IRREDUCIBLE REPRESENTATIONS OF THE SU_3 *GROUP*

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Formulas are found for the irreducible parts and the multiplicities that are contained in the product of two irreducible representations of SU_3. Using the Young diagrams, a formula is derived for the isotopic and hypercharge content of an irreducible representation.

1. INTRODUCTION

It is known [1] that there are four simple Lie groups of rank two—SU_3, C_2, B_2 and G_2, any of which can be used to describe the symmetry of elementary particles. They all contain the isospin group SU_2 as a subgroup 1), so that a complete set contains at least the three commuting operators:

$$H^2 = H_1^2 + {}^1/_4(E_1E_{-1} + E_{-1}E_1), \quad H_1, \quad H_2,$$

whose eigenvalues can be identified with the quantum numbers for the square of the isospin t^2, its projection t_3, and the hypercharge Y.

We note, however, that these operators constitute a complete set only for SU_3. In fact from group theory one knows (cf., for example, [2]), that for a group of rank two the number of additional operators, i.e., operators included in a complete set together with the operators H_1 and H_2, is given by the formula $n = r/2 - 3$, where r is the order of the group. From this formula we see that a complete set for the groups C_2 and B_2 contains, in addition to H_1, H_2 and H^2, one additional operator, while the complete set for G_2 contains three additional operators.2) Thus the groups SU_3, C_2, B_2 and G_2 differ in the number of conserved nonadditive quantum numbers.

From the theoretical point of view, the most attractive group is SU_3, since to it there correspond only the already known conserved quantum numbers t^2, t_3 and Y. Furthermore it is interesting that SU_3 describes the symmetry of the weak and electromagnetic interactions in a completely analogous way to the symmetry of the strong interaction.[3]

The present experimental facts are also best described by SU_3, so we shall consider its representations. This point should not, however, be regarded as definitely established. The accumulation of experimental information may markedly change the situation, bringing to the fore some other group of second or even higher rank, or leading to a complete rejection of the hypothesis of higher symmetries.

1)From the operators of this group we can construct the three operators

$$E_0 = H_1 / r_1(1), \; E_+ = E_1 / i\sqrt{2}r_1(1), \; E_- = E_{-1} / i\sqrt{2}r_1(-1),$$

satisfying the commutation relations for the isospin operators $[E_0, E_\pm] = \pm E_\pm$, $[E_+, E_-] = 2E_0$, while the operator

$$E^2 = E_0^2 + {}^1/_2(E_+E_- + E_-E_+) = H^2 / r_1^2(1)$$

corresponds to the square of the isotopic spin and commutes with H_2 (cf.[1]).

2)For example, among the operators for the groups B_2 and G_2 one can choose three operators:

$$E_0' = H_2 / r_2(\gamma), \; E_\pm' = E_{\pm\gamma} / i\sqrt{2}r_2(\pm\gamma)$$

($\gamma = 4$ for B_2 and $\gamma = 3$ for G_2), satisfying the commutation relations for the isospin operators and commuting with H_1, H_2 and H^2. The operator

$$H'^2 = r_2^2(\gamma)E'^2 = r_2^2(\gamma)\,(E_0'^2 + {}^1/_2(E'_+E'_- + E'_-E'_+))$$

appears in the complete set of commuting operators along with the operators H_1, H_2 and H^2. Thus the groups B_2 and G_2 (and also C_2, which is locally isomorphic to B_2) contain two independent isospin subgroups. The last remark is an obvious consequence of the fact that the group $O_4 = O_2 \times O_2$ is a subgroup of these groups.

2. TWO POSSIBLE APPROACHES TO THE THEORY OF THE REPRESENTATIONS OF SU_3

SU_3 is the group of transformations produced by unitary unimodular matrices A of order three in the three-dimensional complex space C_3. We choose as a basis vector in this space the eigenvector x^i ($i = 1, 2, 3$) of the operators $t^2 = 3H^2$, $t_3 = \sqrt{3}H_1$ and $Y = 2H_2$. From the explicit form of these operators in the representation given in [1], we easily conclude that the first two components $z = (x^1, x^2)$ of the vector x^i correspond to the eigenvalues $t_3 = \pm 1/2$, $t = 1/2$ and $Y = 1/3$, while $s = x^3$ corresponds to the eigenvalues[3)] $t_3 = t = 0$, $Y = -2/3$ (while the eigenvalue of t^2 is $t(t+1)$).

