STRUCTURE OF SHOCK WAVES IN A PLASMA

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The steady-state equations for one-dimensional motion of a plasma perpendicular to a magnetic field are derived with dissipation taken into account. The shock-wave structure is analyzed for collisional dissipation and for the case of an electron two-stream instability. In the first case the wave is of the usual kind and the conditions for field freezing and continuity of material flux, momentum, and energy are satisfied on both sides of the front. In the second case there is a narrow front at which dissipation occurs. In a hot plasma a periodic (but not perfectly sinusoidal) magnetoacoustic wave, which is stable against the two-stream instability, propagates behind the front.

THE subject of collisionless shock waves in plasmas is of great interest in a number of fields, especially cosmic physics. The equations describing steady-state nonlinear motion of a plasma neglecting dissipation (cf. the review in [1]) naturally do not exhibit shock-wave solutions since the latter describe essentially irreversible processes.

At sufficiently large amplitudes, however, this motion becomes unstable.[1,2] In the final analysis the instability must lead to dissipation of the energy of the directed motion of the particles so that one expects shock waves to appear. The role of the instability as the basic mechanism for the formation of shock waves in a low-density plasma was first considered by Sagdeev.[2] In this same work a number of possible instabilities were investigated. A qualitative analysis (cf. [2]) based on the introduction of an effective frictional force in the electron equations of motion was carried out under the assumption that the plasma pressure is small compared to the magnetic pressure.

In the present work we investigate plasma shock waves that propagate perpendicularly to the magnetic field. The basic purpose of the work is the analysis of the fastest dissipative processes. In a low-temperature dense plasma these are collisions; in a low-density plasma the chief dissipative process is the two-stream instability. If the shock waves are not too strong these processes are not dispersive in either time or space, that is to say, the effective frictional force and the heat generated at a given point in space and time are determined by the state of the plasma at the same space-time point. For this reason it is possible to describe the steady-state motion of the plasma by a system of macroscopic equations that take account of dissipation.

Depending on the actual dissipative mechanism that obtains, the requirement that the frictional force and the heat generation be nondispersive imposes a corresponding limitation on the maximum Mach number for which the present analysis is valid (cf. pages 1121 and 1123). Since dissipation leads to plasma heating the equation of motion must contain the frictional force and the pressure gradient and the heat-balance equation must be used.

The equations that are obtained have been investigated for the case of collisional dissipation and for the electron two-stream instability. The principle difference in the two cases is the fact that collisions are always present whereas the two-stream instability can not occur if the beam velocity is smaller than the mean thermal velocity of the electrons.

1. BASIC EQUATIONS

The z-axis is taken along the magnetic field and the x-axis in the direction of propagation of the plane wave in the plasma. The steady-state solutions are described by the single variable $\xi = x - ut$, where u is the wave velocity. It is assumed that the plasma density satisfies

$$H_0^2/8\pi \ll n_0 m_e c^2,$$

where H_0 and n_0 are the unperturbed field and density (for $\xi \to \infty$), m_e is the electron mass and c is the velocity of light. This condition guarantees that the plasma remains neutral[1] so that $n_i = n_e \equiv n$. It then follows that the x component of the ion velocity V_x is equal to the x component of the electron velocity v_x : $V_x = v_x \equiv V$. We denote the y component of the electron velocity by v and the electric field by **E**. In the absence of dissipation the wave equation for a low-pressure plasma reduces to the system:[1]

$$m_i (V - u)\, dV/d\xi = eE_x + eV_y H/c; \quad (1)$$

$$m_i (V - u)\, dV_y/d\xi = eE_y - eVH/c; \quad (2)$$

$$m_e (V - u)\, dV/d\xi = - eE_x - eVH/c; \quad (3)$$

$$m_e (V - u)\, dv/d\xi = - eE_y + eVH/c; \quad (4)$$

$$n_i = n_e = n_0 u/(u - V); \quad (5)$$

$$E_y = uc^{-1} (H - H_0); \quad (6)$$

$$dH'/d\xi = - 4\pi enc^{-1} (V_y - v). \quad (7)$$

These equations give bounded solutions of two kinds—a solitary wave and a periodic wave. The solutions only exist for Mach numbers $M = u/u_a = u\sqrt{4\pi m_i n_0}/H_0 < 2$. When $M > 2$ the flow velocity becomes multiple-valued and the flow becomes unstable. However, solutions do not necessarily exist even when $M < 2$. Collisions and various instabilities cause the motion to decay; in a low-density plasma the basic dissipative mechanism is the instability.

