa',$$

$$S_{ik}R_{ik} \operatorname{Im}\varepsilon \operatorname{Im}\varepsilon' = 2\pi^2 |\varkappa| |\varkappa'| k_3 k_3' \delta(k^2 - \varkappa^2)\delta(k'^2 - \varkappa'^2).$$

Since the time constant of the receiver is large compared to the periods of the radiation incident on it, we can set $\Phi_0(\omega + \omega') = 0$ for $\omega > 0$, $\omega' > 0$. Then, going over in (22) to integration over positive values of $\omega$, $\omega'$, $k_3$ and $k_3'$, we obtain

$$\Psi_I(\tau) = \frac{\hbar^2 c^2}{128\pi^6} \int_0 \left(\operatorname{cth}\frac{\hbar\omega}{2kT}\operatorname{cth}\frac{\hbar\omega'}{2kT} - 1\right) H(\omega, \omega') \cos(\omega - \omega')\tau \times |\Phi(\omega)|^2 |\Phi(\omega')|^2 |\Phi_0(\omega + \omega')|^2 d\omega d\omega', \tag{23}$$

$$H(\omega, \omega') = \int [\varkappa^2\varkappa'^2 + 2\varkappa\varkappa' k_3 k_3' - (\mathbf{k}_0\mathbf{k}_0')^2 + k_3^2 k_3'^2] \times \delta(\varkappa^2 - k^2)\delta(\varkappa'^2 - k'^2) e^{i(\mathbf{k}_0 + \mathbf{k}_0')\rho} \, d\mathbf{k} \, d\mathbf{k}' \, d\mathbf{r} \, d\mathbf{r}'. \tag{24}$$

If the dimensions of the receiver are considerably greater than the wavelengths of the filtered radiation [as was already assumed in (20)], then in expression (24) we can go over to integration over $\rho$ and $\mathbf{r}$ letting the limits on $\rho$ become infinite. Also we have

$$\int e^{i(\mathbf{k}_0 + \mathbf{k}_0')\rho} d\rho d\mathbf{r} = \sigma (2\pi)^2 \delta(\mathbf{k}_0 + \mathbf{k}_0'),$$

where $\sigma$ is the area of the receiver. Then we have

$$H(\omega, \omega') = \sigma(2\pi)^2 \int (\varkappa k_3' + \varkappa' k_3)^2 \delta(\varkappa^2 - k^2)\delta(\varkappa'^2 - k'^2) \times \delta(\mathbf{k} + \mathbf{k}_0') d\mathbf{k} d\mathbf{k}' = \sigma\pi^3 \left[\varkappa\varkappa'(\varkappa^2 + 2\min(\varkappa^2, \varkappa'^2) + \varkappa'^2) - (\varkappa^2 - \varkappa'^2) \ln \frac{\varkappa + \varkappa'}{(|\varkappa^2 - \varkappa'^2|)^{1/2}}\right],$$

$$H(\omega, \omega) = 4\pi^3 \varkappa^4 \sigma. \tag{25}$$

Finally, we shall assume that the frequency characteristic of the filter $|\Phi(\omega)|$ is different from zero and is equal to unity only in the narrow interval $\Delta\omega$ in the neighborhood of the frequency $\omega$, while the characteristic of the low frequency filter $|\Phi_0(\omega)|$ is different from zero is equal to unity in the interval $(0, \Delta\Omega)$. Then for $\Delta\omega \gg \Delta\Omega$ formulas (23) and (25) yield

$$\Psi_I(\tau) = \sigma \frac{\hbar^2\omega^4}{4\pi^3 c^2} n_\omega (n_\omega + 1) \Delta\omega \frac{\sin\Delta\Omega\tau}{\tau}$$

$$\Psi_I(0) = \sigma \frac{\hbar^2\omega^4}{4\pi^3 c^2} n_\omega (n_\omega + 1) \Delta\omega\Delta\Omega. \tag{26}$$

But in the case $\Delta\omega \ll \Delta\Omega$ it is necessary to take into account the "zero-point" radiation from the light filter. Without reproducing the corresponding calculations we shall only point out that in this case

$$\Psi_I(\tau) = \sigma \frac{\hbar^2\omega^4}{8\pi^3 c^2} n_\omega \left[n_\omega \frac{\sin^2(\Delta\omega\tau/2)}{(\tau/2)^2} + 2\frac{\sin\Delta\omega\tau}{\tau} \frac{\sin(\Delta\omega\tau/2)}{\tau/2}\right],$$

$$\Psi_I(0) = \sigma \frac{\hbar^2\omega^4}{8\pi^3 c^2} n_\omega [n_\omega (\Delta\omega)^2 + 2\Delta\omega\Delta\Omega]. \tag{27}$$

From (26) and (27) it can be seen that the correlation time of the wave fluctuations is determined by the smaller of the two intervals $\Delta\omega$ and $\Delta\Omega$ and, in accordance with (23) and (24), for sufficiently narrow intervals $\Delta\omega$ and $\Delta\Omega$ it depends very little on the dimensions of the receiver. Naturally, the results of this section agree completely with those obtained earlier for energy fluctuations. The introduction of a finite transmission band $\Delta\Omega$ in the low frequency channel means that a finite time constant $\tau_0 \approx 1/\Delta\Omega$ is ascribed to the receiver. As a result of this the output signal turns out to be approximately proportional to the energy absorbed during a time $\tau_0$ from a volume $c\tau_0\sigma$. Therefore for large volumes (small values of $\Delta\Omega$) both $\Psi_U(0)$ and $\Psi_I(0)$ are proportional to $\Delta\omega$ [cf. (26)].

The results obtained above can be easily extended to a receiver which, is as before, perfectly black, but at an arbitrary temperature. In this case the expressions for the electromagnetic fields at the surface of the receiver will consist of a sum of two terms one of which is due to waves of radiation being propagated from the left half-space, while the other is due to radiation from the receiver itself. The effect of the second term can be easily taken into account by the method described earlier, by imagining that the right half-space is filled by a rarified medium at a temperature $T'$. Then, in virtue of the statistical independence of the sources situated in the left and the right half-spaces, we find that in order to obtain the correlation function of the output signal it is necessary in this case to add to the expression (26) [or (27)] the same expression but taken at a temperature $T'$.

The situation is more complicated in the case of a receiver which is not perfectly black, since in this case in order to obtain $\Psi_I(\tau)$ we must solve the electrodynamic problem taking into account both the boundary conditions at the surface of the receiver, and also the fact that the receiver itself is a source of a fluctuating field.

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