ON THE COULOMB GREEN'S FUNCTION

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A representation of the Green's function for the Dirac equation, which is a generalization of the Furry-Sommerfeld-Maue approximation, is derived. A closed form for the nonrelativistic Green's function in momentum space is found.

1. INTRODUCTION

KNOWLEDGE of the Green's function of a Dirac particle in the Coulomb field of the nucleus, which plays a role of a propagation function, is required in many problems of quantum electrodynamics (internal bremsstrahlung, radiative capture of electrons and muons by nuclei, etc). If a retarded (advanced) Green's function is available, we can obtain for the outgoing (incoming) wave function of a particle an expression with an asymptotic representation in the form of a plane and divergent (convergent) wave. No closed-form expression has yet been found for the relativistic outgoing Coulomb wave function and the Green's function. It is likewise not clear whether a closed form exists for these functions at all.

The existence of an analytic expression for the non-relativistic outgoing Coulomb function has stimulated the search for a closed form of the corresponding Green's function. Recently Hostler[1] obtained, by summation of a partial-wave Green's function expansion, a closed form constituting a product of two Whittaker functions for the nonrelativistic Coulomb Green's function in coordinate space.

In the present paper we obtain a representation for the relativistic Green's function of a Dirac particle in the form of a series that is a generalization of the Furry-Sommerfeld-Maue (FSM) approximation. The analysis will be in complete analogy with the corresponding procedure for the relativistic outgoing function[2]. The representation is based on the existence of a closed expression for the nonrelativistic Coulomb Green's function in the momentum space. In this connection, we obtain in the third section of the article, using the method developed in[2] for summing perturbation-theory series, a closed form for the nonrelativistic Green's function in momentum space.

A transition to the coordinate representation, made in the last section of the article, yields the result of Hostler[1] for the nonrelativistic Green's function.

2. COULOMB GREEN'S FUNCTION OF THE DIRAC EQUATION

The Green's function G_C of the Dirac equation in a Coulomb field $-\alpha ZV$ is given by [1)]

$$G_c = G - \alpha ZGVG_c, \qquad \hat{V} = \gamma_4 V, \tag{1}$$

where G is the Green's function of the Dirac equation in the absence of a field, and is diagonal in momentum space:

$$\langle \mathbf{k}_2 | G | \mathbf{k}_1 \rangle = G(\mathbf{k}_1)\, \delta(\mathbf{k}_2 - \mathbf{k}_1),$$

$$G(\mathbf{k}) = \frac{-1}{i\hat{k} + m} = \frac{i\hat{k} - m}{\mathbf{k}^2 - \mathbf{p}^2 - i\varepsilon}, \qquad \hat{k} = \mathbf{k}\gamma + iE\gamma_4. \tag{2}$$

In (1) and (2), Z—charge of the nucleus, α—fine-structure constant, E and p—energy and momentum of the particle, and the sign of the infinitesimally small increment ϵ determines the rule for circuiting around the Green's-function poles. When $\epsilon > 0$ we obtain the retarded Green's function, and when $\epsilon < 0$ the advanced function. By retaining the functional dependence on ϵ we can also obtain the circuiting rules for the many-particle Green's function in a Coulomb field (see Sec. 3).

We introduce further the Moller operator φ in accordance with the definition

$$G_c = \varphi G, \qquad \varphi = 1 - \alpha ZG\hat{V}\varphi. \tag{3}$$

Following the procedure used in[2], we represent φ in the form of a sum

$$\varphi = \varphi^0 + \Phi, \tag{4}$$

where φ^0 satisfies the nonrelativistic equation [2)]

[1)]The analysis in the present section is applicable to any electrostatic potential V.

