PROPERTIES OF ROTATING LIQUID He *IN THE VICINITY OF THE LAMBDA POINT*

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Submitted to JETP editor October 30, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) **46**, 1456-1460 (April, 1964)

It is shown that, in general, when rotating liquid He II is heated, vortex lines of two types are formed, in the superfluid and normal parts. The lifetimes of the "normal" vortex lines are estimated. A complete set of hydrodynamic equations is derived for He containing both vortex types. The dispersion law for the elastic oscillations of a system of "normal" vortex lines is found.

As is well known, quantized Onsager-Feynman vortex filaments are produced in the superfluid part of rotating liquid He below the λ point. The presence of these filaments gives rise to unique elastic properties of the liquid, which were investigated in detail both theoretically and experimentally [1,2].

The Tbilisi school [3] has recently carried out experiments showing that the elastic properties of liquid He are retained for a rather long time when the latter is heated above the λ point. The present paper is devoted to a theoretical analysis of the effects connected with heating of rotating liquid He II.

1. We consider first one vortex filament located at the center of a cylindrical vessel of radius R at a temperature T_0. In this case there occurs in the liquid superfluid motion with velocity

$$v_s = \hbar/mr, \tag{1}$$

where m is the mass of the atom of the liquid, r the distance from the vortex axis. The normal part is assumed to be at rest ($v_n = 0$). Assume now that the helium is heated to a temperature $T > T_0$ (for example, with the aid of a gamma source). Let us determine the resultant velocity v_n. We assume first that the viscosity η of the normal part is zero. Then v_n is determined simply from the momentum conservation law

$$\rho_{s_0} v_s = \rho_n v_n + \rho_s v_s, \tag{2}$$

where

$$\rho_{s_0} = \rho_s(T_0), \qquad \rho_s = \rho_s(T),$$

that is,

$$v_n = \frac{\Delta\rho_s}{\rho_n}\frac{\hbar}{mr}. \tag{3}$$

Here $\Delta\rho_s = \rho_{s0} - \rho_s$ — change in the density of the superfluid part upon heating. The velocity v_s remains obviously unchanged by the heating. The presence of finite viscosity causes the velocity distribution (3) to be incorrect for $r \lesssim a(t)$ (t — time elapsed after the instant of heating). For a(t) we obtain from dimensionality considerations

$$a(t) \sim \sqrt{\eta t/\rho_n}. \tag{4}$$

Thus, as a result of heating the vortex part in the superfluid part turns into two vortex filaments — an ordinary "superfluid" filament with circulation $2\pi\hbar/m$, and a "normal" filament, the circulation of v_n around which is, as follows from (3), equal to $(\Delta\rho_s/\rho_n)2\pi\hbar/m$. In this case a(t) plays the role of a time-dependent radius of the core of the "normal" vortex. Let us calculate the energy per unit length of the "normal" filament.

$$\varepsilon_n = \int_a^R \frac{\rho_n v_n^2}{2} 2\pi r dr.$$

Substituting (3), we get

$$\varepsilon_n = \pi\rho_n\left(\frac{\Delta\rho_s}{\rho_n}\frac{\hbar}{m}\right)^2 \ln\frac{R}{a(t)}. \tag{5}$$

The value of $\Delta\rho_s$ becomes appreciable (of the order of the density of He) only if the temperature T is not too far from the λ point. We shall therefore consider the region of temperatures close to the point, where $\rho_s \lesssim \rho_n$ at $T < T_\lambda$. We note that if the temperature T is above the λ point, then only a "normal" vortex filament occurs during heating.

2. Assume now that a cylinder containing He II and rotating with angular velocity ω is heated. The number N of vortex filaments per unit cross section area prior to heating is determined by [1]

$$N = m\omega/\pi\hbar. \tag{6}$$

As a result of heating, the same number of "normal'' filaments is produced in the liquid. Owing to the finite viscosity of the normal part, this system of vortices vanishes ultimately, but the corresponding relaxation time τ is quite large. Indeed, when $t \sim \tau$ we should have $a(t) \sim N^{1/2}$. Substituting here (4) and (6), we get

$$\tau \sim \pi\hbar\rho_n/m\omega\eta. \tag{7}$$

When $\eta/\rho_n \sim 10^{-4}$ and $\omega \sim 10^{-7}$ we get $\tau \sim 7$ minutes. This is precisely the order of relaxation time observed by Andronikashvili et al[3].

