ON THE CHOICE OF INVARIANT VARIABLES FOR THE AMPLITUDES OF PARTICLE PRODUCTION PROCESSES

A. FRENKEL[1)]

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It is shown that 3n − 10 double invariants can be chosen in a simple manner as independent variables in the amplitude for the transition of two particles into (n − 2) particles in such a way that all other invariants can be expressed in terms of chosen ones by second-order equations.

LET us consider an interaction in which n particles with specified masses $m_1, \ldots, m_n$ participate. We can construct from the energy-momentum four-vectors $p_1, \ldots, p_n$ a total of $C_n^2 = n(n-1)/2$ double invariants, but only $3n-10$ of them will be independent. The vectors are related as follows: first, in four-dimensional space there exists between any five vectors $k_1, \ldots, k_5$ a geometrical connection which leads to the vanishing of the determinant det $k_r k_j$ (i, j = 1, ..., 5); second, the energy and momentum conservation law imposes on the vectors the "kinematic" constraint $\sum_{i=1}^{n} p_i = 0$.

Unlike the method proposed by Asribekov [1], we first choose $3n-6$ geometrically-independent invariants, expressing all the remaining ones in terms of those chosen with the aid of geometrical optics. We next impose the kinematic constraint on the vectors. Then the number of independent invariants reduces to $3n-10$, and all the dependent invariants are expressed in terms of the independent ones. We deal in this case only with a second-order equation, and the choice of the independent $3n-10$ double invariants is carried out by a simple method which does not call for the diagram technique proposed by Asribekov.

We choose the $3n-6$ geometrically independent invariants in the following manner:

$$\begin{array}{ll} p_1p_2,\ p_1p_3,\ p_1p_4,\ \ldots,\ p_1p_{n-1},\ p_1p_n, & n-1 \\ p_2p_3,\ p_2p_4,\ \ldots,\ p_2p_{n-1},\ p_2p_n, & n-2 \\ p_3p_4,\ \ldots,\ p_3p_{n-1},\ p_3p_n, & \underline{n-3} \\ & 3n-6 \end{array}.$$

We have chosen here all the double invariants p_ip_k, in which at least one of the indices i or $k \leq 3$. We regard as dependent all the invariants in which both indices i, k > 3. We can readily see that such a choice is possible. Indeed, no internal geometrical constraint arises between the chosen $3n-6$ invariants, so that any fifth-order determinant det p_ip_k contains at least one unchosen invariant (with indices i, k > 3). On the other hand, our choice is convenient because any invariant p_ip_k which was not chosen as independent (i, k > 3) is expressed in simple fashion in terms of those chosen with the aid of the determinant constructed from the vectors p_1, p_2, p_3, p_i, p_k:

$$\begin{vmatrix} m_1^2 & p_1p_2 & p_1p_3 & p_1p_i & p_1p_k \\ p_2p_1 & m_2^2 & p_2p_3 & p_2p_i & p_2p_k \\ p_3p_1 & p_3p_2 & m_3^2 & p_3p_i & p_3p_k \\ p_ip_1 & p_ip_2 & p_ip_3 & m_i^2 & p_ip_k \\ p_kp_1 & p_kp_2 & p_kp_3 & p_kp_i & m_k^2 \end{vmatrix} = 0; \qquad i, k > 3.$$

We have obtained a second-order equation for p_kp_i with coefficients that contain only independent invariants and masses. With the aid of the kinematic constraint $\sum_{i=1}^{n} p_i = 0$ the three invariants p_1p_n, p_2p_n, and p_3p_n are expressed in terms of the remaining ones in linear fashion:

$$p_ip_n = -(p_ip_1 + \ldots + p_ip_{n-1}), \qquad i = 1, 2, 3.$$

To exclude the last dependent invariant, we make use of the constraint

$$p_n^2 = m_n^2 = (p_1 + \ldots + p_{n-1})^2. \tag{1}$$

Let us show that p_3p_{n-1} is expressed in terms of the remaining $3n-10$ independent invariants with the aid of a second-order equation. We write (1) in the form

$$m_n^2 = \sum_{i=1}^{n-1} m_i^2 + 2(p_1p_{n-1} + p_2p_{n-1}) + 2p_3p_{n-1} + 2(p_4 + \ldots + p_{n-2})\,p_{n-1} + 2\sum_{\substack{i,\,k=1 \\ i\neq k}}^{n-2} p_ip_k.$$

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1)Central Scientific-Research Physics Institute of the Hungarian Academy of Sciences, Budapest.

