THE OPTICAL THEOREM FOR SCATTERING OF PARTICLES WITH ARBITRARY SPINS

P. WINTERNITZ

Joint Institute for Nuclear Research

Submitted to JETP editor December 13, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) 46, 2108-2111 (June, 1964)

An expression is obtained for the imaginary part of the forward scattering matrix for arbitrarily polarized particles with arbitrary spin. The types of initial polarization states which can influence the total cross section are discussed. The scattering of nucleons on deuterons is treated as an illustration.

In connection with the fact that polarized targets have been realized recently, it seems interesting to discuss processes in which such targets can be utilized. One of the simplest experiments consists in measuring the total scattering cross-section. As has been shown in the work of Phillips [1] and of Bilen'kiĭ and Ryndin [2], the measurement of only the total cross section can yield useful information about the coefficients of the scattering matrix. In the indicated papers, the scattering of nucleons and deuterons on spin zero nuclei has been considered. We will generalize the results of these authors to the case of arbitrary spins.

1. DERIVATION OF THE GENERALIZED OPTICAL THEOREM

According to Puzikov [3] one can write the scattering matrix for particles of arbitrary spins in the following (slightly modified) form:

$$M(\mathbf{k}_i, \mathbf{k}_f) = \sum_{q\varkappa q_1 q_2} (-1)^{q_1+q_2-q+\varkappa}$$
$$\times \sum_{\varkappa_1\varkappa_2} (q_1 q_2 \varkappa_1 \varkappa_2 \mid q -\varkappa)\, T_{q_1\varkappa_1}^{s_1} T_{q_2\varkappa_2}^{s_2} \sum_{\lambda=-r}^{r} a_\lambda^q (q_1 q_2)$$
$$\times \sum_{m_1 m_2} (r+\lambda, r-\lambda, m_1 m_2 \mid q\varkappa)\, Y_{r+\lambda}^{m_1}(\mathbf{k}_i)\, Y_{r-\lambda}^{m_2}(\mathbf{k}_f), \qquad (1)$$

where $T_{q\kappa}^{S}$ are the spin operators for a particle of spin s; $0 \le q \le 2s$; $-q \le \kappa \le q$, $(q_1 q_2 \kappa_1 \kappa_2 \mid q-\kappa)$ are Clebsch-Gordan coefficients, $a_\lambda^q(q_1 q_2)$ are the coefficients of the scattering matrix and $r = q/2$ for even q and $(q+1)/2$, for odd q.

For forward scattering $M(\mathbf{k}_i, \mathbf{k}_f)$ simplifies, and selecting the z axis along the beam direction **k**, we obtain

$$M(0) = \frac{1}{4\pi} \sum_{qq_1q_2\varkappa} \sum_{\lambda=-q/2}^{q/2} (-1)^{q_1+q_2} \sqrt{(q+1+2\lambda)(q+1-2\lambda)}$$
$$\times (q_1 q_2 \varkappa - \varkappa \mid q0)\,(r+\lambda, r-\lambda, 00 \mid q0)$$
$$\times [a_\lambda^q (q_1 q_2)]_0\, T_{q_1\varkappa}^{s_1} T_{q_2-\varkappa}^{s_2}, \qquad (2)$$

where $[a_\lambda^q(q_1q_2)]_0$ are the coefficients of the scattering matrix for zero angle (this is a function of energy). We have taken into account the fact that the terms with odd q disappear due to the properties of the Clebsch-Gordan coefficients.

Considering the initial polarizations to be independent, we write the density matrix of the system in the form

$$\rho = \frac{1}{(2s_1+1)(2s_2+1)} \sum_{k_1 k_2 \mu_1 \mu_2} \langle T_{k_1\mu_1}^{s_1}\rangle \langle T_{k_2\mu_2}^{s_2}\rangle\, T_{k_1\mu_1}^{+s_1} T_{k_2\mu_2}^{+s_2}, \qquad (3)$$

where $\langle T_{k\mu}^{S}\rangle$ are the expectation values of the spin operators in the initial state.

It is known (cf. e.g., [1,4,5]), that for particles with spin the optical theorem can be written in the following form

$$\operatorname{Im} \operatorname{Sp}[\rho M(0)] = (k/4\pi)(\operatorname{Sp}\rho)\,\sigma_T(\rho), \qquad (4)$$

where $\sigma_T(\rho)$ is the total scattering cross section in the state defined by the density matrix ρ. Substituting Eqs. (2) and (3) into (4), we obtain

$$\operatorname{Im}\Big\{ \sum_{qq_1q_2\varkappa\lambda} (-1)^{q_1+q_2} \sqrt{(q+1+2\lambda)(q+1-2\lambda)}$$
$$\times (q_1 q_2 \varkappa - \varkappa \mid q0)\,(r+\lambda, r-\lambda, 00 \mid q0)$$
$$\times [a_\lambda^q (q_1q_2)]_0 \langle T_{q_1\varkappa}^{s_1}\rangle \langle T_{q_2-\varkappa}^{s_2}\rangle \Big\} = k\sigma_T(\rho). \qquad (5)$$

This is the required result. Selecting definite states of initial polarization $\langle T_{q_1\kappa_1}^{S_1}\rangle$ and $\langle T_{q_2\kappa_2}^{S_2}\rangle$ and measuring $\sigma_T(\rho)$, one can obtain relations among the imaginary parts of the coefficients $[a_\lambda^q(q_1q_2)]_0$.

