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REFLECTION AND TRANSFORMATION OF SOUND WAVES INCIDENT ON A VAPOR--LIQUID He II BOUNDARY

D. M. CHERNIKOVA

Moscow Physico-technical Institute

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Transformation and reflection of sound waves incident on a vapor—liquid He II boundary is considered. The general form of the boundary conditions is found and the transformation and reflection coefficients for the sound waves are determined. It is shown that second sound and sound propagating in vapor are mutually transformable, but there is practically no transformation of first sound into second or into sound in vapor, and vice versa.

WE shall consider the transformation and reflection of sound waves obliquely incident on the boundary between He II and its vapor in equilibrium. The sound wave may propagate either from the liquid to the vapor or vice versa, and we determine the general form of the boundary conditions on the surface between the vapor and the liquid He II. The particular case of normal incidence of second sound on the boundary between He II and its vapor, with both media in equilibrium, was considered earlier by Pellam[1] and by Dingle[2].

Usually, when a sound wave is incident on a boundary between two media, a reflected and a refracted wave are produced in addition to the incident wave. In the case of He II and its vapor the number of waves produced is not two but three. The presence of three waves rather than two is connected with the existence of two types of waves in He II, first and second sound.

We confine ourselves to the frequency region $\omega \ll c_2^2/\chi$ (c_2—velocity of second sound, χ—coefficient of temperature conductivity in He II), in which the mean free paths of the vapor molecules and of the elementary excitations in He II are much shorter than the length of the sound wave; the usual hydrodynamic equations (without dispersion) can therefore be used in this frequency region for both the vapor and the liquid. In addition, simple estimates show that equilibrium is sure to exist between the liquid He II and its vapor in the frequency region $\omega \ll c_2^2/\chi$.

We consider sound waves in which all the thermodynamic quantities vary like $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$ (ω—frequency, $\mathbf{k}$—wave vector directed along the wave propagation direction).

1. REFLECTION AND TRANSFORMATION OF SECOND SOUND

In the case when a second-sound wave T'_{2i} is incident on the boundary between He II and its vapor, two waves are produced in the liquid, T'_{2r} (reflected second-sound wave) and P'_1 (first-sound wave), and one more in the vapor, P'—adiabatic wave. Although we have neglected the thermal conductivity of the vapor, we must take into account in the boundary conditions the heat wave in the vapor, which we denote by T' [1)].

We align the interface with the (x, y) plane. It is easy to see that all the waves will have the same frequencies ω and identical wave-vector components k_x and k_y. This leads directly to relations for the propagation directions of the resultant waves. Let (x, z) be the plane of incidence of the wave T'_{2i}, and let θ_{2i}, θ_{2r}, θ_1, and θ be the angles between the directions of propagation of the waves T'_{2i}, T'_{2r}, P'_1 and P' and the z axis. From the equality of k_x and k_y for these waves we then find that T'_{2i}, T'_{2r}, P'_1, and P' all lie in the same plane, and

$$\theta_{2r} = \theta_{2i} = \theta_2, \quad \sin\theta_1 = (c_1/c_2)\sin\theta_2,$$
$$\sin\theta = (c/c_2)\sin\theta_2, \tag{1.1}$$

where c_1—velocity of first sound in He II and c—adiabatic velocity of sound in the vapor.

1)Here and throughout P'_1 denotes the pressure oscillation in the first-sound wave, T'_2 the temperature oscillation in the second-sound wave, P' the pressure oscillation in the sound wave in vapor and T' the temperature oscillation in the heat wave in the vapor.

We introduced a quantity $\eta'(x, t) = \eta^{(0)}\exp(i\omega t + ik \cdot r)$ describing the oscillations of the separation boundary ($\eta^{(0)}$—amplitude of the oscillation of the separation boundary). The connection between the amplitudes of the sound waves is determined by the boundary conditions on the surface separating the He II from its vapor; these conditions call for the equality of the forces acting on the boundary

$$P_1^{(0)} - P^{(0)} = \alpha\omega^2 c_2^{-2} \sin^2\theta_2 \eta^{(0)},$$

equality of the material flux densities through the boundary

$$-\cos\theta_1 P_1^{(0)} / c_1 + i\omega\rho_l \eta^{(0)} = \cos\theta P^{(0)} / c + i\omega\rho_v \eta^{(0)},$$