It is difficult to take account of all possible kinds of unstable perturbations simultaneously. Hence, to simplify the problem we shall consider only the most rapidly growing instability. It can be shown that the solution thus obtained is stable against the other kinds of perturbations.

The instabilities characterized by the highest growth rates are associated with various kinds of electron oscillations and do not have an important effect on the ion motion. Similarly, the electron-electron collision time for electrons with the same energy is $\sqrt{m_i/m_e}$ times smaller than for the ions. Hence, it may be assumed that the most rapid dissipative processes lead to the conversion of energy of the directed motion of the electron component into internal energy without involving the ions.

If the dissipation is nondispersive in space or time and if the ion velocity is much smaller than the electron velocity (conditions that are assumed everywhere in what follows) the dissipation rate per electron can be written in the following way with no loss of generality:

$$\dot{\varepsilon} = - m_e v^2/\tau, \quad (8)$$

where the quantity τ is a function of the other variables and does not depend explicitly on ξ. The dissipation is due to the effect of a frictional force

$$\mathbf{f} = - m_e \mathbf{v}/\tau, \quad (9)$$

which must be included in the electron equations of motion:

$$m_e (V - u)\, v' = - eE_y + eVH/c - m_e v/\tau \quad (10)$$

(the x-component of the frictional force is negligibly small since $v_x \ll v$).

The pressure gradient must also be included in the electron equation of motion:

$$m_e (V - u)\, V' = - p'/n - eE_x - evH/c. \quad (11)$$

In general the introduction of a gradient for a scalar pressure is valid only when the angular distribution of the electron thermal velocities is approximately isotropic. In the region of the shock front the effective scattering time τ must then satisfy the inequality $\omega_H \tau \ll 1$ ($\omega_H = eH/m_e c$ is the electron cyclotron frequency). Behind the front the isotropy condition is well satisfied because the mean velocity is much smaller than the thermal velocity and the spatial gradients are small (quantities do not vary greatly over distances of the order of the Larmor radius).

Finally, it is necessary to introduce the heat balance equation. Neglecting thermal conductivity and using the quantity ϵ to denote the density of heat sources, from the general heat transfer equations[3] we find $T_e dS_e/dt = - \epsilon$, where S_e is the entropy per electron and T_e is the temperature of the electron gas. Expressing d/dt in terms of $d/d\xi$ ($d/dt = (V - u)\, d/d\xi$) and T_e and S_e in terms of p and n we have

$$(V - u)\, (p' - \gamma pn'/n) = (\gamma - 1)\, m_e nv^2/\tau. \quad (12)$$

In the absence of dissipation ($\tau \rightarrow \infty$) Eq. (12) implies adiabatic motion ($dpn^{-\gamma}/dt = 0$). Here γ is the adiabaticity index.

The Maxwell equations and the equations of continuity and ion motion are the same as before. From Eqs. (1), (5), (7), and (11) it is easy to obtain an integral expressing the continuity of the xx component of the momentum flux:

$$H^2/8\pi + p - m_i n_0 uV = \text{const.} \quad (13)$$

An estimate shows that $V_y \sim m_e v/m_i$ so that V_y can be omitted in Eq. (7). Expressing E_y in terms of H and n in terms of V, we finally obtain the following system of equations:

$$m_e (V - u)\, v' = \frac{e}{c} (V - u)\, H + \frac{eu}{c} H_0 - \frac{m_e v}{\tau},$$

$$(V - u)^2 \left(p' - \frac{\gamma pV'}{u - V}\right) = (\gamma - 1)\, \frac{m_e uv^2 n_0}{\tau},$$

$$H' = \frac{4\pi en_0 uv}{c\,(u - V)},$$

$$H^2/8\pi + p - m_i n_0 uV = \text{const.} \quad (14)$$

In the case of a shock wave in an initially cold plasma the constant in Eq. (13) is $H_0^2/8\pi$.