We introduce the vector x_i, complex conjugate to the vector x^i: $x_i = (x^i)^*$, and form the product

$$T^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = x^{i_1} x^{i_2} \ldots x^{i_p} x_{j_1} x_{j_2} \ldots x_{j_q}. \tag{2.1}$$

This product, formed in the three-dimensional complex vector space C_3, we call a mixed tensor with p contravariant and q covariant indices, or simply the tensor of rank $(p) + (q)$. It is obvious that such a tensor transforms according to some representation of SU_3. In fact the aggregate of all tensors of rank $(p) + (q)$, i.e., all systems of 3^{p+q} numbers $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ form a 3^{p+q}-dimensional complex space R which, under the transformation A of SU_3 ($x'^{i'} = A^{i'}_{i} x^i$, $x'_{i'} = A^{i}_{i'} x_i$, where $A^{i'}_{i} = (A_{i'}{}^{i})^*$), transforms according to the matrix B:

$$T'^{i_1' \ldots i_p'}_{j_1' \ldots j_q'} = B^{i_1' \ldots i_p';\, j_1 \ldots j_q}_{i_1 \ldots i_p;\, j_1' \ldots j_q'} T^{i_1 \ldots i_p}_{j_1 \ldots j_q}, \tag{2.2}$$

where

$$B^{i_1' \ldots i_p';\, j_1 \ldots j_q}_{i_1 \ldots i_p;\, j_1' \ldots j_q'} = A^{i_1'}_{i_1} \ldots A^{i_p'}_{i_p} A^{j_1}_{j_1'} \ldots A^{j_q}_{j_q'},$$

which is the matrix of the tensor representation.

We introduce the concept of an irreducible tensor: a tensor (p, q) which transforms according to an irreducible representation of SU_3, or an irreducible tensor, is one whose space contains no invariant subspaces. (A subspace R_1 of R is said to be invariant with respect to the representation given by the matrices B if R_1 is transformed into itself by all the matrices B.)

It is obvious that to understand the properties of group representations we need only consider irreducible representations, since by definition any reducible tensor splits into a sum of irreducible ones (or its space is made up of several invariant subspaces).

We shall prove several theorems which enable us to expand tensors into their irreducible parts:

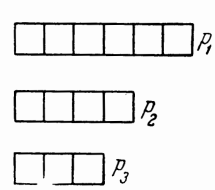
Theorem 1. The tensors $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$, for which the contraction on a particular pair of indices i_1, j_1 is zero, form a subspace of R which is invariant with respect to the tensor representation B (the contraction or trace of the tensor $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ on the indices i_1, j_1 is the sum $T^{k i_2 \ldots i_p}_{k j_2 \ldots j_q}$).

The proof follows immediately from (2.2) if we use the unitarity condition for the matrix A:

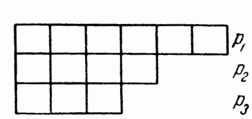
$$A_{i_1}{}^{k'} A_{k'}{}^{j_1} = \delta_{i_1}{}^{j_1}.$$

Theorem 2. The tensors whose upper and lower indices are separately symmetrized according to particular Young diagrams form a subspace of R invariant with respect to the representation B. Let us consider a tensor $T^{i_1 \ldots i_p}$ symmetrized according to the Young diagram (p_1, p_2, p_3). (The proof for a mixed tensor of rank $(p) + (q)$ is analogous, since the upper and lower indices are symmetrized independently.) By definition this means that the tensor $T^{i_1 \ldots i_p}$ is obtained from $T_1^{i_1 \ldots i_p} = x^{i_1} \ldots x^{i_p}$ as the result of a symmetrization operation which is conveniently described as follows: to each index i_k we assign a box, to a set of k symmetric indices we assign a row of k boxes, and to l antisymmetric indices a column of l boxes. Since $i_k = 1, 2, 3$, we must have $l \leq 3$ (otherwise there will be identical indices among the antisymmetric ones, so that the corresponding tensor is zero).