equality of the energy flux densities through the boundary

$$w_l\,(-\cos\theta_1 P_1^{(0)} / c_1 + i\omega\rho_l\eta^{(0)} + \cos\theta_2 (T_{2i}^{(0)} - T_{2r}^{(0)}) \rho_l \rho_s \sigma_l{}^2 T / \rho_n c_2 = w_v(\cos\theta P^{(0)} / c + i\omega\rho_v\eta^{(0)}), \qquad (1.2)$$

equality of the temperatures

$$T_{2i}^{(0)} + T_{2r}^{(0)} = P^{(0)}\beta_v T / \rho_v c_{pv} + T^{(0)},$$

and equality of the chemical potentials

$$\rho_l^{-1} P_1^{(0)} - \sigma_l\,(T_{2i}^{(0)} + T_{2r}^{(0)}) = \rho_v^{-1} P^{(0)} - \sigma_v(P^{(0)}\beta_v T / \rho_v c_{pv} + T^{(0)}),$$

where α—surface tension coefficient; $T_{2i}^{(0)}$, $T_{2r}^{(0)}$, $P_1^{(0)}$, $P^{(0)}$, $T^{(0)}$—the respective amplitudes of the waves T'_{2i}, T'_{2r}, P'_1, P', and T'; ρ_s and ρ_n—superfluid and normal densities of He II; T—temperature on the boundary; ρ_l, σ_l, w_l—density, entropy, and heat function of He II, respectively; ρ_v, σ_v, w_v, c_{pv}, β_v—the density, entropy, heat function, specific heat, and coefficient of thermal expansion of the vapor, respectively.

In the general case, the solutions of the system (1.2) are very cumbersome. However, recognizing that $\sigma_l/\sigma_v \ll 1$ and $\rho_v/\rho_l \ll 1$, and also that surface tension can be neglected in the frequency range in question because the term that contains α is then negligibly small, $\alpha\omega/\rho c_1 c_2^2 \ll 1$, we obtain for the solution of the system (1.2)

$$\frac{P_1^{(0)}}{T_{2i}^{(0)}} = \frac{P^{(0)}}{T_{2i}^{(0)}} = \frac{2\rho_v\sigma_v\gamma\cos\theta_2}{\cos\theta + \gamma\cos\theta_2},$$

$$\frac{T_{2r}^{(0)}}{T_{2i}^{(0)}} = \frac{-\cos\theta + \gamma\cos\theta_2}{\cos\theta + \gamma\cos\theta_2}, \qquad (1.3)$$

where

$$\gamma = (\sigma_l / \sigma_v)^2 \rho_s \rho_l c / \rho_n \rho_v c_2.$$

With the aid of (1.3) we determine the coefficient of reflection of second sound $\Omega_{T'_{2r}}$ from the boundary, and the coefficients of transformation of the second sound into first $\Omega_{P'_1}$ and into ordinary sound in vapor $\Omega_{P'}$

$$\Omega_{T'_{2r}} = \left(\frac{-\cos\theta + \gamma\cos\theta_2}{\cos\theta + \gamma\cos\theta_2}\right)^2,$$

$$\Omega_{P_1'} = \frac{4(\rho_v c/\rho_l c_1)\cos\theta_1\cos\theta_2}{(\cos\theta + \gamma\cos\theta_2)^2}, \quad \Omega_{P'} = \frac{4\gamma\cos\theta\cos\theta_2}{(\cos\theta + \gamma\cos\theta_2)^2}, \qquad (1.4)$$

where

$$\cos\theta_1 = (1 - \sin^2\theta_2 c_1^2 / c_2^2)^{1/2}, \quad \cos\theta = (1 - \sin^2\theta_2 c^2 / c_2^2)^{1/2}.$$

We see from (1.3) and (1.4) that the first-sound wave propagates parallel to the boundary, starting with the angle of incidence $\theta_2 = \theta'_2$ ($\sin\theta'_2 = c_2/c_1$) while the sound wave propagates in the vapor starting with $\theta_2 = \theta''_2$ ($\sin\theta''_2 = c_2/c$), inasmuch as $\cos\theta_1$ and $\cos\theta$ vanish at these angles of incidence. If $\theta_2 > \theta'_2$ and $\theta_2 > \theta''_2$, the values of $\cos\theta_1$ and $\cos\theta$ respectively become pure imaginary, and the sound waves traveling parallel to the boundary begin to attenuate with increasing distance from the boundary, namely P'_1 along the negative z axis and P' along the positive z axis. Since $\theta''_2 > \theta'_2$, the second sound is completely reflected from the boundary when $\theta_2 = \theta''_2$ ($\Omega_{T'_{2r}} = 1$), i.e., total internal reflection of second sound takes place. Figure 1 shows that θ''_2, usually called the critical angle, exists only at temperatures exceeding 0.7°K when c becomes larger than c_2.