We now derive the hydrodynamic equations for liquid helium containing two types of vortices, assuming that $t \ll \tau$. These equations are generalizations of the equations obtained by Hall[1] and by Bekarevich and Khalatnikov[4] in the presence of ''superfluid'' vortex filaments only. We introduce the velocities $\mathbf{v}_s$ and $\mathbf{v}_n$, averaged over regions whose linear dimensions are large compared with the distance between vortices, and also the vector ω_n, the direction of which coincides with the direction of the ''normal'' vortices in the given place, and the magnitude of which is equal to the number of these vortex filaments which cross a unit surface perpendicular to ω_n, multiplied for convenience by the circulation $2\pi\hbar m^{-1}\Delta\rho_s/\rho_n$. We shall start, as usual, from the conservation laws. We write the equations for the conservation of mass, energy, and momentum in the following fashion:

$$\partial\rho/\partial t + \operatorname{div} \mathbf{j} = 0, \tag{8}$$

$$\partial E/\partial t + \operatorname{div} (\mathbf{Q}_0 + \mathbf{q}) = 0, \tag{9}$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial}{\partial x_k}(\Pi_{ik}^{(0)} + \pi_{ik}) = 0, \tag{10}$$

where $\mathbf{Q}_0$ and $\Pi_{ik}^{(0)}$ — energy and momentum fluxes in the absence of vortices and dissipation, ρ — density of the liquid, $\mathbf{j}$ — momentum per unit volume, E — energy per unit volume connected with the internal energy ϵ by the relation

$$E = \tfrac{1}{2}\rho v_s^2 + \mathbf{p}\mathbf{v}_s + \varepsilon. \tag{11}$$

Here $\mathbf{p} = \mathbf{j} - \rho\mathbf{v}_s$ — momentum in the coordinate system moving with velocity $\mathbf{v}_s$. For the differential of ϵ we have (see [4])

$$d\varepsilon = TdS + \mu d\rho + (\mathbf{v}_n - \mathbf{v}_s, d\mathbf{p}) + \lambda_s d\omega_s + \lambda_n d\omega_n, \tag{12}$$

where T, S, and μ are respectively the temperature, entropy, and chemical potential, $\omega_s = \operatorname{curl} \mathbf{v}_s$,

$$\lambda_s = \rho_s \frac{\hbar}{2m}\ln\frac{b}{a_0}, \quad \lambda_n = \rho_n\left(\frac{\Delta\rho_s}{\rho_n}\right)\frac{\hbar}{2m}\ln\frac{b}{a(t)}, \quad b \sim N^{-1/2}, \tag{13}$$

and a_0 — interatomic spacing.

We write down also the equations for superfluidity, growth of entropy, and ω_n:

$$\partial\mathbf{v}_s/\partial t + (\mathbf{v}_s\nabla)\,\mathbf{v}_s + \nabla\mu = \mathbf{f}, \tag{14}$$

$$\partial S/\partial t + \operatorname{div} S\mathbf{v}_n = R/T \ (R > 0), \tag{15}$$

$$\partial\boldsymbol{\omega}_n/\partial t = \operatorname{rot}\{\boldsymbol{\varphi} + [\mathbf{v}_n\boldsymbol{\omega}_n]\}. \tag{16*}$$

The unknown quantities $\mathbf{q}$, π_{ik}, $\mathbf{f}$, R, and $\boldsymbol{\varphi}$ which are contained in (9)—(16) must be determined from the condition that relations (8), (10), and (14)—(16) lead automatically to (9).