It is convenient to introduce the following dimensionless variables:

$$s = \frac{\omega_0}{c}\xi;\quad W = \frac{V}{u};\quad w = v\sqrt{\frac{4\pi n_0 m_e}{H_0^2}};\quad h = \frac{H}{H_0};\quad \psi = \frac{8\pi p}{H_0^2}. \qquad (15)$$

The system now assumes the form

$$(1 - W)w' = (1 - W)h + a\varphi w - 1, \qquad (16)$$

$$(1 - W)^2[\psi' - \gamma\psi W'/(1 - W)] = -2(\gamma - 1)a\varphi w^2, \qquad (17)$$

$$(1 - W)h' = w, \qquad (18)$$

$$h^2 + \psi = 1 + 2M^2 W, \qquad (19)$$

where $M = (4\pi m_i n_0 u^2/H_0^2)^{1/2}$ is the Mach number, $a = c/\omega_0 u\tau_0$, ω_0 is the plasma frequency for $n = n_0$, τ_0 is a scale factor for the constant that appears in the dissipation rate τ and the dimensionless function φ gives the dependence of this constant on ψ, W, w, and h. The constant a gives the order of magnitude of the ratio of the time required for a particle to move through the wave front to the dissipation time. The fluxes of matter, momentum, and energy are continuous across the shock front in ordinary gas dynamics. In magnetohydrodynamics these conditions are supplemented by the field-freezing requirement. In the present problem the general integrals of the system are the fluxes of matter and momentum. In general, the freezing condition is violated. However, if a shockwave solution exists ($h' = \psi' = w' = W' = 0$ when $\xi \to \pm\infty$ but

$$\underset{\xi\to-\infty}{h \neq 1},\quad \underset{\xi\to-\infty}{W \neq 1}),$$

then the freezing is recovered asymptotically. This is evident from Eq. (16): $h = (1 - W)^{-1}$ when $w' = 0$ and $h' = 0$, or, in terms of the dimensional variables,

$$H(-\infty)/H_0 = n(-\infty)/n_0. \qquad (20)$$

The energy flux is also discontinuous in the general case. This is due to the fact that there are electric fields associated with the shock front and it is possible to have periodic hydromagnetic waves behind the front. Hence, in the case of shock waves in a plasma it is not clear that one should use the usual jump conditions, which express the state of the plasma behind the front in terms of the parameters of the unperturbed state and Mach number. To answer this question it is necessary to make a detailed investigation of the equations of the problem.

2. MAGNETOHYDRODYNAMIC SHOCK WAVES

We first consider shock waves in a very cold dense plasma assuming that collisions represent the primary dissipation mechanism. To use Eqs. (16)—(19) we must assume that either the electron Larmor radius or the electron mean free path is much smaller than the width of the transition layer. In this case the energy dissipated in a small region is not transported into another region of space. If the Larmor radii and mean free path become comparable with the width of the layer a peculiar thermal conductivity associated with the Larmor gyration of the electrons appears.

The width of the transition layer is ck/ω_0 where the dimensionless factor k can be large in the present problem. As we shall see later, at small values of the Mach number $k \to \infty$ while the Larmor radii approach zero so that the equations are known before hand to have a region of applicability for the present problem. Quantitative criteria are given in the final section.