We divide the total number of indices p into three groups, in each of which there are p_n boxes ($n = 1, 2, 3$, $p_1 \geq p_2 \geq p_3$, $p_1 + p_2 + p_3 = p$). All the indices within one group are symmetrized. Each such operation is described by the pattern consisting of a row of p_1, p_2 and p_3 boxes, respectively:



We now antisymmetrize on the boxes located vertically above one another, and agree to describe this operation by the socalled Young pattern (p_1, p_2, p_3):



(2.3)

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3)This choice of basis vector corresponds to the quark or ace models proposed by Gell-Mann[4] and Zweig[5].

The tensor $T^{i_1 \cdots i_p}$ obtained from the initial tensor $T_1^{i_1 \cdots i_p}$ by means of the symmetrization operation described by the Young pattern (2.3) is called the tensor symmetrized according to the Young pattern (p_1, p_2, p_3).

Since the Young pattern is completely determined by the numbers p_1, p_2, p_3, which are called its signature, it is also determined by the numbers $p_3, p_2 - p_3, p_1 - p_2$, which give respectively the numbers of antisymmetrized triples of indices, antisymmetrized pairs, and the number of symmetrized indices.[4)] It is easy to show that the operation (2.2) transforms symmetric or antisymmetric indices among themselves, i.e., the transformed tensor $T'^{i_1' \cdots i_p'}$ is characterized by the same numbers $p_3, p_2 - p_3, p_1 - p_2$, and thus by the same Young pattern (2.3), which proves Theorem 2.

From these theorems it follows that an irreducible tensor is one whose upper and lower indices are separately symmetrized according to some Young pattern, and for which all traces are zero. But there is still one additional feature of the group SU_3 which greatly simplifies the properties of irreducible tensors: two antisymmetric co- (contra-) variant indices are equivalent to one contra- (co-) variant index, i.e., one can raise (lower) indices by means of the antisymmetric special tensors ϵ^{ijk} (ϵ_{ijk}).

To prove this we note that the tensors

$$x^{i_1}x^{i_2}x^{i_3}\varepsilon_{i_1 i_2 i_3} \tag{2.4}$$

and

$$x_{j_1}x_{j_2}x_{j_3}\varepsilon^{j_1 j_2 j_3} \tag{2.5}$$

together with

$$x^i x_i \tag{2.6}$$

are scalars for SU_3. In fact, under the transformation $x^i \rightarrow x'^{i'} = A_i^{i'} x^i$, we have, for example, for (2.4)

$$\begin{aligned} \varepsilon_{i_1 i_2 i_3} x^{i_1}x^{i_2}x^{i_3} \rightarrow \varepsilon_{i_1' i_2' i_3'} x'^{i_1'}x'^{i_2'}x'^{i_3'} &= \varepsilon_{i_1' i_2' i_3'} A_{i_1}^{i_1'} A_{i_2}^{i_2'} A_{i_3}^{i_3'} x^{i_1}x^{i_2}x^{i_3} \\ &= \varepsilon_{i_1 i_2 i_3} \det A x^{i_1}x^{i_2}x^{i_3} = \varepsilon_{i_1 i_2 i_3} x^{i_1}x^{i_2}x^{i_3}, \end{aligned} \tag{2.7}$$

since $\det A = 1$ because the matrices A are unimodular.

From a comparison of (2.4), (2.5) and (2.6) we get the statement made above: the antisymmetric product $\epsilon_{j i_1 i_2} x^{i_1} x^{i_2}$ of two contravariant vectors transforms like the covariant vector x_j, and conversely the antisymmetric product $\epsilon^{i j_1 j_2} x_{j_1} x_{j_2}$ of two covariant vectors transforms like the contravariant vector x^i.

