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EXPRESSION FOR THE SPECTRAL FUNCTION IN TERMS OF THE VALUES OF THE AMPLITUDE IN THE PHYSICAL REGION

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A method is described by which one can calculate the spectral function of the double Mandelstam representation, as obtained by continuation of the two-particle unitarity condition in the s channel, in terms of the values of derivatives of the scattering amplitude with respect to the cosine of the scattering angle z at z = 0.

We consider for simplicity the scattering of two particles of equal mass ($\pi\pi$ scattering). The scattering amplitude $A(st)$ as a function of the invariant variables s and t satisfies the Mandelstam double dispersion relation.[1]

Continuation of the two-particle unitarity condition from the s channel gives the following contribution to the spectral function[2] ($z > 1$):

$$\rho(sz) = \left[\frac{s-4m^2}{s}\right]^{1/2}\iint\frac{\theta[z-z^{(+)}]}{K^{1/2}(zz_1z_2)}$$

$$\times[A_t(z_1s)A_t^*(z_2s)+A_u(z_1s)A_u^*(z_2s)]\,dz_1\,dz_2, \qquad (1)$$

$$\rho(s,-z) = \left[\frac{s-4m^2}{s}\right]^{1/2}\iint\frac{\theta[z-z^{(+)}]}{K^{1/2}(zz_1z_2)}$$

$$\times[A_t(z_1s)A_u^*(z_2s)+A_u(z_1s)A_t^*(z_2s)]\,dz_1dz_2;$$

$$t=\frac{s-4m^2}{2}(z-1),\quad t_1,\ u_1=\frac{s-4m^2}{2}(z_1-1),$$

$$K(zz_1z_2)=z^2+z_1^2+z_2^2-2zz_1z_2-1,$$

$$u=\frac{s-4m^2}{2}(-z-1),\quad t_2,\ u_2=\frac{s-4m^2}{2}(z_2-1),$$

$$z^{(+)}=z_1z_2+[(z_1^2-1)(z_2^2-1)]^{1/2};$$

Here $A_t(t_1s)$ and $A_u(u_1s)$ are the absorptive parts in the t and u channels, respectively; the isotopic indices are omitted.

We form the linear combination

$$\rho^{(\pm)}(sz)=\rho(sz)\pm\rho(s,-z)=\left[\frac{s-4m^2}{s}\right]^{1/2}\iint\frac{\theta[z-z^{(+)}]}{K^{1/2}(zz_1z_2)}$$

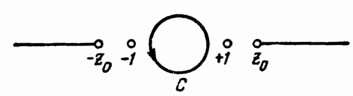
$$\times[A_t(z_1s)\pm A_u(z_1s)]\,[A_t^*(z_2s)\pm A_u(z_2s)]\,dz_1\,dz_2 \qquad (2)$$

and express it in terms of derivatives of $A(sz)$ at $z = 0$. To do this we make use of the analytic properties of $A(sz)$ as a function of z[2]:

$$A(sz)=\frac{1}{\pi}\int_{z_0}^{\infty}\frac{A_t(z's)}{z'-z}dz'+\frac{1}{\pi}\int_{z_0}^{\infty}\frac{A_u(z's)}{z'+z}dz',$$

$$z_0=1+\frac{8m^2}{s-4m^2}. \qquad (3)$$

Then, if the analytic function $\varphi(z)$ has singularities only inside the contour C (see figure), we have



$$\frac{1}{2\pi i}\int_C\varphi^{(\pm)}(z)\,A(sz)\,dz$$

$$=\frac{1}{\pi}\int_{z_0}^{\infty}[A_t(z's)\pm A_u(z's)]\,\varphi^{(\pm)}(z')\,dz', \qquad (4)$$

with $\varphi^{(\pm)}(-z) = \mp\varphi^{(\pm)}(z)$. If we now expand the quantity $\theta/K^{1/2}$ in (2) in a double series in a set of even and odd functions $\varphi_n^{(\mp)}(z_{1,2})$ which are complete for the interval $[1, \infty]$,

$$\left[\frac{s-4m^2}{s}\right]^{1/2}\frac{\theta[z-z^{(+)}]}{K^{1/2}(zz_1z_2)}$$

$$=\sum_{n_1,n_2=0}^{\infty}C_{n_1n_2}^{(\pm)}(sz)\,\varphi_{n_1}^{(\pm)}(z_1)\,\varphi_{n_2}^{(\pm)}(z_2), \qquad (5)$$

we get

$$\rho^{(\pm)}(sz)=\pi^2\sum_{n_1n_2=0}^{\infty}C_{n_1n_2}^{(\pm)}(sz)\,\Phi_{n_1}^{(\pm)}(s)\,\Phi_{n_2}^{(\pm)*}(s),$$

$$\Phi_n^{(\pm)}(s)=\frac{1}{2\pi i}\oint_C A(sz)\,\varphi_n^{(\pm)}(z)\,dz. \qquad (6)$$

A complete set of functions $\varphi_n^{(\pm)}(z)$ having the required analytic properties can be constructed easily. For example, it is convenient to use Chebyshev polynomials of the reciprocal of the argument. We take

$$\varphi_n(z) = z^{-k}[\cos 2n(\arccos z^{-1})]$$
$$= \frac{1}{z^k}\Big[\sum_{i=0}^{n}(-1)^{n-i}\frac{(n+i-1)!}{2i!(n-i)!}\frac{2^{2i}}{z^{2i}}\Big]. \qquad (7)$$

The functions $\varphi_n(z)$ are orthogonal and normalized in the interval $[1, \infty]$ with the weight $4z^{2k}/\pi z(z^2-1)^{1/2}$, where k is an integer which is even or odd, depending on what sort of $\varphi_n(z)$ we wish to obtain. The size of k is chosen so that the integral

$$\int_{z_0}^{\infty}[A_t(z's) \pm A_u(z's)]\,\varphi_n^{(\pm)}(z')\,dz'$$

will have meaning in the case of any limited number of subtractions.