We eliminate w and ψ from the equations and differentiate with respect to h ($ds = dh/q$, where $q = h'$). Then

$$\tfrac{1}{2}d(q\Theta)^2/dh = h\Theta + a\varphi\Theta q - 1, \qquad (21)$$

$$\frac{d\Theta}{dh} = \frac{2}{\gamma}\,\frac{h\Theta - (\gamma - 1)a\varphi\Theta q}{2M^2 + 1 - h^2 - 2M^2(\gamma + 1)\Theta/\gamma}. \qquad (22)$$

The boundary conditions reduce to $\Theta \equiv 1 - W = 1$ and $q = 0$ when $h = 1$.

The thermal energy of the unperturbed plasma is denoted by $\epsilon_0 H^2/4\pi n_0$ ($\epsilon_0 \ll 1$). The function φ can be written in the form $\varphi = \Theta^{-1}[\epsilon_0 + (q\Theta)^2 + \psi\Theta]^{-3/2}$ since the electron collision time in the nonrelativistic case is proportional to the three-halves power of the energy and is inversely proportional to the density. In this case the constant a is given by

$$a = \frac{c}{\omega_0 u}\,\frac{8\pi n_0 e^4}{m_e^2}\left(\frac{4\pi m_e n_0}{H_0^2}\right)^{3/2}\Lambda = \frac{32\pi^2 c}{u}\,\frac{e^3 n_0^2}{H_0^3}\Lambda, \qquad (23)$$

where Λ is the Coulomb logarithm. In what follows we assume that $a \gg 1$. Conditions of this kind obtain in astrophysics (for example in the sun). Thus, when $n_0 = 10^{14}$ cm^{-3}, $H_0 = 50$ G, $u \sim u_a$ and $T_0 = 5000°$ we find $a \approx 10^2$ while $H^2/8\pi p \approx 10$. It is also assumed that the heating is not so strong that $a\varphi_{min} \gg 1$, where φ_{min} is the value of φ corresponding to the most energetic electrons. Under these conditions dissipation is strong everywhere.

We now seek a solution of Eqs. (21) and (22) in a series in inverse powers of a. The expansion of

Θ starts with a zero-order term while the expansion for q starts with the $\sim 1/a$ term. The derivative in Eq. (21) is a quadratic term and can be neglected in the zeroth approximation. The validity of omitting the term with the higher derivative is evident from physical considerations: as a consequence of the strong dissipation the flow densities are small and hence (for the given field jump) the transition region is expanded considerably.

Below we limit ourselves to the zeroth approximation and omit the subscripts on Θ and q. From Eq. (21) we have

$$a\varphi\Theta q = 1 - h\Theta. \tag{24}$$

Substituting Eq. (24) in Eq. (22) we have

$$\frac{d\Theta}{dh} = \frac{2}{\gamma}\frac{\gamma h\Theta - \gamma + 1}{2M^2 + 1 - h^2 - 2M^2(\gamma+1)\Theta/\gamma}. \tag{25}$$

From a formal point of view Eq. (25) can be regarded as the equation for the lines of force of some field **U** in the (h, Θ) plane:

$$U_h = \gamma\,(2M^2 + 1 - h^2 - 2M^2(\gamma+1)\,\Theta/\gamma);$$

$$U_\Theta = 2\,(\gamma h\Theta - \gamma + 1).$$

Since div **U** = $\partial U_h/\partial h + \partial U_\Theta/\partial\Theta = 0$, the quantity U has a one-component vector potential A: $U_h = \partial A/\partial\Theta$, $U_\Theta = -\partial A/\partial h$ while the integral of Eq. (25) is $A(h, \Theta)$ = const. Calculation of A yields

$$A = \gamma\,[(2M^2 + 1 - h^2)\,\Theta - M^2(\gamma+1)\,\Theta^2/\gamma] + 2(\gamma - 1)\,h.$$

Determining A from the condition $\Theta_{h=1} = 1$ we have

$$\gamma\Theta\,[2M^2 + 1 - h^2 - M^2(\gamma+1)\,\Theta/\gamma] + 2(\gamma-1)\,h = (M^2+2)(\gamma-1). \tag{26}$$

The existence of a shock wave requires that all the space derivatives vanish for some $h_0 > 1$. The corresponding value $\Theta = \Theta_0$ must be $1/h_0$ from Eq. (24). Writing $\Theta_0 = 1/h_0$ in Eq. (26) we obtain an equation for the jump h that coincides with the usual magnetohydrodynamic equation. It then follows, in particular, that the energy flux is continuous in the present case.