[2)]The usual nonrelativistic equation is obtained by replacing E with m. The function G^0 (5) differs from the nonrelativistic free Green's function by a factor 2E.

$$\varphi^0 = 1 - \alpha Z G^0 V \varphi^0. \qquad G^0(\mathbf{k}) = \frac{-2E}{\mathbf{k}^2 - \mathbf{p}^2 - i\varepsilon}, \tag{5}$$

Substituting (4) in (3), and using (5) and the identity

$$G\gamma_4 = \left(1 - \frac{\gamma_4}{2E} G^{-1}\right) G^0$$
$$\times \left(G(\mathbf{k})\gamma_4 = \left(1 - \frac{\gamma_4}{2E} G^{-1}(\mathbf{k})\right) G^0(\mathbf{k})\right), \tag{6}$$

we obtain the following expression for Φ[3]:

$$\Phi = \varphi^{\mathrm{I}} - \alpha Z G_c \hat{V} \varphi^{\mathrm{I}}, \qquad \varphi^{\mathrm{I}} = \frac{\gamma_4}{2E} G^{-1}(1 - \varphi^0), \tag{7}$$

$$\langle \mathbf{k}_2 | \varphi^{\mathrm{I}} | \mathbf{k}_1 \rangle = \frac{\gamma_4}{2E}(i\hat{k} + m)\{\langle \mathbf{k}_2 | \varphi^0 | \mathbf{k}_1 \rangle - \delta(\mathbf{k}_2 - \mathbf{k}_1)\}. \tag{7a}$$

With the aid of (4), (6), and (7), we can represent expression (3) for the Green's function G_c in the form

$$G_c = \mathfrak{G}^0 + \mathfrak{G}, \qquad \mathfrak{G} = \mathfrak{G}^{\mathrm{I}} - \alpha Z G_c \hat{V} \mathfrak{G}^{\mathrm{I}}, \tag{8}$$

$$\mathfrak{G}^0 = \varphi^0 G = \varphi^0 G^0 \left(1 - \frac{\gamma_4}{2E} G^{-1}\right)\gamma_4 = G_c{}^0\left(1 - \frac{\gamma_4}{2E} G^{-1}\right)\gamma_4, \tag{8a}$$

$$\mathfrak{G}^{\mathrm{I}} = \frac{\gamma_4}{2E}(1 - G^{-1}\varphi^0 G) = \frac{\gamma_4}{2E}\left\{1 - G^{-1}\gamma_4 G_c{}^0 + \frac{1}{2E} G^{-1}\gamma_4 G_c{}^0 G^{-1}\gamma_4\right\}, \tag{8b}$$

$$G_c{}^0 = \varphi^0 G^0, \qquad \langle \mathbf{k}_2 | G_c{}^0 | \mathbf{k}_1 \rangle = \langle \mathbf{k}_2 | \varphi^0 | \mathbf{k}_1 \rangle G^0(\mathbf{k}_1). \tag{8c}$$

Using (2) and (8c) we can readily rewrite (8a) and (8b) in momentum space.

The functions (7a) and (8b) are proportional to αZ, since $\delta(\mathbf{k}_2 - \mathbf{k}_1)$ is cancelled out by the first term of the expansion of $\langle \mathbf{k}_2 | \varphi^0 | \mathbf{k}_1 \rangle$ in powers of αZ. By integrating Eq. (8) for the function $\mathfrak{G}$, we can represent the Green's function G_c in the form of a series in powers of aZ, in perfect analogy with the procedure used for the wave function in[2]. The first two terms of the expansion, $\mathfrak{G}^0 + \mathfrak{G}^1$, represent the Green's function in the FSM approximation. Indeed, when k_1 is replaced by p, the Moller operator φ goes over in momentum space into an outgoing wave function (footnote 3). The first two terms of the expansion of φ in αZ, $\varphi^0 + \varphi^1$, go over here into the FSM function[2], a feature of which is that terms of order $(\alpha Z)^2/l^2$ have been discarded in the expansion of the exact function in terms of the orbital angular momentum l[3]. The parameters of the expansion of the FSM approximation come from the expansion of the quantities $\gamma_l = \{(j + 1/2)^2 - \alpha^2 Z^2\}^{1/2}$, which do not depend on p or r. By going over to momentum space in each partial wave we can return to the expression for the Green's function by replacing the corresponding p by k_1 (footnote 3), without changing the form of γ_l. It follows therefore that the FSM approximation has the same meaning for the Green's function as for the wave function.

We note further that we have not used the squared Dirac equation, and therefore the FSM approximation has been written out by us in explicit form, in contrast with the results of Hostler[1], where it is necessary to act further on the function by some operator.

We see from (8) that in order to obtain the relativistic Green's function in the FSM approximation and to determine the corrections to this approximation it is necessary to have a closed expression for the nonrelativistic Moller function φ^0. The next section is devoted to a determination of this function in momentum space.