Because the operations of antisymmetrization and raising (or lowering) of indices are equivalent, there are two possible approaches to the theory of the tensor representations of SU_3.

1. Assume that all tensors are contravariant.[6] Then the mixed tensor of rank $(p) + (q)$ is equivalent to a contravariant tensor of rank $p + 2q$, in which $2q$ indices are antisymmetric in pairs. As shown above (Theorem 2), a contravariant tensor is irreducible if it corresponds to some Young pattern, i.e., if it is characterized by a triple of numbers (p_1, p_2, p_3) constituting the signature of a Young pattern. But it follows from (2.7) that a completely antisymmetric third rank tensor transforms like a scalar. To such a tensor there corresponds a Young pattern with three boxes in one column. Since multiplication by a scalar does not change the properties of the tensor under transformations of the group (and these are all that we are interested in), all columns of three boxes in the Young pattern (2.3) can be dropped. Thus, in fact, an irreducible tensor for SU_3 is given by two numbers (f_1, f_2) (with $f_1 = p_1 - p_3$ and $f_2 = p_2 - p_3$), which give the numbers of boxes in the first and second rows of the Young pattern.

2. Assume that all tensors are symmetric in their upper and lower indices separately, i.e., every antisymmetric pair of superscripts (subscripts) has been rewritten as one subscript (superscript).[7] As shown above (Theorem 1), such a tensor is irreducible if the contraction (trace) is zero. (This is sufficient since as a result of the symmetry in all upper and all lower indices the tensor has only one trace.) It is given completely by the numbers of upper and lower indices p and q, and is denoted by (p, q).

There is an obvious connection between the numbers (f_1, f_2) and (p, q) which label the irreducible representations in the two approaches: $p = f_1 - f_2$, $q = f_2$. This follows from the fact that f_2 is the number of antisymmetric pairs of indices in approach 1, and these are represented in approach 2 by the same number of covariant indices. From now on we shall use the same notation (p, q) for both approaches. In this notation the number of boxes in the first row of the Young pattern corresponding to the tensor (p, q) is $q + p$, the number in the second row is q, i.e., p is the number of boxes "overhanging" beyond the second row.

Both approaches are of course equivalent from the point of view of the theory of representations

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4)We note that a tensor symmetrized according to some Young pattern is completely symmetric with respect to a permutation of indices which corresponds to interchanging two columns having the same number of boxes.

of SU_3. But to get results as quickly as possible in representation theory one must know how to use both approaches, since each of them is especially effective in studying certain aspects of the theory. In Secs. 3 and 4 we shall give examples of the application of both approaches.

3. FIRST APPROACH. ISOTOPIC CONTENT OF SUPERMULTIPLETS

Let us find the isotopic content of an irreducible representation (supermultiplet) of SU_3. As we have seen, the basis vector of the group x^i consists of two isotopic components z and s, corresponding to isotopic spins $1/2$ and 0, and having the respective hypercharges $1/3$ and $-2/3$.

Let us choose as the basis vector in the space of the irreducible representation (p, q) the eigenvector of the operators $\mathbf{t}^2$, t_3 and Y, and find the eigenvalues of these operators. To do this we fix the indices placed in the Young pattern (p, q). If the index in a given box is 1 or 2, we replace it by z, if it is 3, we substitute s.

It is easy to see that there are only two possible independent arrangements of symbols in the columns of the Young pattern:

a) the symbol z is in both the first and second boxes of the column of the Young pattern;

b) the symbol z appears in the first box of the column in the Young pattern; the second box contains s (the case where the first box contains s and the second z reduces to this by a change of sign; because of the antisymmetry in the indices in a given column, s cannot appear in both boxes in a column, since the corresponding tensor component is 0).