The calculation is simplified considerably if $\gamma = 2$. The jump H is given by the relation

$$\frac{H(-\infty)}{H_0} = \frac{n(-\infty)}{n_0} = \frac{u}{u - V(-\infty)} = \frac{3M^2}{M^2+2}; \tag{27}$$

the jump in the sum of the magnetic and gas pressures is given by

$$\frac{8\pi p(-\infty) + H^2(-\infty)}{H_0^2} = \frac{4M^2 - 1}{3}. \tag{28}$$

The dissipated energy is transferred completely to the electrons.

However, these waves can only exist for values of M that are not too large. At some critical value of M the relation between h and Θ is no longer unique. Since the derivative q is always negative (as can be shown by calculation) h changes monotonically. Consequently the ambiguity in Θ can only mean that the velocity has two values simultaneously at certain points. Thus, breaking of the shock front occurs even in the presence of strong collisional dissipation. The limiting value M^* at which breaking starts is easily found directly from Eq. (25) by equating to zero the denominator of the right side when $h = h_0$, $\Theta = \Theta_0$. If $\gamma = 2$ then $M^* \approx 2.5$.

The breaking of the front and the formation of a region of multivelocity flow has been obtained by a number of authors in investigations of nonlinear plasma motion that neglect dissipation (cf. review in[1]). The result obtained above indicates that dissipation delays the onset of breaking to some extent (without dissipation $M^* = 2$). It is evident that breaking of the wave does not mean that shock waves cannot be formed when $M > M^*$. However, when $M > M^*$ the structure of the wave is considerably different from that considered above: When $M < M^*$ the main part of the dissipated energy goes to the electrons; when $M > M^*$, however, a region of two-velocity ion flow arises in the front and part of the energy dissipated by virtue of the two-stream instability goes into heating the ions. In principle the structure of this wave can be analyzed similarly, using the equations of multivelocity hydrodynamics.

Returning to the investigation of shock waves characterized by $M < M^*$ we now consider the structure in somewhat greater detail. From Eqs. (24) and (26) we determine $q = dh/ds$:

$$\frac{dh}{ds} = \frac{1 - h\Theta(h)}{a}\frac{(\varepsilon_0 + \psi\Theta)^{-3/2}}{\Theta}. \tag{29}$$

For a low intensity wave ($M - 1 \ll 1$) we can expand Θ in powers of $\mu = M - 1$ and $\eta = h - 1$ with accuracy to terms of second order:

$$\Theta = 1 - \eta + 2\mu\eta - {}^1/_2\,\eta^2. \tag{30}$$

Substituting in Eq. (29) we have

$$q = \frac{\eta(\eta - 2\mu)}{a}\varepsilon_0^{3/2}, \tag{31}$$

$$h = 1 + (M^2 - 1)\left[1 - \tanh s\,\frac{\varepsilon_0^{3/2}(M^2-1)}{a}\right]. \tag{32}$$

When $M - 1 \approx 1$ the profile of the shock wave becomes steeper and the width is reduced to $l \sim ac/\omega_0$. The electron velocity becomes $\sim \sqrt{H_0^2/4\pi n_0 m_e}$ (i.e., $\psi \sim 1$). It is evident that in this case the electron Larmor radius is small compared with the width of the front if $a \gg 1$.

Thus, using Eq. (16)—(19) we can analyze the structure of magnetohydrodynamic shock waves if the amplitude is not too large. The calculations lead to the same relations for the jump and structure in the front as for the macroscopic theory that neglects viscosity and thermal conductivity. The dissipated energy is transferred to the electron component of the plasma and only after this, by virtue of collisions, is it uniformly distributed between the electrons and ions. Since the total pressure of the plasma is not changed during the temperature equalization process, the spatial distribution of H is, as before, determined by the relations that have been given. When $M > M^* \approx 2.5$, a two-velocity ion flow arises at the front and heating of the ions occurs simultaneously with heating of the electrons.