$\Omega_{P'_1}$ is an exceedingly small quantity in the entire temperature interval in question, so that the transformation of second sound into first can be neglected in practice. The largest transformation of second sound into ordinary sound in vapor (Figure 2) occurs at normal incidence in the temperature region 1–2°K. With increasing angle of in-

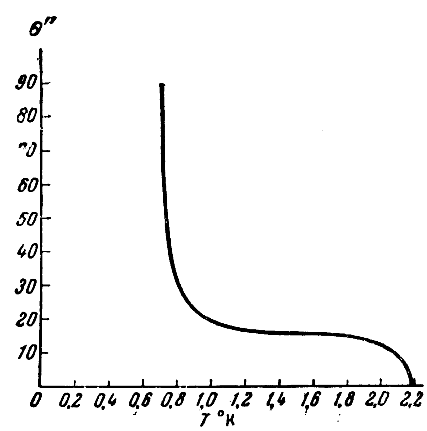

FIG. 1. Temperature dependence of the critical angle θ''_2.

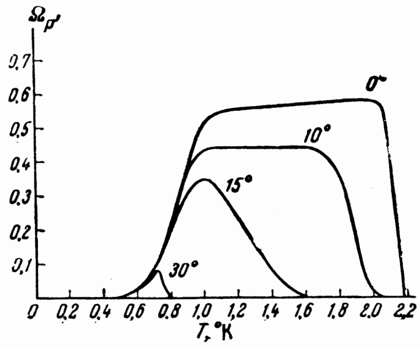

FIG. 2. Temperature dependence of $\Omega_{P'}$ at constant second-sound incidence angle (the figures on the curves denote the angles of second-sound incidence).

cidence, $\Omega_{P'}$ decreases greatly together with the temperature range in which $\Omega_{P'}$ has a maximum. The reason is that the critical angles are small in the temperature region in which the transformation of second sound is most noticeable. At sufficiently low temperatures (below 0.5°K), when all the thermodynamic quantities are determined only by phonons, $\Omega_{P'}$ tends to zero with decreasing temperature, and there is practically no transformation of second sound.

2. REFLECTION AND TRANSFORMATION OF A SOUND WAVE INCIDENT ON THE BOUNDARY FROM THE VAPOR

In the case of incidence of a sound wave P_i' from the vapor on the boundary between the He II and its vapor, two waves are produced in the liquid, P_1' (first sound) and T_2' (second sound); in addition, a reflected wave P_r' is produced in the vapor. In addition to these sound waves, as in the preceding case, we take into account in the boundary conditions the thermal wave in the vapor T'. We denote by θ_i, θ_r, θ_1, and θ_2 the angles between the directions of P_i', P_r', P_1', P_2' and the z axis, respectively. From the equality of k_x for all four waves, we get

$$\theta_r = \theta_i = \theta,\ \sin\theta_1 = \sin\theta c_1/c,\ \sin\theta_2 = \sin\theta c_2/c; \quad (2.1)$$

The connection between the amplitudes of the sound waves is determined by the boundary conditions (1.2), which in this case take the form

$$P_1^{(0)} - (P_i^{(0)} + P_r^{(0)}) = \alpha\omega c^{-2}\sin^2\theta\eta^{(0)},$$

$$-\cos\theta_1 P_1^{(0)}/c_1 + i\omega\rho_l\eta^{(0)} = \cos\theta(-P_i^{(0)} + P_r^{(0)})/c + i\omega\rho_v\eta^{(0)},$$

$$w_l\left(-\frac{\cos\theta_1}{c_1}P_1^{(0)} + i\omega\rho_l\eta^{(0)}\right) - \rho_l\frac{\rho_s}{\rho_n}\frac{\sigma_l{}^2T}{c_2}T_2^{(0)}\cos\theta_2 = w_v\left[\frac{\cos\theta}{c}(-P_i^{(0)} + P_r^{(0)}) + i\omega\rho_v\eta^{(0)}\right],$$

$$T_2^{(0)} = (P_i^{(0)} + P_r^{(0)})\beta_vT/\rho_vc_{pv} + T^{(0)},$$

$$\rho_l^{-1}P_1^{(0)} - \sigma_lT^{(0)} = \rho_v^{-1}(P_i^{(0)} + P_r^{(0)}) - \sigma_l[(\beta_vT/\rho_vc_{pv})(P_i^{(0)} + P_r^{(0)}) + T^{(0)}]. \quad (2.2)$$