The last sum contains in addition to the independent invariants also those which are expressed in terms of determinants that do not contain the invariant p_3p_{n-1}, since the vector p_{n-1} is not involved in the sum. The invariant p_3p_{n-1} appears only in the determinant set up for the expression for the invariant $(p_4+\ldots+p_{n-2})p_{n-1} \equiv pp_{n-1}$:

$$\begin{vmatrix} m_1^2 & p_1p_2 & p_1p_3 & p_1p & p_1p_{n-1} \\ p_2p_1 & m_2^2 & p_2p_3 & p_2p & p_2p_{n-1} \\ p_3p_1 & p_3p_2 & m_3^2 & p_3p & p_3p_{n-1} \\ pp_1 & pp_2 & pp_3 & p^2 & pp_{n-1} \\ p_{n-1}p_1 & p_{n-1}p_2 & p_{n-1}p_3 & p_{n-1}p & m_{n-1}^2 \end{vmatrix}$$

$$= A(pp_{n-1})^2 + B(pp_{n-1}) + C = 0. \tag{2}$$

From the structure of the determinant we can readily see that A does not depend on p_3p_{n-1}, B depends on it linearly, and C quadratically. We note also that the invariant $p^2 = (p_4+\ldots+p_{n-2})^2$ is expressed in turn with the aid of determinants that do not contain the invariant p_3p_{n-1}.

Eliminating the invariant pp_{n-1} from (1) with the aid of (2) we obtain after simple transformations the expression

$$\left\{\left[m_n^2 - \sum_{i=1}^{n} m_i^2 - 2(p_1+p_2+p_3)p_{n-1} - 2\sum_{\substack{i,\,k=1\\ i\neq k}}^{n-2} p_ip_k\right] A + B\right\}^2 = B^2 - 4AC.$$

Taking into consideration the properties established above for the coefficients A, B, and C and for the sum Σp_ip_k contained here, we note that we have a second-order equation in p_3p_{n-1}, the coefficients of which contain only $3n - 10$ independent invariants:

$$\begin{array}{rl} p_1p_2,\ p_1p_3,\ p_1p_4,\ldots,p_1p_{n-1}, & n-2 \\ p_2p_3,\ p_2p_4,\ldots,p_2p_{n-1}, & n-3 \\ p_3p_4,\ldots,p_3p_{n-2}, & \underline{n-5} \\ & 3n-10\,. \end{array}$$

In conclusion we note that our choice is symmetrical with respect to the vectors in the following sense: we can choose as p_1, p_2, and p_3 any three of the n vectors $p_1,\ldots,p_n$. With the aid of the relation $\sum_{i=1}^{n} p_i = 0$ we can eliminate any of the vectors $p_4,\ldots,p_n$, and instead of p_3p_{n-1} we can express any of the remaining invariants with indices $i < 4$, $k > 3$ in terms of the remaining ones.

The proposed set of independent invariants, made up of only the double invariants p_ip_k, can be useful in the study of analytic properties of the scattering amplitudes in which n particles participate based on the unitarity of the S matrix. This was already pointed out by Chan Hong-mo [2], who considered the possible extension of the Chew-Mandelstam method [3] to processes with $n > 4$; in that paper there is also constructed a set of independent double invariants for $n = 6$, but no systematic method for obtaining such a set for an arbitrary number of particles is indicated. The simplicity of the method of constructing our set does not mean at all that it can replace in all cases successfully the set proposed in [1], which is more complicated to construct. Indeed, as shown by Asribekov [4], his system of independent double, triple, etc. invariant variables is used successfully in the investigation of the analytic properties of a definite class of Feynman amplitudes by the Landau method. Although a transition is possible in principle from one set to the other, such a transformation is much more difficult in practice than the construction of the required set of invariants from the vectors $p_1,\ldots,p_n$ themselves. Since the method of choosing the independent invariants greatly influences the possibility of a concrete improvement of the analytic properties of the scattering amplitude, an investigation of systematic methods for constructing different sets of this type is apparently of definite interest.

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