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ASYMPTOTIC EXPRESSION FOR THE SCATTERING AMPLITUDE IN THE COMPTON EFFECT

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The scattering amplitude for the Compton effect is obtained for $t \to \infty$ by summation of a certain class of perturbation-theory diagrams which are improved by aid of the renormalization-group. The calculations are performed in the threshold region $s \to 4m^2$.

MANY authors [1,2] pointed out the effectiveness of the renormalization-group method for the analysis of asymptotic behavior of scattering amplitudes. As to quantum electrodynamics, it was shown in [2], with the electron-positron system as an example, that the summation of a definite class of perturbation-theory diagrams, improved with the aid of the renormalization group, leads in the high-energy limit to a "Regge" behavior of the scattering amplitudes in the crossed channel.

In order to obtain the asymptotic behavior at high energies we consider in the present article, from the same point of view, a second example in quantum electrodynamics—the Compton effect. The asymptotic behavior is considered in the positronium threshold region $s \to 4m^2$, in order to check the correspondence between the values of the energy levels obtained from the expression for the Regge exponent, and the values of the positronium energy levels calculated in the usual manner.

The renormalization group method for finding the asymptotic behavior is applied not to the invariant amplitudes, the choice of which is arbitrary, but to physical amplitudes, the scattering amplitudes in the singlet and triplet states. We therefore first expand the scattering amplitude in the lower orders of perturbation theory in a series of invariant amplitudes, separate the amplitudes corresponding to scattering in the s-channel in the singlet and triplet states, and then employ the method of summation with the aid of the renormalization group.

We write down the scattering matrix in the following well-known fashion:

$$S = 1 - i(2\pi)^{-2}\delta^4(q_1 + p_1 - q_2 - p_2)(4q_1^0 q_2^0)^{-1/2} F. \quad (1)$$

Since F is bilinear in the polarization vectors, we have, say for the t-channel:

$$\langle \gamma_2 N_2 | F | \gamma_1 N_1 \rangle = \sum_{\mu,\nu} \varepsilon_{2\mu} \bar{u}(p_2) F_{\mu\nu} u(p_1) \varepsilon_{1\nu}. \quad (2)$$

The use of both invariance principles leads to an expansion of $F_{\mu\nu}$ into a system of six linearly independent amplitudes [4]

$$F_{\mu\nu} = \sum_i A_i I_{\mu\nu}^i. \quad (3)$$

The normalization is carried out in such a way that there are no kinematic singularities in the invariant amplitudes.

By means of corresponding simple calculations we can separate the invariant amplitudes A_i in the second- and fourth-order matrix elements. In the second order, the contribution to the scattering amplitude is given by pole diagrams for which

$$F_{\mu\nu}^{(2)} = e^2\gamma_\mu \frac{\hat{p}_1 + \hat{q}_1 + m}{t - m^2}\gamma_\nu + e^2\gamma_\mu \frac{\hat{p}_1 - \hat{q}_2 + m}{u - m^2}\gamma_\nu. \quad (4)$$

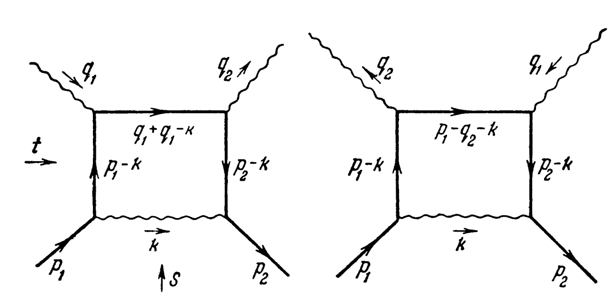
For the invariant amplitudes we obtain the following expression:

$$A_i^{(2)} = R_i e^2 \left\{ \frac{1}{t - m^2} + \frac{\eta_i}{u - m^2} \right\},$$

$$\eta_i = \begin{cases} +1 & \text{for } i = 1, 2, 3, 6 \\ -1 & \text{for } i = 4, 5 \end{cases},$$