Carrying out the time differentiation in (11) we obtain after some transformations, with allowance for (8), (10), (14)—(16) and the explicit forms of $\mathbf{Q}_0$ and $\Pi_{ik}^{(0)}$ (see [4])

$$\begin{aligned}\dot{E} &+ \operatorname{div}\{\mathbf{Q}_0 + (\pi\mathbf{v}_n) + \lambda_s[\boldsymbol{\nu}_s, \mathbf{f} + [\boldsymbol{\omega}_s, \mathbf{v}_n - \mathbf{v}_s]] \\ &+ \lambda_n[\boldsymbol{\nu}_n\boldsymbol{\varphi}]\} = R + \Big(\pi_{ik} - \lambda_s\omega_s\delta_{ik} - \lambda_n\omega_n\delta_{ik} \\ &+ \lambda_s\frac{\omega_{si}\omega_{sk}}{\omega_s} + \lambda_n\frac{\omega_{ni}\omega_{nk}}{\omega_n}\Big)\frac{\partial v_{ni}}{\partial x_k} + (\mathbf{f} + [\boldsymbol{\omega}_s, \mathbf{v}_n - \mathbf{v}_s], \mathbf{j} \\ &- \rho\mathbf{v}_n + \operatorname{rot}\lambda_s\boldsymbol{\nu}_s) + \boldsymbol{\varphi}\operatorname{rot}\lambda_n\boldsymbol{\nu}_n,\end{aligned} \tag{17}$$

where

$$\boldsymbol{\nu}_n = \boldsymbol{\omega}_n/\omega_n, \ \boldsymbol{\nu}_s = \boldsymbol{\omega}_s/\omega_s, \ (\pi\mathbf{v}_n)_i = \pi_{ik}v_{nk}.$$

From a comparison of (17) with (9) we get

$$\mathbf{q} = (\pi\mathbf{v}_n) + \lambda_s[\boldsymbol{\nu}_s, \mathbf{f} + [\boldsymbol{\omega}_s, \mathbf{v}_n - \mathbf{v}_s]] + \lambda_n[\boldsymbol{\nu}_n\boldsymbol{\varphi}], \tag{18}$$

$$\begin{aligned}R = &-\Big[\pi_{ik} - (\lambda_s\omega_s + \lambda_n\omega_n)\delta_{ik} + \lambda_s\frac{\omega_{si}\omega_{sk}}{\omega_s} + \lambda_n\frac{\omega_{ni}\omega_{nk}}{\omega_n}\Big]\frac{\partial v_{ni}}{\partial x_k} \\ &- (\mathbf{f} + [\boldsymbol{\omega}_s, \mathbf{v}_n - \mathbf{v}_s], \mathbf{j} - \rho\mathbf{v}_n + \operatorname{rot}\lambda_s\boldsymbol{\nu}_s) - \boldsymbol{\varphi}\operatorname{rot}\lambda_n\boldsymbol{\nu}_n.\end{aligned} \tag{19}$$

The condition that R be positive enables us to determine the form of the quantities of interest to us

$$\pi_{ik} = (\lambda_s\omega_s + \lambda_n\omega_n)\delta_{ik} - \lambda_s\frac{\omega_{si}\omega_{sk}}{\omega_s} - \lambda_n\frac{\omega_{ni}\omega_{nk}}{\omega_n} + \tau_{ik}, \tag{20}$$

$$\begin{aligned}\boldsymbol{\varphi} = &-\frac{1+\alpha}{\rho_n}[\boldsymbol{\omega}_n \cdot \operatorname{rot}\lambda_n\boldsymbol{\nu}_n] \\ &+ \frac{\beta}{\rho_n}[\boldsymbol{\nu}_n[\boldsymbol{\omega}_n, \operatorname{rot}\lambda_n\boldsymbol{\nu}_n]] - \frac{\gamma}{\rho_n}\boldsymbol{\nu}_n(\boldsymbol{\omega}_n, \operatorname{rot}\lambda_n\boldsymbol{\nu}_n),\end{aligned} \tag{21}$$

where τ_{ik} is the usual tensor of viscous stresses, $\beta > 0$, and $\gamma > 0$. The expression for $\mathbf{f}$ does not differ from the corresponding formula in the paper by Bekarevich and Khalatnikov[4]. Equations (8), (10), and (14)—(16) comprise the sought system of hydrodynamic equations.

