ANALYTIC PROPERTIES OF AMPLITUDES IN THE SECOND ORDER OF NONLINEAR FIELD THEORY

M. K. VOLKOV and G. V. EFIMOV

Joint Institute for Nuclear Research

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Unitarity of the theory is proven in the second order of nonlinear quantum scalar field theory. The asymptotic value of the imaginary part of the amplitudes at high energies is obtained.

1. An attempt was made recently[1-3] to construct a final local theory of the scalar field by introducing an essentially nonlinear interaction Lagrangian that satisfies definite requirements. The investigated class of interaction Lagrangians

$$L_I(\varphi(x)) = -gU(\varphi(x)),$$

where $U(\alpha)$, regarded as a function of the complex variable α, has the following properties:

1) $U(\alpha)$ is an analytic function in the complex α plane with a finite number of cuts, the singularities being such that the integral of $|U(\alpha)|^2$ exists in any bounded region;

2) $U(\alpha)$ is real, has no singularities on the real axis, and can be expanded at the point $\alpha = 0$ in a Taylor series:

$$U(\alpha) = \sum_{n=0}^{\infty} \alpha^n u_n/n!;$$

3) $U(\alpha)$ satisfies at infinity the condition

$$\lim_{|\alpha|\to\infty} \alpha^{-2} U(\alpha) = 0.$$

In [1] the amplitudes of the processes were obtained in second-order perturbation theory, in terms of the powers of the interaction Lagrangian in the Euclidean space of momentum variables, i.e., in the region where they are real. In the present article we investigate the amplitudes in the same order of perturbation theory in the physical region of the momenta. We are interested in the satisfaction of unitarity and in the asymptotic behavior of the imaginary parts of the amplitudes at large values of the momenta.

2. We shall show that the procedure of analytic continuation in the region of physical values of the momentum, indicated in [1], agrees with the unitarity condition. The amplitudes of the physical processes in second order are determined by a sum of integrals in the form (see [1])

$$K_{m_1m_2}(p^2) = 4\pi^2 \int_0^\infty d\beta\beta^2 \frac{J_1(\beta\sqrt{-p^2})}{\sqrt{-p^2}} F_{m_1m_2}(\Delta_2(\beta)), \quad (1)$$

where

$$\Delta_2(\beta) = \mu K_1(\mu\beta)/4\pi^2\beta, \; p^2 < 0.$$

The representation (1) is valid in the space-like region of the momentum values ($p^2 < 0$). We recall that the functions $F(\Delta)$ have an essential singularity at $\Delta = 0$ and for real positive Δ they can be expanded in an asymptotic series

$$F_{m_1m_2}(\Delta) = \sum_{n=n_0}^{\infty} u_{n+m_1}u_{n+m_2}\Delta^n/n!, \quad (2)$$

where u_n—coefficients of the expansion of $U(\alpha)$ in a Taylor series.

The integral (1) is real. It can be continued without difficulty into the region $0 < p^2 < (n_0\mu)^2$

$$K_{m_1m_2}(p^2) = 4\pi^2 \int_0^\infty d\beta\beta^2 \frac{I_1(\beta p)}{p} F_{m_1m_2}(\Delta_2(\beta)) \quad (3)$$

($p = (p^2)^{1/2}$), and remains real in this region.

In the physical region (for $p^2 > (n_0\mu)^2$), the integral (3) begins to diverge at large values of β, i.e., $K_{m_1m_2}(p^2)$ has, as a function of the complex variable p^2, a cut originating at the point $p^2 = (n_0\mu)^2$. Continuation into the region

$$(n_0\mu)^2 < p^2 < (N+1)^2\mu^2 \quad (4)$$

will be realized in the following fashion. We perform in (3) the identical transformation