$$R_1 = 2m, \quad R_2 = 0, \quad R_3 = +m,$$

$$R_4 = -i, \quad R_5 = i, \quad R_6 = -i. \quad (5)$$



a b

In fourth order the diagrams a and b are important. The remaining fourth-order diagrams make contributions which do not depend on s, and are therefore insignificant in the threshold region, as will be shown below. For the fourth-order diagram a we have

$$F_{\mu\nu}^{a(4)} = \frac{e^4}{(2\pi)^4 i} \sum g^{nn} \int d^4k \gamma_n (\hat{p}_2 - \hat{k} + m) \gamma_\mu \times (\hat{p}_1 + \hat{q}_1 - \hat{k} + m) \gamma_\nu (\hat{p}_1 - \hat{k} + m) \gamma_n \Phi \times (p_1, p_2, p_1 + q_1, k), \quad (6)$$

$$\Phi(p_1, p_2, p_1 + q_1, k) = 1 / (k^2 - \lambda^2) [(p_1 - k)^2 - m^2] \times [(p_2 - k)^2 - m^2][(p_1 + q_1 - k)^2 - m^2]. \quad (6a)$$

To find the invariant amplitudes it will be necessary to calculate the following integrals:

$$J_0 = \frac{1}{\pi^2 i} \int d^4k \Phi(p_1, p_2, p_1 + q_1, k),$$

$$J_\alpha = \frac{1}{\pi^2 i} \int d^4k k_\alpha \Phi(p_1, p_2, p_1 + q_1, k),$$

$$J_{\alpha\beta} = \frac{1}{\pi^2 i} \int d^4k k_\alpha k_\beta \Phi(p_1, p_2, p_1 + q_1, k),$$

$$J_{\alpha\beta\gamma} = \frac{1}{\pi^2 i} \int d^4k k_\alpha k_\beta k_\gamma \Phi(p_1, p_2, p_1 + q_1, k). \quad (7)$$

In the asymptotic region $t \to \infty$ we have

$$J_0 = -2f(s) t^{-1} \ln t, \quad (8)$$

where

$$f(s) = \int_{4m^2}^{\infty} \frac{ds'}{(s' - s)[s'(s' - 4m^2)]^{1/2}}. \quad (9)$$

For

$$s < 4m^2, \qquad 4m^2 - s \ll m^2$$

we have

$$f(s) = \pi / [s(4m^2 - s)]^{1/2}. \quad (10)$$

From (10) we see that the terms proportional to J_0 have a singularity in the threshold region $s \to 4m^2$, and from this it follows that the contributions due to the diagrams that do not depend on s are insignificant in the threshold region.

Account of the principal terms in the threshold region yields for J_α, $J_{\alpha\beta}$, and $J_{\alpha\beta\gamma}$

$$J_\alpha = J_0 P_\alpha, \qquad J_{\alpha\beta} = J_0 P_\alpha P_\beta, \qquad J_{\alpha\beta\gamma} = J_0 P_\alpha P_\beta P_\gamma, \quad (11)$$

where $P_\alpha = (p_{1\alpha} + p_{2\alpha})/2$. As a result we obtain for the principal terms of the invariant amplitudes as $t \to \infty$ in the threshold region the following expressions:

$$A_1^{a(4)} = 0, \qquad A_2^{a(4)} = 0, \qquad A_3^{a(4)} = -e^4 smJ_0/16\pi^2,$$
$$A_4^{a(4)} = -4im^2e^4J_0/16\pi^2, \; A_5^{a(4)} = 2im^2e^4J_0/16\pi^2, \; A_6^{a(4)} = 0. \quad (12)$$