3. SHOCK WAVES IN A LOW DENSITY PLASMA

In a low-density plasma collisions no longer play an important role and dissipation is due to instabilities. The most rapidly growing instability is the electron two-stream instability. If the ion plasma frequency $\Omega_0 = \sqrt{4\pi e^2 n_0/m_i}$ is much greater than the electron cyclotron frequency $\omega_H = eH_0/m_e c$ the maximum growth rate is associated with perturbations with wavelengths $\lambda_0 = 2\pi v/\omega_0 \ll c/\omega_0$. The maximum growth rate is very sharp and hence it may be assumed that the alternating fields in the beam have a spatial scale of $\sim \lambda_0$.

The growth rate of the instability for $\lambda \sim \lambda_0$ can be expressed by means of the interpolation formula,

$$\varkappa_0 = \Omega_0 \sqrt{(v^2 - S^2)/(\alpha v^2 + S^2)}, \qquad (33)$$

where Ω_0 is the ion plasma frequency, $\alpha = (m_e/m_i)^{1/3}$, $S = \sqrt{\gamma p/n m_e}$ is the velocity of the electron beam. When $v^2 < S^2$ the instability vanishes. If the dissipation is to be spatially dispersionless we must impose in addition to the condition indicated ($\lambda_0 \ll c/\omega_0$) the requirement that the Larmor radius of the scattered electrons r_L be small compared with the width of the front l. When $w \sim 1$ and $l \sim c/\omega_0$ this condition is violated ($r_L \approx l$) and it cannot be assumed that the energy dissipated at some point by virtue of the instability appears as heat at the same point. It will be shown below that $r_L \ll l$ for shock waves that are not too small ($M^2 \leq 2$). If the condition for the absence of spatial dispersion is satisfied the time dispersion must also vanish since the displacement of electrons along the x axis during the dissipation time $\tau \approx 1/\Omega_0$ is small (the inequality $u/\Omega_0 \ll c/\omega_0$ is an automatic consequence of the quasi-neutrality condition $H^2 \ll 8\pi m_e n_0 c^2$).

Since the electron two-stream instability does not operate when $v^2 < S^2$, the solution can consist of two parts—an inherent shock wave for which $v^2 < S^2$ and dissipation occurs, and a stable periodic magnetic-sound wave propagating in the heated plasma. The field, pressure and velocity must be continuous at the boundary C. Furthermore, if the magnetic-sound wave is to be stable at the boundary the difference $v^2 - S^2$ must vanish or, in dimensionless form,

$$\sigma = (q\Theta)^2 - \tfrac{1}{2}\gamma\Theta\psi. \qquad (34)$$

To the left of the boundary the quantity σ is negative.

At C the following condition must be satisfied in addition to Eq. (34):

$$\frac{d\sigma}{dh} = \frac{d}{dh}\left[(q\Theta)^2 - \frac{\gamma}{2}\Theta\psi\right] = 0, \qquad (35)$$

where the derivative is computed using the equation for the stable wave. In order to show this we first note that since the pressure of the unperturbed plasma is small then in some region of space $\sigma > 0$. We denote by h_0 the value of h at the boundary of this region. The condition $q = 0$ at infinity together with Eq. (34) with $h = h_0$ determines uniquely the solution in the region of the shock wave and makes it possible to compute the boundary values of ψ_0 and Θ_0. In turn, h_0, Θ_0, ψ_0 and Eq. (34) make it possible to determine uniquely the pressure in the magnetoacoustic wave. The choice of the parameter h_0 is as yet arbitrary. If h_0 is sufficiently close to unity, then for the magnetic-sound wave near the point h_0 the quantity σ is again positive so that the instability appears. As h_0 is increased the absolute value of $d\sigma/dh$ at this point will fall off and finally vanish. As h_0 is increased further the derivative changes sign. Thus, the relation in (35) determines the minimum value of h_0 for which the stable magnetic-sound wave can break away. It is clear that an actual break occurs at precisely this point.