$$K_{m_1m_2}(p^2) = A_{m_1m_2}(p^2) + B_{m_1m_2}(p^2), \quad (5)$$

where

$$A_{m_1m_2}(p^2) = \frac{4\pi^2}{p}\Big\{\int_0^a d\beta\beta^2 I_1(p\beta) F_{m_1m_2}(\Delta_2(\beta))$$

$$+ \int_a^\infty d\beta\beta^2 I_1(p\beta)$$

$$\times\left[F_{m_1m_2}(\Delta_2(\beta))-\sum_{n=n_0}^{N}u_{n+m_1}u_{n+m_2}\Delta_2^n(\beta)/n\right]\Bigg\} \qquad (6)$$

$$B_{m_1m_2}(p^2)=\frac{4\pi^2}{p}\int_a^{\infty}d\beta\beta^2 I_1(p\beta)\sum_{n=n_0}^{N}u_{n+m_1}u_{n+m_2}\Delta_2^n(\beta)/n! \qquad (7)$$

($a > 0$). The integral $A_{m_1m_2}(p^2)$ can be readily continued into the region (4) and remains real in this region.

To continue the integral $B_{m_1m_2}(p^2)$ into the region (4), we rotate the integration contour upward by an angle $\pi/2$, since it must be recalled that the mass μ, contained in $\Delta_2(\mu\beta)$, has a small imaginary negative increment ($\mu \to \mu - i\epsilon$). We have

$$B_{m_1m_2}(p^2)=\frac{4\pi^2}{p}\int_a^{a+i\infty}d\beta\beta^2 I_1(p\beta)\sum_{n=n_0}^{N}u_{n+m_1}u_{n+m_2}\Delta_2^n(\beta)/n! \qquad (8)$$

In this expression we can already regard p as an arbitrary positive quantity.

The imaginary part of $B_{m_1m_2}(p^2)$, as shown in the appendix, is equal to

$$\operatorname{Im}B_{m_1m_2}(p^2)=8\pi^4\sum_{n=n_0}^{N}\frac{u_{n+m_1}u_{n+m_2}}{n!}\frac{\Omega_n(p^2)}{(16\pi^3)^n}, \qquad (9)$$

where

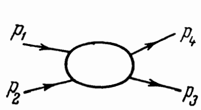
$$\Omega_n(p^2)=\int\frac{d\mathbf{k}_1}{\omega_1}\dots\int\frac{d\mathbf{k}_n}{\omega_n}\delta^{(4)}(p-k_1-\dots-k_n), \qquad (10)$$

$\Omega_n(p^2)$—phase volume of n particles at energy p. Ultimately the imaginary part of $K_{m_1m_2}(p^2)$ for arbitrary $p^2 > 0$ is given by the expression

$$\operatorname{Im}K_{m_1m_2}(p^2)=8\pi^4\sum_{n=n_0}^{[p]}\frac{u_{n+m_1}u_{n+m_2}}{n!}\frac{\Omega_n(p^2)}{(16\pi^3)^n}, \qquad (11)$$

where [p] denotes the closest integer smaller than p.

The proof of unitarity will be presented using as an example the amplitude of the elastic scattering of scalar particles (see the figure).



The amplitude of elastic scattering of scalar particles is given, accurate to second order of perturbation theory, by the expression

$$\langle 0|a_{\mathbf{p}_3}a_{\mathbf{p}_4}Sa_{\mathbf{p}_1}^{+}a_{\mathbf{p}_2}^{+}|0\rangle=[\delta(\mathbf{p}_1-\mathbf{p}_3)\delta(\mathbf{p}_2-\mathbf{p}_4)$$
$$+\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3)]$$
$$+i\frac{\delta^{(4)}(p_1+p_2-p_3-p_4)}{(2\pi)^2(2\omega_1 2\omega_2 2\omega_3 2\omega_4)^{1/2}}T(s,t,u), \qquad (12)$$

where[1]

$$T(s,t,u)=-gu_4+g^2K_{40}(0)+4g^2K_{31}(\mu^2)$$
$$+g^2[K_{22}(s)+K_{22}(t)+K_{22}(u)],$$
$$s=(p_1+p_2)^2,\quad t=(p_1-p_3)^2,$$
$$u=(p_1-p_4)^2,\quad \omega_j=\sqrt{p_j^2-\mu^2}. \qquad (13)$$