Thus the coefficients $C_{n_1n_2}(sz)$ are simply the components of the expansion in double Fourier series of the function

$$[(s-4m^2)/s]^{1/2}z_1^k z_2^k\theta[z-z^{(+)}]$$
$$\times\theta[z_1-z_0]\theta[z_2-z_0]/K^{1/2}(zz_1z_2),$$

in which we have made the change of variables $z_1 = 1/\cos\varphi_1$, $z_2 = 1/\cos\varphi_2$. Since the only singularities of the $\varphi_n^{(\pm)}(z)$ are poles at $z = 0$, it follows from (6) that $\Phi_n^{(\pm)}(s)$ can be expressed in terms of a sum of derivatives of the amplitude $A(sz)$ with respect to z at $z = 0$.

The series (6) converges absolutely, since it can be shown that

$$C_{n_1n_2}^{(\pm)}(sz) = O(1/n_1n_2^{1/2} + 1/n_1^{1/2}n_2), \quad \Phi_n^{(\pm)}(s) = O(1/n);$$

because the first quantity is a Fourier component of a function that has a square-root singularity, and the second is a Fourier component of a function with limited variation.

We now expand $A(sz)$ as a function of z on the interval $[-1, +1]$ in terms of a complete set of functions $\mathscr{P}_l(z)$:

$$A(sz) = \sum_{l=0}^{\infty}\mathscr{P}_l(z)f_l(s). \qquad (8)$$

Let $\mathscr{P}_l(z)$ be analytic functions of z in the z plane cut from $z = 1$ to $z = \infty$ and from $z = -\infty$ to $z = -1$, and have the properties $\mathscr{P}_l(-z) = (-1)^l\mathscr{P}_l(z)$. We substitute (8) in (6), and get

$$\rho^{(\pm)}(sz) = \pi^2\sum_{n_1n_2}C_{n_1n_2}^{(\pm)}(sz)\Big(\sum_{l_1}K_{n_1l_1}^{(\pm)}f_{l_1}(s)\Big)\Big(\sum_{l_2}K_{n_2l_2}^{(\pm)}f_{l_2}^*(s)\Big), \qquad (9)$$

where

$$K_{nl}^{(+)} = \begin{cases}\frac{1}{2\pi i}\oint_C\varphi_n^{(+)}(z)\mathscr{P}_l(z)\,dz, & l\text{ — even},\\ 0, & l\text{ — odd}.\end{cases}$$

The definition of $K_{nl}^{(-)}$ is analogous to this. In (9) we carry out a formal interchange of the summations over n and l:

$$\rho^{(\pm)}(sz) \sim \sum_{l_1,l_2=0}^{\infty}D_{l_1l_2}^{(\pm)}(sz)f_{l_1}(s)f_{l_2}^*(s); \qquad (10)$$

$$D_{l_1l_2}^{(\pm)}(sz) = \lim_{\substack{x_1\to1\\x_2\to1}}D_{l_1l_2}^{(\pm)}(sz,\ x_1x_2),$$

$$D_{l_1l_2}^{(\pm)}(sz,\ x_1x_2) = \sum_{n_1n_2}C_{n_1n_2}^{(\pm)}(sz)K_{n_1l_1}^{(\pm)}x_1^{n_1}K_{n_2l_2}^{(\pm)}x_2^{n_2}. \qquad (11)$$

Then it can be shown that

$$D_{l_1l_2}^{(\pm)}(sz) = \frac{-1}{4}\Big[\frac{s-4m^2}{s}\Big]^{1/2}\iint\frac{\theta[z-z^{(+)}]}{K^{1/2}(zz_1z_2)}$$
$$\times[\Delta\mathscr{P}_{l_1}(z_1)\mp\Delta\mathscr{P}_{l_1}(-z_1)][\Delta\mathscr{P}_{l_2}(z_2)\mp\Delta\mathscr{P}_{l_2}(-z_2)]\,dz_1\,dz_2,$$

$$\Delta\mathscr{P}_l(z) = \mathscr{P}_l(z+i\varepsilon) - \mathscr{P}_l(z-i\varepsilon),$$
$$\Delta\mathscr{P}_l(z) = \Delta\mathscr{P}_l(-z)(-1)^{l+1}. \qquad (12)$$

In the case of expansion of the amplitude $A(sz)$ in Legendre polynomials $[\mathscr{P}_l(z) = P_l(z)]$ we have $\Delta P_l(z) = 0$, and consequently $D(sz) = 0$; that is, the interchange of the summations in (9) is illegitimate. This indicates that in the practical calculation of $\rho(sz)$ by the formula (9) it is necessary first to fix $n_1, n_2 < N$, and then approximate this partial sum over n with sums over l_1, l_2.

Using the asymptotic behavior of the partial amplitudes $f_l(s) \sim 1/z_0^l$, one can show that a satisfactory approximation to $\rho(sz)$ in a sufficiently wide range of s and z can be achieved only for $n_1, n_2 \sim 5$ and $l_1, l_2 \sim 20$; i.e., one must conclude in the treatment ~ 10 individual even and odd amplitudes.

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

[2] V. N. Gribov, JETP 41, 1962 (1961), Soviet Phys. JETP 14, 1395 (1962).

Translated by W. H. Furry
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