Let us determine the order of magnitude of the coefficients α, β, and γ in (21). To this end we note that since the vortex filaments in the normal part are macroscopic formations $[a(t) \gg a_0]$, we can calculate α, β, and γ in principle by carry-

*$[\nu_n\omega_n] = \nu \times \omega_n$, rot = curl.

ing out explicit averaging of the ordinary (that is, unaveraged) hydrodynamic equations over volumes that are large compared with $N^{-3/2}$. In this case these coefficients will be expressed in terms of the viscosity η, the density, and the circulation, the order of magnitude of which is h/m. It is now clear from dimensionality considerations that α, β, $\gamma \sim \eta m/\rho h \sim 10^{-1}$. We note that (21) is written out in such a way that in an ideal liquid ($\eta = 0$) we have $\alpha = \beta = \gamma = 0$.

3. Let us consider on the basis of the equations obtained the elastic oscillations of the liquid. Since the superfluidity equation (14) is not changed by the presence of "normal" vortices, weakly damped oscillations, corresponding to the bending of the "superfluid" vortex filaments, can occur in the system when $\rho \ll \rho_s$. The frequency Ω of these oscillations is connected with the wave vector k by the relation[1]

$$\Omega = \lambda_s k^2 / \rho_s, \tag{22}$$

where it is assumed that $\Omega \gg \omega$.

Let us stop to discuss the inverse limiting case $\rho_s \ll \rho_n$. This includes, in particular, the case $T > T_\lambda$, when $\rho_s = 0$. Assuming the liquid to be incompressible and neglecting the small terms, we transform the equations (10) and (14) with allowance for (16):

$$\dot{\mathbf{v}}_n + (\mathbf{v}_n \nabla)\, \mathbf{v}_n = -\rho_n^{-1} \nabla p_n + \lambda_n \rho_n^{-1} (\boldsymbol{\omega}_n \nabla)\, \boldsymbol{\nu}_n, \tag{23}$$

$$\dot{\boldsymbol{\omega}}_n = \operatorname{rot} \{[\mathbf{v}_n \boldsymbol{\omega}_n] + \lambda_n \rho_n^{-1} (\boldsymbol{\omega}_n \nabla)\, \boldsymbol{\nu}_n\}. \tag{24}$$

It is necessary to add to these equations the continuity equation

$$\operatorname{div} \mathbf{v}_n = 0. \tag{25}$$

Putting

$$\mathbf{v}_n = [\boldsymbol{\omega} \mathbf{r}] + \mathbf{v}_n', \quad \boldsymbol{\omega}_n = \boldsymbol{\omega}_{n0} + \boldsymbol{\omega}_n',$$

where $\mathbf{v}_n'$ and $\boldsymbol{\omega}_n'$ are small increments proportional to $\exp[i(\mathbf{k} \cdot \mathbf{r} - \Omega t)]$, and eliminating the pressure by taking the curl of (23), we obtain for $\mathbf{k} \parallel \boldsymbol{\omega}$ and $\Omega \gg \omega$

$$\Omega\, [\mathbf{k} \mathbf{v}_n'] + \lambda_n \rho_n^{-1} k\, [\mathbf{k}, \boldsymbol{\omega}_n'] = 0, \tag{26}$$

$$-i\Omega \boldsymbol{\omega}_n' + \lambda_n \rho_n^{-1} k\, [\mathbf{k}, \boldsymbol{\omega}_n'] = 0. \tag{27}$$

The system (26) and (27) has non-trivial solutions only if the corresponding determinant vanishes:

$$\Omega^2 k^2\, [\Omega^2 - (\lambda_n \rho_n^{-1} k^2)^2] = 0. \tag{28}$$

From (28) with allowance for (13) we obtain the law for the dispersion of the oscillations

$$\Omega = \frac{\Delta\rho_s}{\rho_n} \frac{\hbar k^2}{2m} \ln \frac{b}{a(t)}. \tag{29}$$

With increasing time t, the frequency Ω decreases logarithmically and vanishes when $t \gg \tau$.

We note in conclusion that the presence of oscillations with a dispersion law (29) can be observed in experiments on the vibration of a disc in rotating liquid helium if $t < \tau$.

I am grateful to I. M. Khalatnikov and L. P. Pitaevskiĭ for a discussion of the results of the present work.

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Translated by J. G. Adashko
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