3. MOMENTUM REPRESENTATION OF NON-RELATIVISTIC FUNCTIONS

In this section we shall consider only nonrelativistic functions, identified in the preceding section by a zero superscript. In order not to clutter up the notation, this superscript will be left out if used in this sense.

We introduce first some definitions. The Yukawa potential will be denoted by the symbol $V_{i\eta}$ (i—imaginary unit). In momentum and coordinate space, the operator of the Yukawa potential takes the form

$$\langle \mathbf{k}_2 | V_{i\eta} | \mathbf{k}_1 \rangle = \frac{1}{2\pi^2} \frac{1}{(\mathbf{k}_2 - \mathbf{k}_1)^2 + \eta^2},$$
$$\langle \mathbf{r}_2 | V_{i\eta} | \mathbf{r}_1 \rangle = \frac{e^{-\eta r_1}}{r_1} \delta(\mathbf{r}_2 - \mathbf{r}_1). \tag{9}$$

When $\eta = 0$ we obtain the Coulomb potential V_0. We note one important identity

$$-\frac{\partial}{\partial \eta} V_{i\eta}\Big|_{\eta \to 0} = 1, \quad -\frac{\partial}{\partial \eta}\langle \mathbf{k}_2 | V_{i\eta} | \mathbf{k}_1 \rangle\Big|_{\eta \to 0} = \delta(\mathbf{k}_2 - \mathbf{k}_1). \tag{10}$$

The Coulomb Moller function of interest to us is the solution of equation

$$\varphi = 1 - \alpha Z G V_0 \varphi. \tag{11}$$

We introduce the function $T_{i\eta} = V_{i\eta}\varphi$, satisfying the equation

$$T_{i\eta} = V_{i\eta} - \alpha Z V_{i\eta} G T_0. \tag{12}$$

[3]All the corresponding expressions for the wave function are obtained by making the substitution $k_1 = p$ and by premultiplying the Dirac spinor u_p. For example, by making the indicated operations in (7a) and by taking into account the Dirac equation $(i\hat{p} + m)u_p = 0$, we obtain the second term of the FSM function (formula (26) of[2]).

From the definition of $T_{i\eta}$ and from the identity (10) we obtain the following two representations for φ:

$$\varphi = 1 - \alpha ZGT_0, \qquad \varphi = -\frac{\partial}{\partial\eta} T_{i\eta}|_{\eta\to 0}, \tag{13}$$

$$\langle \mathbf{k}_2|\varphi|\mathbf{k}_1\rangle = \delta(\mathbf{k}_2 - \mathbf{k}_1) - \alpha ZG(\mathbf{k}_2)\langle \mathbf{k}_2|T_0|\mathbf{k}_1\rangle. \tag{13a}$$

Expanding (12) in powers of aZ, we obtain

$$\langle \mathbf{k}_2|T_{i\eta}|\mathbf{k}_1\rangle = \sum_{n=0}^{\infty} T_{i\eta}{}^n, \quad T_{i\eta}{}^n = (-\alpha Z)^n \langle \mathbf{k}_2|V_{i\eta}(GV_0)^n|\mathbf{k}_1\rangle. \tag{14}$$

The summation of the series (14) is in perfect analogy with the procedure in [2], with the aid of an identity valid for arbitrary a and k:

$$-\alpha Z\langle \mathbf{s}|V_{i\eta}GV_{pa}|\mathbf{k}\rangle = i\xi\int_0^1 \frac{dy}{\Lambda}\langle \mathbf{s}|V_{p\Lambda+i\eta}|\mathbf{B}\rangle, \tag{15}$$

$$\Lambda = +\{(1-n^2y)(1-y) + a^2y\}^{1/2}, \quad \mathbf{B} = \mathbf{k}y, \quad \mathbf{n} = \mathbf{k}/p, \tag{15a}$$

$$\xi = \alpha ZE/p, \qquad p = +(p^2 + i\varepsilon)^{1/2}. \tag{15b}$$

The expression for $T_{i\eta}$ is obtained from (15) by putting $\mathbf{s} = \mathbf{k}_2$, a = 0, and $\mathbf{k} = \mathbf{k}_1$. The integration variables y, Λ, and $\mathbf{B}$ will be tagged in this case by a subscript 1. We now put $a = \Lambda_1$ and $\mathbf{k} = \mathbf{B}_1$; we then obtain in the right side of (15)