Let us consider the channel $s > 4\mu^2$, $t < 0$, $u < 0$. The imaginary part of the amplitude $T(s, t, u)$ is, according to (11),

$$\operatorname{Im}T(s,t,u)=g^2\operatorname{Im}K_{22}(s)=g^2 8\pi^4\sum_{n=1}^{[\sqrt{s}]}\frac{u_{n+2}^2}{n!}\frac{\Omega_n(s)}{(16\pi^3)^n}. \qquad (14)$$

From the unitarity condition $SS^+ = 1$ we can readily obtain, putting $S = 1 + iT$,

$$\frac{\delta^{(4)}(p_1+p_2-p_3-p_4)}{(2\pi)^2(2\omega_1 2\omega_2 2\omega_3 2\omega_4)^{1/2}}2\operatorname{Im}T(s,t,u)$$
$$=\sum_n\langle 0|a_{\mathbf{p}_3}a_{\mathbf{p}_4}T|n\rangle\langle n|T^{+}a_{\mathbf{p}_1}^{+}a_{\mathbf{p}_2}^{+}|0\rangle. \qquad (15)$$

Since we are checking unitarity in second order of perturbation theory, it is necessary to retain in the right side of (15) only first-order matrix elements, i.e.,

$$\langle 0|a_{\mathbf{p}_3}a_{\mathbf{p}_4}T|n\rangle=-\frac{(2\pi)^4}{\sqrt{n!}}g\frac{u_{n+2}}{(2\pi)^{3(n+2)/2}}$$
$$\times\frac{\delta^{(4)}(p_3+p_4-k_1-\dots-k_n)}{(2\omega_3 2\omega_4 2\omega_{k_1}\dots 2\omega_{k_n})^{1/2}}. \qquad (16)$$

Substituting (16) in (15) we readily obtain (14). This proves unitarity in second order of perturbation theory.

We can analogously verify unitarity for the amplitudes of other physical processes.

3. We now turn to the asymptotic expression for the imaginary part of $K_{m_1m_2}(p^2)$ as $p^2 \to +\infty$. We replace the sum in (11) by an integral:

$$\operatorname{Im}K_{m_1m_2}(p^2)\approx 8\pi^4\int_{n_0}^{p}dn\frac{u_{n+m_1}u_{n+m_2}}{\Gamma(n+1)}\frac{\Omega_n(p^2)}{(16\pi^3)^n}. \qquad (17)$$

This does not change the main character of the asymptotic expression.

As $p^2 \to \infty$, the principal role in the integral of (17) will be played by the coefficients u_{n+m} at large values of ($n + m$). Let us find the explicit expression for these coefficients. If we assume that the interaction function $U(\alpha)$, which enters into the class of Lagrangians under consideration, has in the complex α plane ν cuts originating at the points $\alpha_1, \dots, \alpha_\nu$, then we can show that

$$u_n\approx\text{const}\frac{\Gamma(n)}{|\alpha_0|^n}\left(1+O\left(\left|\frac{\alpha_j}{\alpha_0}\right|^n\right)\right)\quad(n\gg 1), \qquad (18)$$

where $|\alpha_0| = \min\{|\alpha_j|\}$; $1 \le j \le \nu$. Thus,

$$\text{Im}\, K_{m_1m_2}(p_2) \approx \text{const} \int_0^p dn\, \Gamma(n)\, \frac{\Omega_n(p^2)}{a^n}, \qquad (19)$$

where $a = 16\pi^3 |\alpha_0|^2$.

