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EQUILIBRIUM SHAPE OF ATOMIC NUCLEI

Yu. K. KHOKHLOV

P. N. Lebedev Physics Institute, Academy of Sciences U.S.S.R.

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A system of nucleons in the potential of an anisotropic harmonic oscillator is considered, the interaction between the nucleons being assumed due to pairing forces. The energy of such a model is calculated as a function of deformation of the potential. The case when the energy minimum is in the region of large non-axial deformations is studied specially. The existence of the minimum and its position are found to be practically independent of the magnitude of the pairing interaction.

1. We consider the energy of the nucleus as a function of its deformation. Confining ourselves to quadrupole deformations and assuming that the energies of the proton and neutron components enter additively in the total energy, we can write

$$E = E(\beta, \gamma; Z_1) + E(\beta, \gamma; Z_2). \qquad (1)$$

Here E —energy of the nucleus, $E(\beta, \gamma; Z)$ —energy of Z nucleons of the same species, Z_1 and Z_2 —numbers of protons and neutrons, β and γ —quadrupole deformation parameters defined in Sec. 3 below.

We note, to avoid misunderstanding, that the additivity of the proton and neutron component energies in (1) is the consequence of the superfluid model used in the present work. This additivity denotes not the presence of interaction between the neutrons and the protons, but merely the absence of pairing between them. The quadrupole interaction responsible for the deformation is assumed to be the same for both like and unlike nucleons, and is taken into account by introducing a deformed potential common to particles of both species. This is expressed formally in the fact that the deformations of the proton and neutron subsystems are set equal to each other in (1).

In the collective model of the nucleus, the function (1) plays the role of the potential energy of the β and γ oscillations. The position of the absolute minimum of this function determines the equilibrium shape of the nucleus. In particular, an investigation of the function (1) should disclose whether the equilibrium shape of the nucleus is axially symmetrical. This question arose in connection with the model proposed by Davydov and Filippov for γ-deformed nuclei [1] and was considered in many papers [2-6] without [2-5] and with [6] account of pairing.

In the present paper we follow Belyaev [6] and use for the calculation of the function (1) a superfluid model with pure oscillator single-particle potential

$$U(x, y, z) = M(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2) / 2. \qquad (2)$$

We introduce no corrections whatever in the potential (2) [1)]. Unlike Belyaev, the calculation is carried out within the framework of the model with full rigor, without any supplementary assumptions.

The results of the calculations confirm the already made statement (see, for example [6]) that a system consisting of nucleons of one species has an axially symmetrical equilibrium shape. This denotes that the reason for the non-axiality of the nucleus as a whole can be only the effect of the p-n interaction, observed by Filippov [2] with unpaired particles as an example. On the other hand, it turns out that the pairing interaction influences relatively little the effect of the p-n interaction. The latter circumstance makes more interesting the calculations in [4,5] carried out with the aid of the model of unpaired particles for a realistic potential. According to these calculations, some nuclei in the rare-earth region are characterized by an especially weak dependence of the energy on γ (weak γ-stability) and can have a gently sloping maximum corresponding to stable γ-deformation. If this picture is due essentially to the p-n interaction, then it should hold true also when pairing is taken into account. Of course, the final solution

[1)]Introduction of Nilsson corrections in the potential (2) and subsequent account of these by means of perturbation theory does not render the potential move realistic. The point is that the application of perturbation theory to this case is unjustified because the unperturbed system is close to being degenerate (cf.[7]).

of the question of equilibrium deformations calls for an investigation of concrete nuclei.