$$y = y_2, \quad \Lambda^2 = \Lambda_2^2 = (1 - n_1^2y_1^2y_2)(1-y_2) + \Lambda_1^2y_2 = (1 - n_1^2y_1y_2)(1 - y_1y_2),$$

$$\mathbf{B} = \mathbf{B}_2 = \mathbf{k}_1y_1y_2, \qquad \mathbf{n}_1 = \mathbf{k}_1/p. \tag{16}$$

Making a change of variables $y_1 = x_1$ and $y_1y_2 = x_2$ ($dx_2/x_1 = dy_2$), and pre-multiplying both sides of (15) by x_1, we obtain

$$-\alpha Zx_1\langle \mathbf{s}|V_{i\eta}GV_{p\Lambda_1}|\mathbf{B}_1\rangle = i\xi\int_0^{x_1}\frac{dx_2}{x_2\Lambda_2}x_2\langle \mathbf{s}|V_{p\Lambda_2+i\eta}|\mathbf{B}_2\rangle,$$

$$\Lambda_i^2 = (1 - n_1^2y_i)(1-y_i), \qquad \mathbf{B}_i = \mathbf{k}_1x_i, \qquad \mathbf{n}_1 = \mathbf{k}_1/p. \tag{17}$$

The expression for $T_{i\eta}{}^2$ can be obtained from (17) by replacing $\mathbf{s}$ with $\mathbf{k}_2$ and operating on both sides of the equation with the operator $\int_0^1 dx_1/x_1\Lambda_1$. Replacing further in (17) the indices 2 and 1 by n and n − 1, and operating on both sides of the equation with the operator

$$\int_0^1 dx_1/x_1\Lambda_1 \ldots \int_0^{x_{n-2}} dx_{n-1}/x_{n-1}\Lambda_{n-1},$$

we obtain for $T_{i\eta}{}^n$ an expression in the form of the following recurrence relation (see the corresponding procedure for the function B in [2]):

$$T_{i\eta}{}^n \equiv T_{i\eta}{}^n(1), \qquad T_{i\eta}{}^n(x) = i\xi\int_0^x \frac{dx_1}{x_1\Lambda_1}T_{i\eta}{}^{n-1}(x_1), \tag{18}$$

$$T_{i\eta}{}^0(x) = x\langle \mathbf{k}_2|V_{p\Lambda+i\eta}|\mathbf{k}_1x\rangle, \qquad \Lambda^2 = (1 - n_1^2x)(1-x). \tag{18a}$$

Summing both sides of (18) over n from n = 1 to infinity, we obtain

$$\langle \mathbf{k}_2|T_{i\eta}(x)|\mathbf{k}_1\rangle = T_{i\eta}{}^0(x) + i\xi\int_0^x\frac{dx_1}{x_1\Lambda_1}\langle \mathbf{k}_2|T_{i\eta}(x_1)|\mathbf{k}_1\rangle,$$

$$\langle \mathbf{k}_2|T_{i\eta}|\mathbf{k}_1\rangle = \langle \mathbf{k}_2|T_{i\eta}(1)|\mathbf{k}_1\rangle. \tag{19}$$

The solution of the integral equation (19) can be obtained either by summing an iteration series, as in Sec. 2 of [2], or by a reduction to a differential equation via differentiation with respect to x. This solution is of the form

$$\langle \mathbf{k}_2|T_{i\eta}|\mathbf{k}_1\rangle = \int_0^1 \exp\left\{i\xi\int_x^1\frac{dx_1}{x_1\Lambda_1}\right\}\frac{\partial}{\partial x}(x\langle \mathbf{k}_2|V_{p\Lambda+i\eta}|\mathbf{k}_1x\rangle)\,dx \tag{20a}$$

$$= \langle \mathbf{k}_2|V_{i\eta}|\mathbf{k}_1\rangle + i\xi\int_0^1\frac{dx}{\Lambda}\langle \mathbf{k}_2|V_{p\Lambda+i\eta}|\mathbf{k}_1x\rangle\exp\left\{i\xi\int_x^1\frac{dx_1}{x_1\Lambda_1}\right\}. \tag{20b}$$