The phase volume $\Omega_n(p^2)$, taken as a function of n with fixed p^2, has a maximum approximately in the middle of the interval, and falls off towards the ends of the integration interval. Since $\Gamma(n)$ increases very rapidly towards the end of the interval, and $\Omega_n(p^2)$ varies smoothly, the entire integrand function has a sharp maximum near the end of the interval. Let us calculate the integral (19) by the saddle point method, using the asymptotic representation of the phase volume for $n \lesssim p$ (see, for example, [4]):

$$\Omega_n(p^2) = \frac{(2\pi)^{3(n-1)/2}}{\Gamma(3(n-1)/2)\, n^{3/2}} (p-n)^{(3n-5)/2} \qquad (p - n \ll n). \qquad (20)$$

The saddle point can be obtained in the usual fashion:

$$\bar{n} = \frac{p}{\mu}\left[1 - \frac{3}{2\ln(p/\mu)}\left(1 + O\left(\frac{1}{\ln(p/\mu)}\right)\right)\right]$$

$$(\ln(p/\mu) \gg 1). \qquad (21)$$

Ultimately

$$\text{Im}\, K_{m_1m_2}(p^2) \approx \exp\left\{\frac{p}{\mu}\ln\frac{p}{\mu}\right\} f(p), \qquad (22)$$

where f(p)—certain function that grows more slowly than the exponential function given here.

We call attention to the following:

1) The principal asymptotic term in (22) is the same for all Lagrangians in the class of interaction functions $U(\alpha)$ under consideration.

2) The principal asymptotic term in (22) does not depend on the physical process in question.

3) The asymptotic expression for the imaginary part is determined by the particle mass, and not by the new parameters (with dimension of length) which enter into the interaction Lagrangian.

4. An asymptotic expression for $K_{m_1m_2}(p^2)$ for large space-like values of the momentum $p^2 \to -\infty$ can be readily obtained from (1):

$$|K_{m_1m_2}(p^2)| \le \frac{\text{const}}{|p^2|} \qquad (-p^2 \gg \mu^2, \quad \text{const} > 0). \qquad (23)$$

Thus, $K_{m_1m_2}(p^2)$ as a function of the complex variable p^2 has an essential singularity at $p^2 = \infty$.

In conclusion, let us discuss the behavior of the amplitude for large values of the momentum in the physical region. The growth deduced for (22) runs counter to nature. In all probability it will be compensated for by inclusion of the higher perturbation-theory approximations, if unitarity is satisfied in each order of this theory. Thus, only a study of the higher approximations of perturbation theory can answer this question.

On the other hand, attempts have been made recently to find limitations on the growth of the amplitudes of physical processes that follow from the causality requirement. It was found, thus, that the growth in the amplitude of the elastic scattering of a meson by a nucleon should be slower than exponential in the energy, in the Breit system of coordinates (we recall that in this system $E \sim s$). If this conclusion is true, then the behavior of the amplitudes in the nonlinear theory under consideration agrees with the causality requirement, at least in second order of perturbation theory, since it follows from (22) that

$$\text{Im}\, T(s, t, u) < e^s.$$

In conclusion the authors thank L. G. Zastavenko for a discussion.

APPENDIX

We seek the imaginary part of the integral

$$B_n(p^2) = \frac{4\pi^2}{p} \int_a^{a+i\infty} d\beta\, \beta^2 I_1(\beta p) \left(\frac{\mu K_1(\mu\beta)}{4\pi^2\beta}\right)^n, \qquad (A.1)$$

where $p > n\mu$.