Neglecting surface tension, as before ($\alpha\omega/\rho_lc^3 \ll 1$), and recognizing that $\sigma_l/\sigma_v \ll 1$ and $\rho_v/\rho_l \ll 1$, we obtain for the solution of the system (2.2)

$$\frac{P_1^{(0)}}{P_i^{(0)}} = \frac{2\cos\theta}{\cos\theta + \gamma\cos\theta_2}, \quad \frac{T_2^{(0)}}{P_i^{(0)}} = \frac{2(\rho_l\sigma_l)^{-1}\cos\theta}{\cos\theta + \gamma\cos\theta_2},$$

$$\frac{P_r^{(0)}}{P_i^{(0)}} = \frac{\cos\theta - \gamma\cos\theta_2}{\cos\theta + \gamma\cos\theta_2}, \quad (2.3)$$

where

$$\cos\theta_2 = (1 - \sin^2\theta c_2^2/c^2)^{1/2}.$$

With the aid of (2.3) we determine the coefficient $\Omega_{P_r'}$ of reflection of the incident wave from the boundary, and the coefficients $\Omega_{P_1'}$ and $\Omega_{T_2'}$ of transformation of the incident wave into first and second sound, respectively:

$$\Omega_{P_1'} = \frac{4(\rho_vc/\rho_lc_1)\cos\theta_1\cos\theta}{(\cos\theta + \gamma\cos\theta_2)^2}, \quad \Omega_{T_2'} = \frac{4\gamma\cos\theta\cos\theta_2}{(\cos\theta + \gamma\cos\theta_2)^2},$$

$$\Omega_{P_r'} = \left(\frac{\cos\theta - \gamma\cos\theta_2}{\cos\theta + \gamma\cos\theta_2}\right)^2, \quad (2.4)$$

where

$$\cos\theta_1 = [1 - (c_1/c)^2\sin^2\theta]^{1/2}.$$

We see from (2.3) and (2.4) that when the sound is incident from the vapor the critical angle θ'' is determined from the relation $\sin\theta'' = c/c_2$ and, unlike the preceding case of incidence of second sound, the angle exists only in a temperature region lower than 0.7°K, in which c is smaller than c_2 (Fig. 3).

For the same reasons as before, we neglect the transformation of the incident wave into first sound. As can be seen from (1.4) and (2.4), $\Omega_{P'}$

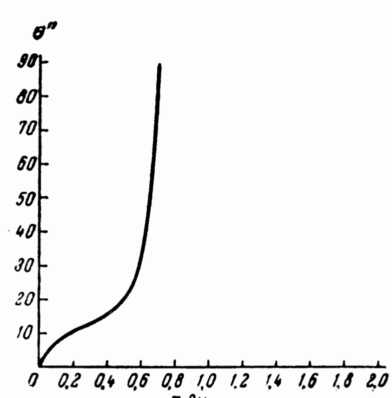

FIG. 3. Temperature dependence of the critical angle θ''.

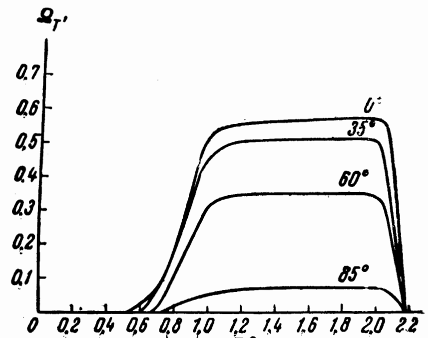

FIG. 4. Temperature dependence of $\Omega_{T_2'}$ on the temperature at a constant angle of incidence of the sound wave from the vapor. (The numbers on the curves denote the soundwave angles of incidence.)

and $\Omega_{T_2'}$ are given by the same expressions, except that θ and θ_2 depend differently on the angle of incidence in the two cases. Therefore in the case of normal incidence $\Omega_{P'}$ and $\Omega_{T_2'}$ coincide (Figs. 2 and 4), and $\Omega_{T_2'}$ increases much less with increasing angle of incidence than $\Omega_{P'}$. The reason is that critical angles now exist at low temperatures, at which the transformation of the incident wave into second sound is insignificant. At sufficiently low temperatures (below 0.5°K), when all the thermodynamic quantities are determined only by the phonons, there is practically no transformation of the incident wave, as in the preceding case.