We now show that the solution (20) satisfies the fundamental equation (12). To this end we set up the expression $aZ\langle \mathbf{k}_2|V_{i\eta}GT_0|\mathbf{k}_1\rangle$ and use the identity obtained by differentiating both sides of (17) with respect to x:

$$-\alpha Z\langle \mathbf{k}_2|V_{i\eta}GT_0|\mathbf{k}_1\rangle$$

$$= -\alpha Z\int_0^1 dx\frac{\partial}{\partial x}(x\langle \mathbf{k}_2|V_{i\eta}GV_{p\Lambda}|\mathbf{k}_1x\rangle)\exp$$

$$= i\xi\int_0^1\frac{dx}{\Lambda}\langle \mathbf{k}_2|V_{p\Lambda+i\eta}|\mathbf{k}_1x\rangle\exp$$

$$= \langle \mathbf{k}_2|T_{i\eta}|\mathbf{k}_1\rangle - \langle \mathbf{k}_2|V_{i\eta}|\mathbf{k}_1\rangle, \quad \exp = \exp\left\{\int_x^1 dx_1/x_1\Lambda_1\right\}. \tag{21}$$

The solution (20) can be transformed to a simpler form by making the change of variable

$$x = \frac{\zeta^2 - 1}{\zeta^2 - n_1^2}, \qquad \Lambda = \zeta(1-x), \qquad \frac{dx}{x\Lambda} = \frac{2d\zeta}{\zeta^2 - 1}. \tag{22}$$

Let us write out the expression (20b) for the case $\eta = 0$ [4)]:

$$\langle \mathbf{k}_2|T_0|\mathbf{k}_1\rangle = \frac{1}{q^2}\left\{1 + i\xi\int_1^\infty d\zeta\left(\frac{\zeta+1}{\zeta-1}\right)^{i\xi}\frac{2}{\zeta^2 - \Delta^2}\right\},$$

$$\xi = \frac{\alpha ZE}{p}, \qquad \Delta^2 = 1 + (1 - n_1^2)(1 - n_2^2)p^2/q^2,$$

$$\mathbf{q} = \mathbf{k}_2 - \mathbf{k}_1, \qquad \mathbf{n}_i = \mathbf{k}_i/p. \tag{23}$$

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4) The integral representation (23) is not valid for all values of the parameters ξ and Δ, so that it is more convenient to use the representation in the form of a contour integral along the cut [1]. We note that the change of variable $z = (\xi - 1)/(\xi + 1)$ leads to the integral representation obtained by Brattsev and Trifonov [4].

Formula (23) can be represented in analytic form in terms of hypergeometric functions:

$$\langle \mathbf{k}_2 | T_0 | \mathbf{k}_1 \rangle = \frac{1}{q^2}\left\{1 + \frac{1}{2\Delta}\left[{}_2F_1\left(1, -i\xi; 1-i\xi; \frac{\Delta-1}{\Delta+1}\right) - {}_2F_1\left(1, -i\xi; 1-i\xi; \frac{\Delta+1}{\Delta-1}\right)\right]\right\}. \quad (24)$$

Using the recurrence relations for the hypergeometric functions, we can obtain many other representations, on which we can not dwell here.

We obtain the needed expression for the Moller function by substituting (24) in (13a) and using the definition (5) of G(k).

As can be seen from (23) and (24), the function T_0, and consequently also the functions φ and G_C, contain a dependence on only two parameters, ξ (15b) and Δ (23). This circumstance can be very helpful in the calculation and tabulation of the functions. This property disappears on going over to coordinate space. However, it is precisely the momentum space which is most frequently used in applications.

The function (24) contains an infrared singularity when $\Delta \to 0$, corresponding to k_2 or $k_1 \to p$. It is precisely at these points that the expression for φ corresponds to an outgoing wave function of a particle. When k_2 and k_1 tend to p simultaneously, the function T goes over into the scattering amplitude. To explain the behavior of φ near these singularities it is necessary to introduce an infinitesimally small screening parameter. Such an analysis was made in[2] (for the wave function) and in the third section of[5] (for the amplitude).