2. According to the superfluid model of the nucleus, the energy Z of like nucleons is equal to

$$E(\beta, \gamma; Z) = -\frac{\Delta^2}{G} + \sum_n \varepsilon_n \left[1 - \frac{\varepsilon_n - \lambda}{((\varepsilon_n - \lambda)^2 + \Delta^2)^{1/2}}\right]. \quad (3)$$

Here G —constant of the pairing forces, λ —chemical potential, Δ —half-width of the gap in the single-particle excitation spectrum. The quantities Δ and λ are in turn determined by the system of equations

$$\sum_n \frac{1}{((\varepsilon_n - \lambda)^2 + \Delta^2)^{1/2}} = \frac{2}{G},$$

$$\sum_n \left[1 - \frac{1}{((\varepsilon_n - \lambda)^2 + \Delta^2)^{1/2}}\right] = Z. \quad (4)$$

The summation in formulas (3) and (4) is carried out in accordance with the single-particle levels ϵ_n, from the lowest ones up to some upper limit as indicated in the next section. When writing down the formulas account was already taken of the double degeneracy of the states with respect to the spin projections (summation should be only over the orbital states). In Solov'ev's paper [8] these formulas are numbered (3.8), (3.9), and (3.10)

Let us make a few remarks concerning expression (3).

The condition that the energy be stationary under small variations of the potential-deformation parameters takes the form

$$\delta E(\beta, \gamma; Z) = 0. \quad (5)$$

Substituting here expression (3) and carrying out differentiation with account of (4), we obtain

$$\sum_n \left[1 - \frac{\varepsilon - \lambda}{((\varepsilon_n - \lambda)^2 + \Delta^2)^{1/2}}\right] \delta\varepsilon_n = 0, \quad (6)$$

where $\delta\epsilon_n$ —change in the level ϵ_n under small variations of the parameters β and γ. In the limit $\Delta \to 0$ the expression in the square brackets goes over into a steplike function, which characterizes the filling of the levels in the non-interacting particle model. As to the quantity $\delta\epsilon_n$, it is independent of Δ by definition. Thus, the effect of pairing on the stationarity condition (8) reduces to a "smearing" of the population of the single-particle levels, and in all other respects condition (6) is the same as in the noninteracting-particle model.

It is easy to verify (see, for example, [6]) that the stationarity condition (6) is simultaneously a self-consistency condition, according to which the parameters of the deformation potential should coincide with the parameters of the nucleon-cloud deformation. This circumstance, on the one hand, confirms the correctness of the choice of the initial expression (3) as energy of the system under consideration, and on the other hand signifies that the single-particle potential of the present model is consistent only at the extremum point.

3. The surface on which the potential (2) is constant is obviously an ellipsoid, the semiaxes of which are inversely proportional to the corresponding frequencies ω_1, ω_2, and ω_3. The variation of the shape of the ellipsoid corresponds to the variation in the frequencies ω_1, ω_2, and ω_3 under the additional condition of volume conservation

$$\omega_1\omega_2\omega_3 = \omega_0^3 = \text{const.} \quad (7)$$

The condition (7) will be satisfied simultaneously if two independent parameters β and γ are introduced, related with the frequencies by the formula

$$\omega_k = \omega_0(1 + \beta/3)^{-2\cos(\gamma - 2\pi k/3)}, \qquad k = 1, 2, 3, \quad (8)$$

and varying within the limits

$$0 \leqslant \beta < \infty, \qquad 0 \leqslant \gamma \leqslant 60°. \quad (9)$$

The thus introduced parameters β and γ differ somewhat from the usual Bohr parameters, but this difference is quite insignificant at small deformations.

The state of a nucleon in a potential (2) is determined by three positive integers n_1, n_2, n_3, and also by the spin projection on an arbitrary direction. The energy is independent of the spin projection and is equal to

$$\varepsilon_n = \hbar[\omega_1(n_1 + 1/2) + \omega_2(n_2 + 1/2) + \omega_3(n_3 + 1/2)]. \quad (10)$$

Here n denotes the set of numbers n_1, n_2, and n_3.

Each shell in the potential (2) is characterized by a number $N = n_1 + n_2 + n_3$. We shall consider systems in which the shell N = 5 is being filled, and with an even number of nucleons of each species. This means that the number of nucleons of one species, designated above by Z, can range from Z = 70 (shell N = 5 empty) to Z = 112 (shell N = 5 filled with 42 nucleons).