We make the substitution $\beta = e^{i\pi/2}x$ and use the relations

$$K_1(e^{i\pi/2}\mu x) = -\tfrac{1}{2}\pi H_1^{(2)}(\mu x); \qquad I_1(e^{i\pi/2}px) = iJ_1(px),$$

Then

$$B_n(p^2) = \frac{4\pi^2}{p} \int_{-ia}^{-ia+\infty} dx\, x^2 J_1(xp) \left(-\frac{\mu H_1^{(2)}(\mu x)}{8\pi i x}\right)^n. \qquad (A.2)$$

Using the equality

$$J_1(x) = \tfrac{1}{2}[H_1^{(2)}(x) + H_1^{(1)}(x)],$$

we represent $B_n(p^2)$ in the form

$$B_n(p^2) = C_n(p^2) + D_n(p^2), \qquad (A.3)$$

where

$$C_n(p^2) = \frac{2\pi^2}{p} \int_{-ia}^{-ia+\infty} dx\, x^2 H_1^{(2)}(xp) \left(-\frac{\mu H_1^{(2)}(\mu x)}{8\pi i x}\right)^n \qquad (A.4)$$

$$D_n(p^2) = \frac{2\pi^2}{p} \int_{-ia}^{-ia+\infty} dx\, x^2 H_1^{(1)}(xp) \left(-\frac{\mu H_1^{(2)}(\mu x)}{8\pi i x}\right)^n. \qquad (A.5)$$

The integrand in (A.4) is similar and decreases exponentially in the lower half-plane, so that the integration contour can be rotated from the line $(-ia, -ia+\infty)$ to $(-ia, -i\infty)$. We obtain (after making the substitution $x = e^{-\pi i/2}y$)

$$C_n(p^2) = -\frac{4\pi}{p} i \int_a^\infty dy\, y^2 K_1(yp) \left(\frac{\mu K_1(\mu y)}{4\pi^2 y}\right)^n. \quad (A.6)$$

Let us consider $D_n(p^2)$, which we represent in the form of a sum

$$D_n(p^2) = E_n(p^2) + F_n(p^2), \quad (A.7)$$

where

$$E_n(p^2) = \frac{2\pi^2}{p} \int_{-ia-\infty}^{-ia+\infty} dx\, x^2 H_1^{(1)}(xp) \left(-\frac{\mu H_1^{(2)}(\mu x)}{8\pi i x}\right)^n. \quad (A.8)$$

$$F_n(p^2) = \frac{2\pi^2}{p} \int_{-ia}^{-ia-\infty} dx\, x^2 H_1^{(1)}(xp) \left(-\frac{\mu H_1^{(2)}(\mu x)}{8\pi i x}\right)^n. \quad (A.9)$$

As shown in [4], when $p^2 > (n\mu)^2$ the function $E_n(p^2)$ is equal to

$$E_n(p^2) = \pi i (16\pi^3)^{-n+1} \Omega_n(p^2), \quad (A.10)$$

where $\Omega_n(p^2)$—phase volume of n particles at energy p.

After substituting $x = e^{-i/2}y$ in the integral of (A.9) and making the substitution

$$H_1^{(1)}(e^{-\pi i/2}yp) = -2iI_1(yp) + \frac{2}{\pi} K_1(yp); \quad H_1^{(2)}(e^{-\pi i/2}\mu y) = -\frac{2}{\pi} K_1(\mu y),$$

we obtain

$$F_n(p^2) = \frac{4\pi^2}{p} \int_a^{a-i\infty} dy\, y^2 I_1(yp) \left(\frac{\mu K_1(\mu y)}{8\pi^2 y}\right)^n + i\frac{4\pi}{p} \int_a^\infty dy\, y^2 K_1(py) \left(\frac{\mu K_1(\mu y)}{4\pi^2 y}\right)^n. \quad (A.11)$$

It is easy to note that the first integral in the right side of (A.11) is equal to $B_n^*(p^2)$, and the second is equal to $-C_n(p^2)$.

We have thus obtained

$$B_n(p^2) = \pi i (16\pi^3)^{-n+1} \Omega_n(p^2) + B_n^*(p^2), \quad (A.12)$$

$$\mathrm{Im}\, B_n(p^2) = \tfrac{1}{2}\pi (16\pi^3)^{-n+1} \Omega_n(p^2), \quad (A.13)$$

from which (9) follows directly.

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