We see from (24) that the functions T, φ, and G_C have poles at the points $i\xi = n$, n = 1, 2, 3, ..., corresponding to bound states of the particle. The rules for circuiting these poles are set by the sign of the imaginary increment $i\epsilon$ (15b). Since the momentum p is encountered in (12) only in the free Green's function, in the form of the combination $p^2 + i\epsilon$, it is necessary that p have in the final expression (24) the same meaning as in (15b). Putting $\epsilon = \epsilon'(n - n_0)$, $\epsilon' > 0$, $\epsilon' \to 0$, we obtain the circuiting rules corresponding to the many-particle Green's function with n_0 non-interacting particles in a Coulomb field.

4. COORDINATE REPRESENTATION

To obtain the coordinate representation of the non-relativistic Moller function φ and Green's function G_C it is convenient to make use of (13b). With the aid of this formula, using the definition of the Green's function (3) and the expression for $T_{i\eta}$ (20a), we obtain

$$G_c = \varphi G = -\frac{\partial}{\partial\eta} T_{i\eta} G = 2E \int_1^\infty d\zeta \left(\frac{\zeta+1}{\zeta-1}\right)^{i\xi} g, \quad (25)$$

where

$$\langle \mathbf{k}_2 | g | \mathbf{k}_1 \rangle = -\frac{\partial}{\partial\eta}\frac{\partial}{\partial\zeta}\frac{x}{p^2 - k_1^2} \langle \mathbf{k}_2 | V_{p\Lambda+i\eta} | \mathbf{k}_1 x \rangle, \quad (26)$$

$$x = \frac{\zeta^2 - 1}{\zeta^2 - n_1^2}, \qquad \Lambda = \zeta(1 - x), \qquad \mathbf{n}_1 = \frac{\mathbf{k}_1}{p}. \quad (27)$$

The function (26) is conveniently represented in the symmetrical form

$$\langle \mathbf{k}_2 | g | \mathbf{k}_1 \rangle = -ip\left(-\frac{\partial}{\partial\eta_1}\right)\left(-\frac{\partial}{\partial\eta_2}\right)\frac{x}{a^2} \langle \mathbf{k}_2 | V_{z_2 - z_1 x} | \mathbf{k}_1 x \rangle \quad (28)$$

$$= -ip\left(-\frac{\partial}{\partial\eta_1}\right)\left(-\frac{\partial}{\partial\eta_2}\right) \times \frac{1}{2\pi^2}\frac{1}{a^4 + 2(\mathbf{k}_1\mathbf{k}_2 - z_1 z_2)a^2 + (k_1^2 - z_1^2)(k_2^2 - z_2^2)}, \quad (28a)$$

where

$$x = \frac{a^2}{z_1^2 - k_1^2}, \quad a^2 = p^2(\zeta^2 - 1),$$
$$z_l = p\zeta + i\eta_l \quad (\eta_l \to 0). \quad (29)$$

In these formulas η has been replaced by η_1, and it is easy to verify by direct differentiation that the derivative with respect to η_2 in (28) is equivalent to the derivative with respect to ζ in (26).

To go over into coordinate space in (28), we make use of an equation that follows from the definition (9):

$$-\frac{\partial}{\partial\eta}\int e^{i\mathbf{k}'\mathbf{r}} \langle \mathbf{k}' | V_{pa+i\eta} | \mathbf{k} \rangle d^3k' = e^{i\mathbf{k}\mathbf{r}} e^{i(pa+i\eta)r}. \quad (30)$$

Applying this formula to (28) ($k' = k_2$), we obtain

$$\langle \mathbf{r}_2 | g | \mathbf{k}_1 \rangle = -ip\left(-\frac{\partial}{\partial\eta_1}\right)\frac{x}{a^2} e^{i\mathbf{k}_1\mathbf{r}_2 x} e^{i(z_2 - z_1 x)r_2}. \quad (31)$$