We denote the number of times z appears in the first row by k, and the number of times it appears in the second row by l. Obviously $q \leq k \leq q + p$ and $0 \leq l \leq q$ (for if k were less than q, the same symbol would appear in both boxes of the column, and this is impossible). Since the hypercharge is an additive quantum number, it is taken into account very simply: the total number of entries z is equal to $k + l$, the total number of s entries is $p + 2q - (k + l)$, so that

$$Y = {}^1\!/_3(k + l) - {}^2\!/_3[p + 2q - (k + l)]$$
$$= k + l - {}^2\!/_3(p + 2q).$$

Thus the representations (p, q) correspond to a particle with integer hypercharge only when $p + 2q$ is divisible by 3.

Now let us find t. From spinor theory we know that the antisymmetric combination of two spinors corresponds to spin 0, while the symmetric combination gives spin 1. Thus a column containing the symbol z in the first and second boxes gives no contribution to the spin t of the tensor component corresponding to the given arrangement of indices in the Young pattern. But symbols z appearing in the first box of columns having s in the second box are in general not symmetric under permutation (this operation on the Young pattern is an antisymmetrization), so that they do not give an irreducible representation of the spinor group SU_2 and do not correspond to a definite spin. (It is easily shown, in the same way as above for SU_3, that the Young pattern corresponding to an irreducible representation of SU_2 contains a single row, so that all the irreducible representations (except the scalar) are symmetric under interchange of spinor indices.)

To obtain the basis vector of the representation having definite values of t, t_3 and Y, we must form a linear combination of components with the same Y and t_3 which is symmetric under interchange of all the $k - l$ symbols z that are antisymmetrized with symbols in the second row; in addition this linear combination must remain antisymmetric under permutation of two z's that are in the same column. Such a linear combination has the form:

$$\Sigma \begin{array}{|c|c|c|c|c|c|} \hline z & z_1 & z_2 & z & z_3 & s_3 \\ \hline z & s_1 & s_2 & z & \multicolumn{2}{c}{} \\ \cline{1-4} \end{array} \quad ,$$

where we sum over all possible permutations of the symbols z_i, and this symbol corresponds to the component of the basis vector with

$$t = {}^1\!/_2(k - l), \quad q \leqslant k \leqslant q + p, \quad 0 \leqslant l \leqslant q,$$
$$Y = k + l - {}^2\!/_3(p + 2q). \qquad (3.1)$$

It is easily seen that the quantities

$$t_{max}, \quad Y(t_{max}); \qquad t_{min}, \quad Y(t_{min});$$
$$Y_{max}, \; t(Y_{max}); \quad Y_{min}, \; t(Y_{min})$$

are uniquely determined for any representation, and have the form

$$t_{max} = (p + q)\,/\,2, \qquad Y(t_{max}) = (p - q)\,/\,3;$$
$$t_{min} = 0, \qquad Y(t_{min}) = -2(p - q)\,/\,3;$$
$$Y_{max} = (p + 2q)\,/\,3, \quad t(Y_{max}) = p\,/\,2;$$
$$Y_{min} = -(q + 2p)\,/\,3, \quad t(Y_{min}) = q\,/\,2. \qquad (3.2)$$

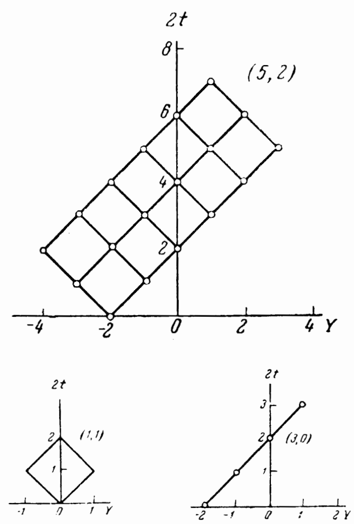
The isotopic content of the irreducible representation (p, q) is conveniently shown by points in the Y, t plane. From (3.1) it follows that these points are the intersections of the lines

$$Y - 2t = 2l - {}^2\!/_3(p + 2q)$$

and

$$Y + 2t = 2k - {}^2/_3(p + 2q),$$

i.e., lines passing through the points $t = 0$, $Y = 2[l - Y_{max}]$ and $t = 0$, $Y = 2(k - Y_{max})$ at angles of $\pi/2$ and $3\pi/2$ to the Y axis, respectively. Graphs of the isotopic content[5)] of the representations (5, 2), (1, 1) and (3, 0) are shown in the figure.