Let us recall some known properties of the deformation as a function of the shell filling. For Z = 70 the ground state of the system is, of course, spherically symmetrical. With increasing number of the filled states, the system stretches and reaches maximum deformation for the nearly half-filled shell (formally at Z = 91). In this region the prolate shape is replaced by an oblate form, and the equilibrium value of β remains practically unchanged. The deformation then decreases in reverse order and vanishes as the shell is nearly

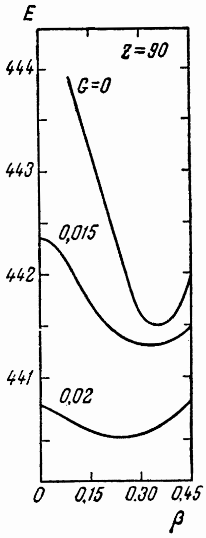

FIG. 1. Energy of a system of 90 like nucleons, for $\gamma = 0$. It is seen that as G increases from zero to 0.015 the position at the maximum shifts slightly (on the other hand the slope changes greatly). The value G = 0.015 is the most realistic (Δ is equal to 0.135 at the minimum point); the value G = 0.02 must already be regarded as too high (Δ is equal to 0.342 at the minimum point).

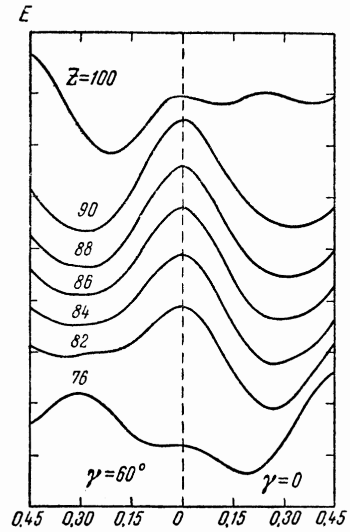

FIG. 2

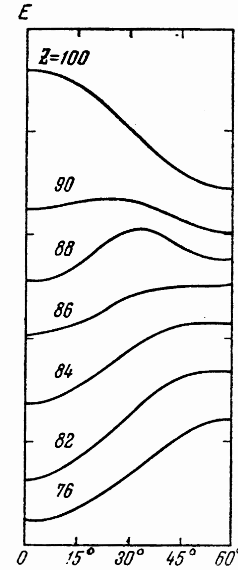

FIG. 3

FIG. 2. Energy of the system of Z like nucleons for $\gamma = 0°$ and $\gamma = 60°$; G = 0.015. Different origins are chosen for each curve on the ordinate axis.

FIG. 3. Energy of a system of Z like nucleons as a function of γ for an equilibrium value of β; G = 0.015. Scale twice as large as in Fig. 2. We see that there are no minima corresponding to nonaxial deformations. The shape of the curve for Z = 88 is characteristic of the region of transition from prolate to oblate nuclei.

filled. There is thus a definite correspondence in the properties of the systems that are symmetrical relative to the half-filled shell (for example, the curves Z = 76 and Z = 100 in Fig. 2 are practically images of each other); although this correspondence is not absolute (see [6]) it allows us to disregard the large value Z > 90. We shall likewise disregard systems with a nearly empty shell.

The final calculations were made for Z values of 76, 82, 84, 86, 88, 90, and 100. For each given Z the function $E(\beta, \gamma, Z)$ was calculated in the intervals $0 \leq \beta \leq 0.45$ and $0 \leq \gamma \leq 60°$ in steps of $\Delta\beta = 0.03$ and $\Delta\gamma = 3°$. The calculations were made with an electronic computer by means of formulas (3) and (4); the upper limit of summation was determined in these formulas by the inequality $n_1 + n_2 + n_3 \leq 6$.