Considering the solution in the region of the magnetic sound wave we see that the limiting value $\sigma = 0$ is a maximum (since $d\sigma/dh|_{h=h_0} = 0$ while σ is negative near h_0). Hence, $\sigma \leq 0$ everywhere in the region of the magnetic sound wave and the wave is stable.

In the region of the shock wave the equations have the form given in (21) and (22). The constant $a = c\Omega_0/u\omega_0$ when $u \sim u_a$ is of order $\sqrt{8\pi m_e n_0 c^2/H_0^2}$ and by virtue of the neutrality condition is a large quantity. However, since the function $\varphi \sim \sqrt{\sigma}$ the expansion in $1/a$ will be considerably different from that given in Sec. 2. Specifically, we must now require that the expansion of the quantity σ (and not q) start with terms $\sim 1/a^2$. In the zeroth approximation in $1/a$ this gives

$$(q\Theta)^2 = \tfrac{1}{2}\gamma\psi\Theta. \quad (36)$$

The physical meaning of this result is clear: since the instability develops very rapidly the difference $v^2 - S^2$ is always close to the stability boundary.

We now determine $a\varphi q\Theta$ from Eq. (36):

$$a\varphi q\Theta = 1 - h\Theta + \tfrac{1}{4}\gamma\,[(2M^2 + 1 - h^2 - 4M^2\Theta)\,d\Theta/dh - 2h\Theta]. \quad (37)$$

Substituting Eq. (37) in Eq. (22) we have

$$\frac{d\Theta}{dh} = \frac{2h\Theta - 4(\gamma-1)/\gamma(\gamma+1)}{2M^2 + 1 - h^2 - 4M^2(\gamma^2+1)\Theta/\gamma(\gamma+1)}. \quad (38)$$

The integration of Eq. (38) is carried out in the same way as in Sec. 2 and yields ($\gamma = 2$)

$$(2M^2+1)\,\Theta - h^2\Theta - \tfrac{5}{3}M^2\Theta^2 + \tfrac{2}{3}h = \tfrac{1}{3}M^2 + \tfrac{2}{3}. \quad (39)$$

Since we are considering shock waves that are not too strong h and Θ are approximately unity. We expand Θ in powers of $h - 1$ keeping third-order terms:

$$\Theta = 1 - \frac{h-1}{M^2} - \frac{3M^2-1}{4M^4}(h-1)^2 + \frac{1}{4M^6}(h-1)^3. \quad (40)$$

The dimensionless pressure ψ is a third-order quantity in $h - 1$ and $M^2 - 1$:

$$\psi = \frac{M^2-1}{2M^2}(h-1)^2 - \frac{1}{2M^4}(h-1)^3. \quad (41)$$

In particular, Eqs. (40) and (41) determine the boundary values of $\Theta = \Theta_0$ and $\psi = \psi_0$ when $h = h_0$.

In the region of the magnetic sound wave the dissipation is no longer effective and ψ satisfies the adiabaticity condition: $\psi = \psi_0(\Theta_0/\Theta)^2$. The motion is described by the equations

$$\tfrac{1}{2}\,d\,(q\Theta)^2/dh = h\Theta - 1, \quad (42)$$

$$h^2 = h_0^2 + \psi_0\,(1 - \Theta_0^2/\Theta^2) - 2M^2(\Theta - \Theta_0) \quad (43)$$