Isotopic content of the representations (5,2), (1,1) and (3,0)

We note that the graph for the conjugate representation is gotten by reflecting in the 2t axis. (The conjugate of a representation consists of the matrices that are complex conjugate to the matrices of the given representation. It is then easy to show that the tensor conjugate to the irreducible tensor (p, q) has the indices (q, p). In approach 1 it corresponds to the Young pattern (q, p) which is obtained from the pattern (p, q) by replacing all single-box columns by two-box columns and conversely.) If this is actually so, then $Y_{max}(p, q)$ should be equal to $-Y_{min}(q, p)$, which is clear from formula (3.2). The self-conjugate representation (p, p) thus has a graph in the form of a square whose diagonal is the 2t axis. We also note that only a self-conjugate representation contains the component $t = Y = 0$ (since $Y(0) = 0$ implies $p = q$).

The dimensionality of the (p, q) representation is given by the formula

$$N = \sum_{t=0}^{t_{max}} (2t + 1)n(t), \qquad (3.3)$$

where n(t) is the number of multiplets with isospin t and arbitrary hypercharge in the irreducible representation (p, q). From the figure it is easily seen that n(t) has the following form:

$$n(t) = \begin{cases} 2t + 1, & 0 \leqslant t \leqslant t(Y_{min}), \\ 2t(Y_{min}) + 1, & t(Y_{min}) \leqslant t \leqslant t(Y_{max}), \\ 2(t_{max} - t) + 1, & t(Y_{max}) \leqslant t \leqslant t_{max}. \end{cases} \qquad (3.4)$$

Substituting (3.4) in (3.3) and using (3.2), we get[6)]

$$N = (p + 1)(q + 1)((p + q)/2 + 1). \qquad (3.5)$$

4. SECOND APPROACH. DERIVATION OF FORMULA ENABLING ONE TO FIND THE IRREDUCIBLE COMPONENTS APPEARING IN A PRODUCT OF IRREDUCIBLE REPRESENTATIONS TOGETHER WITH THEIR MULTIPLICITIES

We shall now find the formula for resolving a product of irreducible representations into irreducible components (cf. also [9]). Such a formula is easily gotten using approach 2.

As shown in Sec. 2, in approach 2, for the representation (p, q) we form a traceless tensor $T^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ which is completely symmetric in all upper and lower indices. Let us consider a product of two such tensors, corresponding to the representations (p_1, q_1) and (p_2, q_2):

$$T^{i_1 \ldots i_{p_1}}_{j_1 \ldots j_{q_1}} \times T^{r_1 \ldots r_{p_2}}_{s_1 \ldots s_{q_2}}. \qquad (4.1)$$

Constructing from (4.1) all tensors having the properties enumerated above, we obtain all the irreducible tensors contained in the expansion of the product of the tensors (p_1, q_1) and (p_2, q_2) into irreducible components.

It is logical to carry out the construction of irreducible tensors from (4.1) in the following sequence:

1) lowering an index (using the special tensor ϵ_{nir});

2) raising an index (using the special tensor ϵ^{hjs});

3) symmetrization on all upper and all lower indices;

4) contraction (using the special tensors δ^s_i and δ^j_r);

5) subtraction of the trace.

Operations 3) and 5) do not change the rank of the tensor (i.e., the number of upper and lower in-

5)Formulas (3.1) and the corresponding graphs were first obtained by Rashid and Yamanaka, [8] and later by a different method by V. V. Sudakov (private communication).

6)This method for deriving the formula for the dimensionality of an irreducible representation is due to V. V. Sudakov (private communication).

dices). The numbers (p, q) characterizing the tensors obtained, and their multiplicity, are therefore determined by operations 1), 2) and 4).