3. REFLECTION AND TRANSFORMATION OF FIRST SOUND

In the case of incidence of first sound P_{1i}' on the boundary between He II and its vapor, two waves are produced in the liquid, P_{1r}' (reflected first sound) and T_2' (second sound), and one wave in the vapor P'. In addition to these sound waves we include in the boundary conditions, as in the preceding cases, the thermal wave in the vapor T'. We denote by θ_{1i}, θ_{1r}, θ_2, θ the angles between the directions P_{1i}', P_{1r}', T_2', P' and the z axis, respectively. From the equality of k_x for all four waves, we get

$$\theta_{1r} = \theta_{1i} = \theta_1,$$

$$\sin\theta_2 = \sin\theta_1 c_2 / c_1, \qquad \sin\theta = \sin\theta_1 c / c_1. \quad (3.1)$$

The connection between the amplitudes of the sound waves is determined by the boundary conditions (2.1), which in this case take the form

$$P_{1i}^{(0)} + P_{1r}^{(0)} - P^{(0)} = \alpha\omega^2 c_1^{-2}\sin^2\theta_1\eta^{(0)},$$

$$\cos\theta_1 (P_{1i}^{(0)} - P_{1r}^{(0)}) / c_1 + i\omega\rho_l\eta^{(0)} = \cos\theta P^{(0)} / c + i\omega\rho_v\eta^{(0)},$$

$$w_l\,[\cos\theta_1 (P_{1i}^{(0)} - P_{1r}^{(0)}) / c_1 + i\omega\rho_l\eta^{(0)}] - \cos\theta_2 T_2^{(0)}\rho_l\rho_s\sigma_l^{\,2} T / \rho_n c_2 = w_v(\cos\theta P^0 / c + i\omega\rho_v\eta^{(0)}),$$

$$T^{(0)} = P^{(0)}\beta_v T / \rho_v c_{pv} + T^{(0)},$$

$$\rho_l^{-1}(P_i^{(0)} + P_r^{(0)}) - \sigma_l T^{(0)} = \rho_v^{-1} P^{(0)} - \sigma_v(P^{(0)}\beta_v T / \rho_v c_{pv} + T^{(0)}), \quad (3.2)$$

where

$$\cos\theta_2 = (1 - \sin^2\theta_1 c_2^2 / c_1^2)^{1/2},$$

$$\cos\theta = (1 - \sin^2\theta_1 c^2 / c_1^2)^{1/2}.$$

Since c_1 is larger than c_2 or c, there are no critical angles in this case. Neglecting, the surface tension ($\alpha\omega/\rho_l c_1^3 \ll 1$), as before, and recognizing that $\sigma_l/\sigma_v \ll 1$ and $\rho_v/\rho_l \ll 1$, we obtain the coefficients of reflection of the first sound from the separation boundary, $\Omega_{P_{1r}'}$ of transformation of first sound into second sound $\Omega_{T_2'}$, and into ordinary sound in vapor $\Omega_{P'}$

$$\Omega_{P'_{1r}} = 1 - \frac{4(\rho_v c/\rho_l c_1)\cos\theta_1}{\cos\theta + \gamma\cos\theta_2},$$

$$\Omega_{T'} = \frac{4(\rho_v c/\rho_l c_1)\gamma\cos\theta_1\cos\theta_2}{(\cos\theta + \gamma\cos\theta_2)^2},$$

$$\Omega_{P'} = \frac{4(\rho_v c/\rho_l c_1)\cos\theta_1\cos\theta}{(\cos\theta + \gamma\cos\theta_2)^2}. \quad (3.3)$$

Recognizing that $\sigma_l/\sigma_v \ll 1$ and $\rho_v/\rho_l \ll 1$, we can put $\Omega_{P_r'} = 1$, and $\Omega_{T_2'} = \Omega_{P'} = 0$, i.e., the first sound incident on the boundary between He II and its vapor is reflected almost completely.

I consider it my pleasant duty to express deep gratitude to Professor I. M. Khalatnikov for a discussion of the work and for valuable remarks.

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Translated by J. G. Adashko
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