THE NUMBER OF DIAGRAMS FOR THE ONE-PARTICLE AND TWO-PARTICLE GREEN'S FUNCTIONS IN THE CASE OF DIRECT INTERACTION OF FERMIONS

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The numbers of topologically nonequivalent Feynman diagrams for the one-particle and two-particle Green's functions are calculated for arbitrary order of perturbation theory, with both the symmetrized and the nonsymmetrized techniques.

In statistical physics there is now wide application of Green's-function methods together with the Feynman diagram technique. In this technique all of the terms of perturbation theory that make equal contributions can be summed easily. Terms that make different contributions correspond to topologically different diagrams. Therefore it is of great interest to count up the number of topologically nonequivalent connected diagrams in an arbitrary order of perturbation theory. This offers a possibility for studying the convergence of the series of perturbation theory. Here we shall count up the numbers of topologically nonequivalent connected diagrams for the one-particle and two-particle Green's functions in arbitrary order in perturbation theory.

We shall construct the perturbation theory for the one-particle Green's function $G^{I}_{\alpha\beta}(x, x')$ in the usual way.[1] Then in the term of n-th order in H_{int} there will be $2n + 1$ creation operators $\Psi^{+}_{\alpha}(x)$ and this same number of annihilation operators $\Psi_{\beta}(x')$. Since chronological contractions of the types

$$\Psi_{\alpha}(x)\Psi_{\beta}(x'), \qquad \Psi_{\alpha^+}(x)\Psi_{\beta^+}(x')$$

are identically equal to zero, in the n-th order there are $(2n + 1)!$ ways of making the pairings, which correspond to all possible permutations of the creation operators relative to the "stationary" annihilation operators. Thus in n-th order for the one-particle function the total number of diagrams (both connected and unconnected) is $(2n + 1)!$

We denote the number of connected n-th order diagrams by t_n and count up the total number of diagrams in which the part connected with the external $G^{(0)}$ lines contains $n - m$ interaction operators. This number is

$$\binom{n}{m}(2m)!\,t_{n-m},$$

since the m "disconnected" interaction operators, which can be chosen in $\binom{n}{m}$ ways from the n operators present, give $(2m)!$ diagrams. When we take successively 1, 2, ..., n disconnected interaction operators, we arrive at the equation

$$(2n+1)! = \sum_{m=0}\binom{n}{m}(2m)!\,t_{n-m}. \tag{1}$$

Among the t_n connected diagrams, however, there are sets which are topologically equivalent; these are obtained by taking all possible permutations of the interaction operators, which gives $n!$ equivalent diagrams, and also by interchanging the places of two vertices, which gives 2^n topologically equivalent diagrams. Thus the number of topologically nonequivalent diagrams in n-th order is

$$l_n = t_n / 2^n n!.$$

Using (1), we find a system of equations for the l_n:

$$\sum_{m=0}^{n}[2(n-m)-1]!!\,l_m = (2n+1)!!, \tag{2}$$

where we have set $(-1)!! = 1$. The solution of this system is (see [2], formulas 0.313 and 0.430)

$$l_{m-1} = \sum_{n_i}(-1)^{\sum_i n_i - 1} \times \delta\Big(\sum_i n_i i - m\Big)\Big(\sum_i n_i\Big)!\prod_i\frac{[(2i-1)!!]^{n_i}}{n_i!}. \tag{3}$$

For very large n

$$l_n = (2n+1)!!(1+O(1/n)). \tag{4}$$

Now let us consider the two-particle Green's function $G^{II}_{\alpha\beta,\gamma\delta}(x_1x_2, x_3x_4)$.[1] The counting-up of the number of diagrams for the two-particle function is analogous to the procedure for the one-particle function. In this case, however, for each diagram there is a topologically equivalent one which makes a different contribution. The outcoming lines of these diagrams are interchanged ($x_3 \rightleftarrows x_4$). An example of such diagrams is given in Fig. 1, which shows the zeroth-order diagrams. Although the contributions from such diagrams are different, we shall regard them as topologically equivalent.