Let us first consider operations 1) and 2). As a result of l operations of the type 1), the tensor (4.1) with $p_1 + p_2$ upper and $q_1 + q_2$ lower indices becomes a tensor with $p_1 + p_2 - 2l$ upper indices and $q_1 + q_2 + l$ lower indices. As the result of m operations of type 2), the tensor (4.1) becomes a tensor with $p_1 + p_2 + m$ upper indices and $q_1 + q_2 - 2m$ lower indices. It is obvious that

$$0 \leqslant l \leqslant \min(p_1, p_2), \qquad 0 \leqslant m \leqslant \min(q_1, q_2).$$

But because of the identity

$$\varepsilon_{nir}\varepsilon^{hjs} = \delta_n^h\delta_i^j\delta_r^s + \delta_r^h\delta_n^j\delta_i^s + \delta_i^h\delta_r^j\delta_n^s - \delta_n^h\delta_r^j\delta_i^s - \delta_i^h\delta_n^j\delta_r^s - \delta_r^h\delta_i^j\delta_n^s$$

the simultaneous raising and lowering of an index reduces to a contraction. Thus we shall characterize operations 1) and 2) by the number $k = l - m$, where $-\min(q_1, q_2) \leq k \leq \min(p_1, p_2)$. A positive k means that operations 1) and 2) have resulted in the lowering of k indices, while negative k means that operations 1) and 2) have resulted in the raising of $|k| = -k$ indices.

The tensor thus obtained has $p_1 + p_2 - 2k$ upper and $q_1 + q_2 + k$ lower indices if $k > 0$, and $p_1 + p_2 - k$ upper and $q_1 + q_2 + 2k$ lower indices if $k < 0$.

We shall restrict ourselves to nonnegative k. The treatment for negative k is done in exactly the same way. Because of the symmetry of each of the tensor factors in its upper (lower) indices, it is irrelevant which two indices of the set i, r (j, s) are lowered (raised). Thus the multiplicity of each of the tensors with $p_1 + p_2 - 2k$ upper and $q_1 + q_2 + k$ lower indices that are obtained by operations 1) and 2) when $k > 0$ is equal to one.

Now let us consider the operation of contraction. As result of n contraction operations on the tensor obtained after operations 1), 2) and 3), we get a new tensor with $p_1 + p_2 - 2k - n$ upper and $q_1 + q_2 + k - n$ lower indices. In general this tensor may be obtained in several different ways, i.e., its multiplicity may be greater than unity. In fact the n contractions may be carried out by contracting among themselves t indices of the set r, j and v indices of the set i, s (contraction of indices from the sets i, j and r, s with one another will give zero since the tensors (p_1, q_1) and (p_2, q_2) are traceless), where t and v are related by the condition $t + v = n$. Thus there are different t's and v's corresponding to the same n. It is easy to show that t and v vary within the range

$$0 \leqslant t \leqslant \min(p_1 - k, q_2), \qquad 0 \leqslant v \leqslant \min(p_2 - k, q_1),$$

so that

$$0 \leqslant n \leqslant \min(p_1 - k, q_2) + \min(p_2 - k, q_1).$$

The expression for the multiplicity $\rho(k, n)$ of the tensor with $p_1 + p_2 - 2k - n$ upper and $q_1 + q_2 + k - n$ lower indices is obtained in the following way. We fix n and vary t and v so that their sum is n. We first increase t and decrease v:

$$\begin{array}{l} t, \quad t+1, \ldots, \min(p_1 - k, q_2); \\ v, \quad v-1, \ldots, 0. \end{array}$$

It is obvious that then t and v run through

$$\min(\min(p_1 - k, q_2) - t + 1; v + 1) = \min(p_1 - k, q_2, n) - t + 1$$

values. For decreasing t and correspondingly increasing v, t and v run through

$$\min(\min(p_2 - k, q_1) - v, t) = \min(p_2 - k, q_1, n) - v$$

values. Altogether t and v run through

$$\rho(k, n) = \min(p_1 - k, q_2, n) + \min(p_2 - k, q_1, n) - n + 1$$

values for a fixed n.