2. SCATTERING OF NUCLEONS ON DEUTERONS

As an illustration we consider separately the important case of scattering of particles of spin ½ on particles of spin 1. In this case the scattering matrix has the form [6]

$$M(E, \vartheta) = a_1 + a_2(\mathbf{Sn}) + a_3(\mathbf{Sl})^2 + a_4(\mathbf{Sm})^2$$
$$+ (\sigma\mathbf{n})\{b_1 + b_2(\mathbf{Sn}) + b_3(\mathbf{Sl})^2 + b_4(\mathbf{Sm})^2\}$$
$$+ (\sigma\mathbf{l})\{c_1(\mathbf{Sl}) + c_2[(\mathbf{Sl})(\mathbf{Sn}) + (\mathbf{Sn})(\mathbf{Sl})]\}$$
$$+ (\sigma\mathbf{m})\{d_1(\mathbf{Sm}) + d_2[(\mathbf{Sm})(\mathbf{Sn}) + (\mathbf{Sn})(\mathbf{Sm})]\},$$

where σ and $\mathbf{S}$ are Pauli matrices for particles of spin $1/2$ and 1, respectively, and $\mathbf{n}$, $\mathbf{m}$, $\mathbf{l}$ are unit vectors in the directions $\mathbf{k}_i \times \mathbf{k}_f$, $\mathbf{k}_i - \mathbf{k}_f$, $\mathbf{k}_i + \mathbf{k}_f$.

We choose the z axis along the beam, and obtain for forward scattering

$$M(E, 0) = a_1(0) + a_3(0)S_z^2$$
$$+ [c_1(0) - b_2(0)]\sigma_z S_z + b_2(0)(\sigma\mathbf{S}).$$

In this case the Eq. (5) yields

$$\mathrm{Im}\{a_1(0) + \tfrac{2}{3}a_3(0) + \tfrac{1}{3}\sqrt{2}a_3(0)\langle T_{20}{}^1\rangle$$
$$+ [c_1(0) - b_2(0)]\langle\sigma_z\rangle\langle S_z\rangle$$
$$+ b_2(0)\langle\sigma\rangle\langle\mathbf{S}\rangle\} = k\sigma_T(\rho)/4\pi. \tag{6}$$

We consider the following convenient particular spin states.

1. Both particles are unpolarized:

$$\mathrm{Im}\left(a_1(0) + \frac{2}{3}a_3(0)\right) = \frac{k}{4\pi}\sigma_T \text{ (unpolar.)}$$

2. The nucleon is unpolarized, the deuteron is tensorially polarized so that $\langle T_{20}^1\rangle \neq 0$:

$$\mathrm{Im}\left\{a_1(0) + \frac{2}{3}a_3(0) + \frac{1}{3}\sqrt{2}a_3(0)\langle T_{20}{}^1\rangle\right\}$$
$$= \frac{k}{4\pi}\sigma_T \quad (\langle T_{20}{}^1\rangle \neq 0).$$

3. Both particles are polarized vectorially, so that $\langle\sigma\rangle \parallel \langle\mathbf{S}\rangle \perp \mathbf{k}$, $\langle T_{20}^1\rangle = 0$:

$$\mathrm{Im}\left\{a_1(0) + \frac{2}{3}a_3(0) + b_2(0)\langle\sigma\rangle\langle S\rangle\right\} = \frac{k}{4\pi}\sigma_T \quad (\perp).$$

4. Both particles are vectorially polarized so that $\langle\sigma\rangle \parallel \langle\mathbf{S}\rangle \parallel \mathbf{k}$, $\langle T_{20}^1\rangle = 0$:

$$\mathrm{Im}\left\{a_1(0) + \frac{2}{3}a_3(0) + c_1(0)\langle\sigma\rangle\langle S\rangle\right\} = \frac{k}{4\pi}\sigma_T \quad (\parallel).$$

From these relations one can determine the imaginary parts of the four coefficients of the M-matrix for the angle $\vartheta = 0$. It can be seen from (6) that the other components of the initial polarization do not contribute anything essentially new (to the total cross section).

3. DISCUSSION OF THE GENERAL FORMULA

In the general case one cannot analyze Eq. (5) in such detail as has been done for the special case (6), but one still can derive some results. We note that the general expression has been derived under the assumption of conservation of spatial parity. Time reversal invariance imposes the supplementary condition[3]:

$$a_{-\lambda}^{q}(q_1, q_2) = (-1)^{q_1+q_2+q}\, a_\lambda{}^q(q_1, q_2).$$

It is very easy to generalize Eq. (5) also for the case of interactions which do not conserve parity.

Let us consider several concrete initially polarized states.