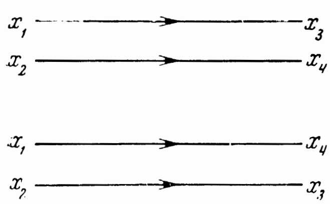

FIG. 1

All topologically nonequivalent connected diagrams can be divided into two types. Diagrams of the first type break apart into two parts. These are diagrams like those of Fig. 1 with proper-energy parts strung along the lines. Diagrams of the second type are those that do not break apart in this way. If we denote by a_n the number of connected diagrams of the second type in n-th order, then the number of topologically nonequivalent n-th order diagrams of the second type is

$$d_n = a_n / 2^{n+1}n!,$$

and we get a system of equations for the d_n:

$$\sum_{m=1}^{n} [2(n-m)-1]!!\, d_m = -(n+2)(2n+1)!! + l_{n+1}. \quad (5)$$

The solution of this system is of the form

$$d_n = \frac{1}{2} l_{n+1} - \sum_{m=0}^{n} l_m l_{n-m}. \quad (6)$$

On the other hand the total number of topologically nonequivalent n-th order diagrams for the two-particle function is $l_{n+1}/2$.

Let us now go on to the consideration of diagrams in the symmetrized technique.[1] In this technique the interaction operator is written as a function of four variables, $\Gamma^{(0)}_{\gamma_1\gamma_2\gamma_3\gamma_4}(x_1x_2, x_3x_4)$, which is antisymmetric with respect to the interchanges $1 \rightleftarrows 2$ and $3 \rightleftarrows 4$. In a diagram $\Gamma^{(0)}$ is represented as a square. In the symmetrized technique the number of topologically nonequivalent diagrams is smaller than in the unsymmetrized technique. For example, to the two topologically different connected first-order diagrams in the unsymmetrized technique there corresponds only one diagram in the symmetrized technique. For the one-particle Green's function a symmetrized diagram replaces 2^n topologically nonequivalent unsymmetrized diagrams. We can regard each square as giving two unsymmetrized diagrams. Diagrams with the element shown in Fig. 2 are an exception. To these two squares there correspond only two unsymmetrized diagrams, since the interchange of the two output lines of the first square, as shown in Fig. 3, is equivalent to interchange of the two input lines of the second square. We call such an element a couple. Thus to each symmetrized diagram with n interaction operators there correspond 2^{n-m} topologically different diagrams in the unsymmetrized technique, where m is the number of couples, and we must affix to each symmetrized diagram the factor $(1/2)^m i^n$. This result differs from the rule given by Abrikosov, Gor'kov, and Dzyaloshinskiĭ (see [1], page 108).

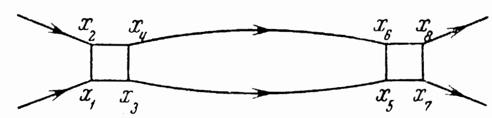

FIG. 2

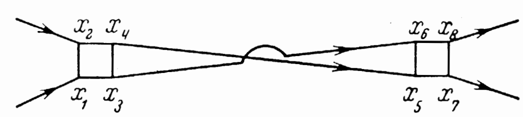

FIG. 3

It is now easy to count up the number of topologically different symmetrized diagrams in the n-th order of perturbation theory. We denote by L_n^0 the number of n-th order symmetrized diagrams without couples. Any other symmetrized n-th order diagram is obtained from an m-th order diagram ($m < n$) without couples by adding $n - m$ couples. The number of such diagrams is equal to the number of ways we can distribute $n - m$ balls in m boxes, multiplied by L_m^0. To all such diagrams there will correspond in the unsymmetrized technique a total of

$$2^m \frac{(n-1)!}{(n-m)!(m-1)!} L_m{}^0$$

topologically different unsymmetrized diagrams. Thus we have the formula

$$l_n = \sum_{m=1}^{n} 2^m \binom{n-1}{m-1} L_m{}^0. \quad (7)$$

The solution of the system (7) is of the form (see [3], Problem 390)

$$L_n^0 = \frac{1}{2^n} \sum_{m=1}^{n} (-1)^{n-m} \binom{n-1}{m-1} l_m. \tag{8}$$

On the other hand the total number of topologically different symmetrized diagrams in nth order is given by

$$L_n = \sum_{m=1}^{n} \binom{n-1}{m-1} L_m^0. \tag{9}$$

In an analogous way one can count up the topologically nonequivalent connected diagrams for the two-particle function in the symmetrized technique, but because of their cumbersomeness we do not give these formulas.

I regard it as my pleasant duty to express my gratitude to A. A. Abrikosov and É. I. Rashba for a discussion of this work.

[1] Abrikosov, Gor'kov, and Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Quantum Field Theory Methods in Statistical Physics), Fizmatgiz, 1962.

[2] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov, i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), Fourth edition, Fizmatgiz, 1963.

[3] I. V. Proskuryakov, Sbornik zadach po lineĭnoĭ algebre (Collection of Problems in Linear Algebra), Second edition, Fizmatgiz, 1962.

Translated by W. H. Furry
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