Since operation 5), without changing the rank of the tensor, makes it irreducible, we finally get the following theorem (where the formulas also include the case of negative k).

The expansion of the product of irreducible representations (p_1, q_2) and (p_2, q_2) contains the irreducible representations

$$(p, q) = \begin{cases} (p_1 + p_2 - 2k - n, & q_1 + q_2 + k - n), \quad k \geqslant 0, \\ (p_1 + p_2 - k - n, & q_1 + q_2 + 2k - n), \quad k < 0, \end{cases} \tag{4.2}$$

where

$$-\min(q_1, q_2) \leqslant k \leqslant \min(p_1, p_2),$$

$$0 \leqslant n(k) \leqslant \begin{cases} \min(p_1 - k, q_2) + \min(p_2 - k, q_1), & k \geqslant 0, \\ \min(p_1, q_2 + k) + \min(p_2, q_1 + k), & k < 0. \end{cases} \tag{4.3}$$

Their multiplicities are given by the formula

$$\rho(k, n) = \begin{cases} \min(p_1 - k, q_2, n) + \min(p_2 - k, q_1, n) - n + 1, & k \geqslant 0, \\ \min(p_1, q_2 + k, n) + \min(p_2, q_1 + k, n) - n + 1 & k < 0. \end{cases} \tag{4.4}$$

We note that

$$\rho(k, 0) = 1. \tag{4.5}$$

Formulas (4.2)—(4.4) can be written somewhat differently using the half-integers $i = k/2$ and

$j = n(k) + |k|/2$. In this notation the preceding theorem is formulated as follows.

The expansion of the product of irreducible representations (p_1, q_1) and (p_2, q_2) contains the irreducible representations

$$(p, q) = (p_1 + p_2 - 3i - j, \quad q_1 + q_2 + 3i - j), \quad (4.2')$$

where

$$-\tfrac{1}{2}\min(q_1, q_2) \leqslant i \leqslant \tfrac{1}{2}\min(p_1, p_2),$$

$$|i| \leqslant j(k) \leqslant \min(p_1 - i, q_2 + i) + \min(p_2 - i, q_1 + i) - |i|. \quad (4.3')$$

The multiplicity of the representation corresponding to given values of i and j is given by the formula

$$\rho(i, j) = \min(p_1 - i, q_2 + i, j) + \min(p_2 - i, q_1 + i, j) - j - |i| + 1. \quad (4.4')$$

Let us consider a few simple examples:

1. We take the product $(f, 0) \times (0, f)$. From (4.3) and (4.4) we get: $k = 0$, $0 \leq n \leq f$, $\rho(0, n) = 1$. Thus the product of two conjugate representations of the type $(f, 0)$ contains only self-conjugate representations (p, p), where $p = 0, 1, \ldots f$, and each representation appears only once:

$$(f, 0) \times (0, f) = \sum_{p=0}^{f} (p, p). \quad (4.6)$$

2. We take the product $(f, 0) \times (f, 0)$. For it, $0 \leq k \leq f$, $n = 0$, $\rho(k, 0) = 1$, so that the product of two identical representations $(f, 0)$ contains $f + 1$ representations of the form $(2[f - k], k)$, $k = 0, 1, \ldots f$, each with unit multiplicity:

$$(f, 0) \times (f, 0) = \sum_{k=0}^{f} (2[f - k], k). \quad (4.7)$$

3. Now we consider a product $(f, g) \times (g, f)$, with $f > g$. Using (4.4) we easily find that the multiplicity $\rho(p)$ of the self-conjugate representations (p, p) appearing in this product is given by the formula (cf.[8]):

$$\rho(p) = \begin{cases} f + g - p + 1, & f + g \geqslant p \geqslant f, \\ g + 1, & f \geqslant p \geqslant g, \\ p + 1, & g \geqslant p \geqslant 0. \end{cases} \quad (4.8)$$

From this we get the theorem: the representation (p, p) appears in the product of two conjugate representations no more than $p + 1$ times and no less than once.

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267