The interaction constant G was selected beforehand such as to make the value of Δ at the point of absolute minimum energy $E(\beta, \gamma; Z)$ equal to approximately 0.1 (we express all quantities in units of $\hbar\omega_0$) for all values of Z under consideration. It was found that such a constant exists and is equal to 0.015 (see Fig. 1). At the equilibrium points $\Delta \approx 0.13$ in this case. Away from the equilibrium point Δ can be approximately double its equilibrium value (see the table). These values of G and Δ agree with calculations made with the Nilsson

Values of E, Δ, and λ in the case Z = 86 for G = 0.015

	β			
	0.00	0.15	0.30	0.45
$E(\beta, 0; 86)$	416.51	415.82	415.42	415.74
Δ	0.242	0.180	0.133	0.115
λ	6.439	6.439	6.434	6.394

potential in the axially symmetrical case [8]. Other values of G were also used for the calculation.

Simultaneously with calculating the function $E(\beta, \gamma; Z)$, we checked this function for an absolute minimum in the entire range of variation of the variables β and γ, and printed-out two sections of the function $E(\beta, \gamma; Z)$ drawn through the minimum point. The summary energy (1) was investigated analogously.

4. Figures 2 and 3 show the sections of the functions $E(\beta, \gamma; Z)$ drawn through the point of absolute minimum. The coordinate γ of the absolute minimum was in all cases equal to zero or 60°, corresponding to an axially-symmetrical equilibrium shape. This circumstance is the result of the pairing interaction of the nucleons. In the absence of pairing, for certain values of Z the system would certainly be axially-asymmetrical. We have made an additional investigation of the system Z = 86, in order to ascertain under what value of G the nonaxial deformation vanishes. For G = 0.012 the system Z = 86 has a very small nonaxial deformation ($\gamma = 3°$), which vanishes at G = 0.013. The values G = 0.012 and G = 0.013 are certainly too low (Δ is equal to 0.054 and 0.081 respectively at the minimum points).

As is well known, the oscillator potential (2)

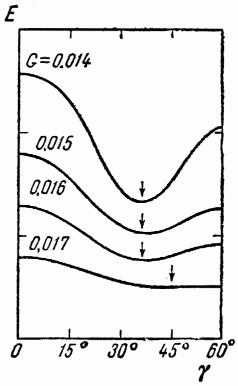

FIG. 4. Energy of a system consisting of two species of nucleons, at equilibrium values of β; $Z_1 = 76$, $Z_2 = 100$. The scale is 20 times larger than on Fig. 2 and 10 times larger than on Fig. 3. It is seen that as G varies over a wide range the position of the minimum remains practically unchanged. The curves were calculated for β values 0.18, 0.15, and 0.15 for G = 0.014, 0.016, and 0.017 respectively (the minimum in γ is exceedingly weak).

affords the most favorable conditions for deformation, compared with any realistic potential. Therefore the system of like particles will also be axially symmetrical in a realistic potential.

Figure 4 represents the results of an investigation of the influence of p-n interactions. From simple geometrical considerations (which incidentally are not connected at all with the presence of particle pairing) it follows that the total energy (1) may turn out to be very weakly dependent on γ (γ-instability), or may even have a weak minimum at $\gamma \neq 0°$ or $60°$ if the terms in (1) have opposite slopes with respect to γ (with each term taken by itself having an arbitrarily large slope). It is directly obvious that upon simple addition of two curves such as shown in Fig. 3 a minimum can be obtained only if at least one of the added curves crosses the line joining its ends at a point lying to the right of 30° for a positive slope and to the left of 30° for a negative slope. This condition is not satisfied by systems with nearly half-filled shells. From among the more remote systems, we investigated the typical pair $Z_1 = 76$ and $Z_2 = 100$. It is seen from Fig. 4 that the influence of pairing reduces to equalization of the total energy as a function of γ without any noticeable displacement of the minimum position. Thus, pairing in this model does not disturb the picture characteristic of the p-n interaction effect.

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