We now calculate the matrix elements corresponding to the physical singlet and triplet states in the s channel. As is well known [5], the following states of the electron-positron system with definite parity and angular momentum are possible:

$$|j\,{}^1/_2\,{}^1/_2\rangle - |j - {}^1/_2 - {}^1/_2\rangle \qquad P = (-1)^{j+1}, \quad S = 0,$$
$$|j\,{}^1/_2 - {}^1/_2\rangle - |j - {}^1/_2\,{}^1/_2\rangle, \qquad P = (-1)^{j+1}, \quad S = 1,$$
$$\left.\begin{matrix}|j\,{}^1/_2 - {}^1/_2\rangle + |j - {}^1/_2\,{}^1/_2\rangle \\ |j\,{}^1/_2\,{}^1/_2\rangle + |j - {}^1/_2 - {}^1/_2\rangle\end{matrix}\right\}, \qquad P = (-1)^j \quad S = 1, \quad (13)$$

and accordingly the states of the $\gamma\gamma$ system are:

$$|j\,1\,1\rangle + |j - 1 - 1\rangle, \qquad P = (-1)^j,$$
$$|j\,1\,1\rangle - |j - 1 - 1\rangle, \qquad P = (-1)^{j+1},$$
$$|j1 - 1\rangle - |j - 1\;1\rangle, \qquad P = (-1)^{j+1},$$
$$|j1 - 1\rangle + |j - 1\;1\rangle, \qquad P = (-1)^j.$$

By virtue of parity conservation, the transitions possible between these states correspond to the following helicity amplitudes [5]:

singlet amplitude:

$$\varphi_{00+}^{\;j} - \varphi_{00-}^{\;j}; \quad (14a)$$

triplet amplitudes:

$$\varphi_{12}^{\;j} - \varphi_{1-2}^{\;j}, \; \varphi_{10}^{\;j}, \; \varphi_{00+}^{\;j} + \varphi_{00-}^{\;j}, \; \varphi_{02}^{\;j}, \; \varphi_{12}^{\;j} + \varphi_{1-2}^{\;j}. \quad (14b)$$

We introduce furthermore a system of six independent helicity amplitudes and, writing down the explicit form of the representation with the spinors included, we reduce these helicity amplitudes to invariant ones [4]. From these formulas and from (14a) we see that the matrix element of the transition in the singlet state, for a given choice of the structures, receives a contribution only from the invariant amplitude A_3.

Thus, for the matrix element of the transition in the singlet state we obtain the following asymptotic expression in the threshold approximation:

$$M^0 = \frac{e^2m}{t}\left\{1 + \frac{e^2}{2\pi^2} m^2 f(s) \ln t + \ldots\right\} + \frac{e^2m}{u}\eta_i\left\{1 + \frac{e^2}{2\pi^2} m^2 f(s) \ln u + \ldots\right\}. \quad (15)$$

Summing this series with the aid of the renormalization group, we obtain the "Regge" asymptotic expression

$$M^0 = +e^2mt^{-1+e^2m^2f(s)/2\pi^2} + \eta_i e^2mu^{-1+e^2m^2f(s)/2\pi^2} \quad (16)$$

and an equation for the determination of the bound states in the region $0 < 4m^2 - s \ll 4m^2$

$$l = -1 + \frac{e^2}{2\pi}\frac{m^2}{[s(4m^2 - s)]^{1/2}}, \qquad l = 0, 1, \ldots, \quad (17)$$

which in the nonrelativistic approximation $(m \to \infty)$ goes over into the equation for the Coulomb levels of the electron-positron system with zero radial quantum number:

$$l = -1 + \alpha(m / -2E)^{1/2}. \quad (18)$$

Thus, as in [2], the scattering amplitude in the singlet state has "Regge" asymptotic behavior with respect to t in the threshold region in the s channel, making it possible to determine the energy levels. It can be seen that in the threshold region the perturbation-theory series is expanded in powers of $\alpha/(-2E)^{1/2}$, which corresponds in this approximation to obtaining the energy levels accurate to e^2. In this order, the spin effects make no contribution to the energy, and therefore one would hope to obtain in the threshold region expressions for the triplet states, with the same asymptotic behavior as for the singlet state. However, this hope, as follows from (5), (12), (15), and from the abovementioned formulas of Hearn and Leader [4], is not fulfilled for the triplet state, at least in this method of summing the series.

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