(Equation (43) follows from the integral of momentum flux). Using Eqs. (42) and (43) we can reduce Eq. (35) to the form

$$2\,(h_0\Theta_0 - 1)\,(M^2\Theta_0 - \psi_0) = h_0\Theta_0\psi_0. \quad (44)$$

Now, substituting Θ_0 and ψ_0 from Eqs. (40) and (41) we have

$$h_0 = 1 + \frac{4M^2(M^2-1)}{7M^2-1} + \ldots. \quad (45)$$

Thus, the pressure jump at the shock front is

$$\psi_0 = \frac{8M^2(3M^2-1)(M^2-1)^3}{(7M^2-1)^3}. \quad (46)$$

It is evident from (45) that the convergence is good when $M \leq 1.5$ (the third-order term ≤ 0.004). Hence, the results that have been obtained need not be restricted to very weak waves ($M - 1 \ll 1$). We note that the irreversible heating of the plasma in the present case is somewhat weaker than follows from the macroscopic conditions at the jump. This verifies the remark made above concerning the violation of field freezing.

We now consider the spatial distribution of the field. In the shock region the derivative q is determined from Eq. (36):

$$q = -(h-1)\left[\frac{M^2-1}{2M^2} - \frac{1}{2M^4}(h-1)\right]^{1/2}, \quad (47)$$

whence

$$h = 1 + M^2(M^2-1)\,\mathrm{ch}^{-2}\sqrt{\frac{M^2-1}{8}}\,s. \quad (48)^*$$

It is then obvious that the width of the front l is of order $c\sqrt{8}/\omega_0\sqrt{M^2-1}$. Even when $M = 1.5$ the Larmor radius is

$$\sqrt{H_0^2\psi_0/8\pi m_e n_0}\,m_e c\,/\,eH_0 = \sqrt{\psi_0/2}\,c/\omega_0$$

and is approximately 20 times smaller than l. Hence, the absence of spatial and temporal dispersion in the friction forces and the heat sources follows automatically from the limitations that have been imposed on the Mach number.

In analyzing the structure of the magnetic-sound wave it is necessary, first of all, to express Θ as a function of h from Eq. (43). The term containing ψ_0 is fourth order in $M^2 - 1$ and can be neglected. The result is now substituted in Eq. (42) and third-order terms and higher are neglected; then, integrating, taking account of the boundary conditions $(q\Theta)^2_{h=h_0} = \psi_0\Theta_0$, we find

$$\frac{d(h-h_0)}{ds} \equiv \frac{d\eta}{ds} = \pm\left[\frac{8M^2(3M^2-1)(M^2-1)^3}{(7M^2-1)^3} - \frac{(M^4-1)(5M^2+1)\eta^2}{M^2(7M^2-1)} - \frac{\eta^3}{2M^2}\right]^{1/2}, \quad (49)$$

or writing $\eta = h - h_0 = (M^2 - 1)y$

*ch = cosh.

$$\frac{dy}{ds} = \pm \sqrt{M^2 - 1}\left[\frac{8M^2(3M^2-1)}{7(M^2-1)^3} - \frac{(M^2+1)(5M^2+1)}{M^2(7M^2-1)}y^2 - \frac{y^3}{2M^2}\right]^{1/2}. \quad (50)$$

It is evident that the wave is not harmonic even when $M^2 \rightarrow 1$. As $M^2 \rightarrow 1$ the period of the wave approaches infinity. The function y(s) is expressed in terms of elliptic functions.

Thus, we have shown, that the steady-state equations for one-dimensional motion of a low-density plasma perpendicular to a magnetic field with dissipation taken into account yield shock solutions. These results verify and extend the considerations given in [2]. The shock waves are actually magnetic sound waves; however, dissipation occurs only at the narrow leading edge. Even when $M \rightarrow 1$ the magnetic sound wave is not strictly sinusoidal.

It is possible that dissipation occurs in two stages. The first stage is treated above and is related to the electron two-stream instability. In this case the main part of the dissipated energy goes into heating of the electron component of the plasma. The second stage might be associated with the decay of the magnetic sound wave (as is well known, many kinds of nonlinear periodic oscillations of a plasma are unstable with respect to decay into waves at multiple frequencies [5]).

Since the method given here imposes substantial limitations on the value of the Mach number, it as yet cannot be used to analyze stronger shock waves, even qualitatively. In particular, it is not clear whether pileup occurs at large M or whether the two-stream instability inhibits this process.

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