To go over to coordinate space with respect to the variable k_1, we make use of the Cauchy formula

$$f(\mathbf{k}^2 + A\,\mathbf{ks} + B) = \frac{1}{2\pi i}\oint \frac{f(t)\,dt}{t - t_0},$$
$$t_0 = \mathbf{k}^2 + A\,\mathbf{ks} + B, \quad (32)$$

where the closed contour goes around the point t_0. Applying formula (32) to (31), we obtain

$$\langle \mathbf{r}_2 | g | \mathbf{k}_1 \rangle = -ipe^{iz_2 r_2}\left(-\frac{\partial}{\partial\eta_1}\right) \times \frac{x}{a^2}\frac{1}{2\pi i}\oint \frac{e^t}{t - ix(\mathbf{k}_1\mathbf{r}_2 - z_1 r_2)}dt = ipe^{iz_2 r_2}\frac{1}{2\pi i}\oint\frac{dt}{t} \times \left(-\frac{\partial}{\partial\eta_1}\right) 2\pi^2 \langle \mathbf{k}_1 | V_{z_1 + i\varkappa} | i\boldsymbol{\varkappa} \rangle, \quad \boldsymbol{\varkappa} = \mathbf{r}_2 a^2/2t. \quad (33)$$

Applying again formula (30) to (33) ($k' = k_1$) and reversing the order of integration, we obtain

$$\langle \mathbf{r}_2 | g | \mathbf{r}_1 \rangle = \frac{ip}{4\pi} \frac{1}{2\pi i} \oint \frac{dt}{t} e^t e^{i(z_1 r_1 + z_2 r_2)} e^{-(\varkappa r_1 + \boldsymbol{\varkappa} \mathbf{r}_1)}, \quad (34)$$

wherein the condition for uniform convergence of the internal integral (with respect to k_1), which is needed to permit reversal of the order of integration, imposes the following limitations on the contour of integration with respect to t:

$$\mathrm{Re}(\varkappa r_1 \pm \boldsymbol{\varkappa} \mathbf{r}_1) < \eta_1 \quad \text{for} \quad |t| > r_1 r_2 a^2 / \eta_1. \quad (35)$$

Thus, the point t = 0 must be inside the integration contour, as well as the principal pole in the integral (33). Letting η_l go to zero and using the definitions (29) and the integral representation of the Bessel function[6], we obtain ultimately

$$\langle \mathbf{r}_2 | g | \mathbf{r}_1 \rangle = \frac{ip}{4\pi} e^{ip\zeta(r_1+r_2)} \frac{1}{2\pi i} \oint \frac{dt}{t} e^t e^{-a^2u^2/4t}$$

$$= \frac{ip}{4\pi} e^{ip\zeta v} J_0(au),$$

$$v = r_1 + r_2, \quad u^2 = 2(r_1 r_2 + \mathbf{r}_1 \mathbf{r}_2), \quad a^2 = p^2(\zeta^2 - 1). \quad (36)$$

Substituting (36) in (25), we arrive at a formula that differs from the corresponding formula (1.13) of[1] by a factor 2E, in accordance with our definition (5) of the function G (see footnote 2). Another difference is that the integral along the real ζ axis is replaced by a contour integral along the cut, which can readily be reduced to a product of Whittaker functions[1].

The coordinate representation for the relativistic functions (8a) and (8b) is obtained by using (36) and (25) and by recalling that the relativistic operators G^{-1} can be written in coordinate space in diagonal form

$$\langle \mathbf{r}_2 | G^{-1} | \mathbf{r}_1 \rangle = -(i\gamma_4 E + i\nabla_{r_1} \boldsymbol{\gamma} + m)\delta(\mathbf{r}_2 - \mathbf{r}_1). \quad (37)$$

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[1] L. Hostler, Preprint ITP-89. L. Hostler and R. H. Pratt, Phys. Rev. Lett. **10**, 469 (1963).

[2] V. G. Gorshkov, JETP **40**, 1481 (1961), Soviet Phys. JETP **13**, 1037 (1961).

[3] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954). W. R. Johnson and R. T. Deck, J. Math. Phys. **3**, 319 (1962).

[4] V. F. Brattsev and E. D. Trifonov, Vestnik, Leningrad State University, Physics-Chemistry, **16**, 36 (1962).

[5] V. G. Gorshkov, JETP **43**, 1714 (1962), Soviet Phys. JETP **16**, 121 (1963).

[6] E. T. Whittaker and G. N. Watson, A Course in Modern Analysis, Cambridge Univ. Press, 1940.

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