1. Both particles are unpolarized; then we obtain one parameter of the matrix M(0):

$$\mathrm{Im}\,[a_0{}^0(0, 0)]_0 = k\sigma_T.$$

2. One of the particles is unpolarized, e.g.,

$$\langle T_{q_1\varkappa_1}^{s_1}\rangle = \langle T_{00}{}^{s_1}\rangle\,\delta_{q_1 0}\delta_{\varkappa_1 0}, \qquad \langle T_{q_2\varkappa_2}^{s_2}\rangle \text{ — (unpolar.);}$$

then only the tensor polarization of the other particle, of type $\langle T_{q_2 0}^{S_2}\rangle$, with q_2 even, will influence the total cross-section. For scattering of spin $1/2$ particles there are no such components (i.e., the vectorial polarization in this case cannot contribute to the total cross section); for spin 1 one can obtain one supplementary coefficient, if $\langle T_{20}^1\rangle \neq 0$.

3. Both particles are only vectorially polarized, i.e., the following components do not vanish:

$$\langle T_{00}{}^{s_1}\rangle, \qquad \langle T_{1\varkappa_1}^{s_1}\rangle, \qquad \langle T_{00}{}^{s_2}\rangle, \qquad \langle T_{1\varkappa_2}^{s_2}\rangle.$$

In this case Eq. (5) yields

$$\mathrm{Im}\,\{[a_0{}^0(00)]_0\langle T_{00}{}^{s_1}\rangle\langle T_{00}{}^{s_2}\rangle$$
$$- 2a_1a_2\Big[[a_0{}^0(11)]_0$$
$$+ \sqrt{\frac{1}{2}}\sum_{\lambda=-1}^{1}\sqrt{9-4\lambda^2}(1+\lambda, 1-\lambda, 00\,|\,20)[a_\lambda{}^2(11)]_0\Big]$$
$$\times [\langle S_x{}^1\rangle\langle S_x{}^2\rangle + \langle S_y{}^1\rangle\langle S_y{}^2\rangle] + 2a_1a_2\Big[-[a_0{}^0(11)]_0$$
$$+ \sqrt{2}\sum_{\lambda=-1}^{1}\sqrt{9-4\lambda^2}(1+\lambda, 1-\lambda, 00\,|\,20)[a_\lambda{}^2(11)]_0\Big]$$
$$\times\langle S_z{}^1\rangle\langle S_z{}^2\rangle\} = k\sigma_T;$$

$$T_{11} = -a(S_x + iS_y), \quad T_{1-1} = a(S_x - iS_y), \quad T_{10} = a\sqrt{2}S_z.$$

Here only mutually perpendicular components of the polarizations can influence the total cross section. One can obtain different information by directing these polarizations parallel or perpendicular to the beam. We note that these ideas can be applied for instance to the case of elastic scattering of neutrinos (or antineutrinos) on leptons (or on nucleons, if there exist neutral currents). Owing to the longitudinal character of the neutrinos one can obtain some additional information only if the lepton is also polarized parallel to the neutrino beam (taking into account parity nonconservation does not modify the conclusions).

4. One particle has nonvanishing components of the polarization only along the beam. Then only the components of type $\langle T^{S}_{q0} \rangle$ of the second particle contribute to the total cross section.

5. One of the particles is a photon. For a photon there exist only the following components of the polarization[7]:

$$\langle T_{00}{}^{1}\rangle, \quad \langle T_{10}{}^{1}\rangle, \quad \langle T_{22}{}^{1}\rangle, \quad \langle T_{2-2}{}^{1}\rangle.$$

Consequently one can obtain nontrivial information if some of the following components of the target polarization are not zero:

$$\langle T^{s_2}{}_{q_20}\rangle \ (q_2 \neq 0), \quad \langle T^{s_2}{}_{q_2-2}\rangle, \quad \langle T^{s_2}{}_{q_22}\rangle.$$

In particular, one cannot obtain any additional information by scattering a photon on an unpolarized target.

Obviously, experiments in which the total cross section is measured are not the most important and interesting ones among those which can be performed if one has polarized beams and targets. However, these experiments are considerably simpler than for instance the performance of the complete experiment[8] and can still yield interesting information.

In conclusion I would like to express my profound gratitude to L. I. Lapidus and Ya. A. Smorodinskiĭ for useful discussions and interest in this work.

[1] R. J. N. Phillips, Nucl. Phys. **43**, 413 (1963).

[2] S. M. Bilenky and R. M. Ryndin, Phys. Lett. **6**, 217 (1963).

[3] L. D. Puzikov, JETP **34**, 947 (1958), Soviet Phys. JETP **7**, 655 (1958).

[4] R. M. Ryndin and Ya. A. Smorodinskiĭ, JETP **32**, 1584 (1957), Soviet Phys. JETP **5**, 1294 (1957).

[5] L. I. Lapidus, JETP **31**, 1099 (1956), Soviet Phys. JETP **4**, 937 (1957).

[6] P. Winternitz, JETP **39**, 1476 (1960), Soviet Phys. JETP **12**, 1025 (1961).

[7] A. Simon, Phys. Rev. **93**, 1050 (1953).

[8] Puzikov, Ryndin, and Smorodinskiĭ, JETP **32**, 592 (1957), Soviet Phys. JETP **5**, 489 (1957).

Translated by M. E. Mayer
314