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GYROSCOPE MOTION IN THEORIES OF GRAVITATION

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The precession of the axis of a gyroscope in the general theory of relativity is compared with the analogous effect in linear theories of gravitation. It is shown that the angular velocity of the precession in the linear theories differs from the result of the general theory of relativity, and therefore this effect can serve as an experiment which will provide a possibility for distinguishing the linear theories from the general theory of relativity.

THE problem of the experimental confirmation of the general theory of relativity is one of present interest.[1-4] One of the new possibilities in this direction was recently studied by Schiff,[3] who considered the motion of a gyroscope in the general theory of relativity. Schiff showed[3] that a gyroscope placed, say, on an artificial satellite will precess with the angular velocity $\boldsymbol{\Omega}$ given by Eq. (10) of the present paper. As we shall see, the observation of the precession of such a gyroscope can be an experiment which will provide a possibility of distinguishing the general theory of relativity from other theories of gravitation.

It is well known that definite attempts are being made at present to approach the problem of gravitation by a different path, distinct from the general theory of relativity. In particular, there are various linear theories of gravitation[5-7] based on the ordinary pseudoeuclidean metric of space-time. An important point is that it is rather difficult to distinguish experimentally between the linear theories and the general theory of relativity; for all three of the critical effects—the deflection of a light ray in a gravitational field, the motions of the perihelia of planets, and the red shift—the linear theories give the same values as the general theory of relativity.[6,7] It has been possible to find a difference only in a fourth effect,[8] namely an additional displacement of the perihelia of planets (or satellites) caused by rotation of the central body. This effect is extremely small, however, and one can scarcely hope to detect and measure it in the near future.

Therefore it is of some interest to examine the motion of a gyroscope in the linear theories and then compare it with the results of the general theory of relativity.

1. GYROSCOPE MOTION IN THE GENERAL THEORY OF RELATIVITY

The method that Schiff[3] used to determine the angular velocity of the precession of a gyroscope is rather complicated and cumbersome. The starting point in Schiff's work[3] is the equations of motion for the spin of a test particle in the Schwartzschild field as found by Papapetrou,[9] who based his work on Fock's dynamical principle.[10] The number of equations obtained in this way is smaller than the number of unknown functions, and therefore it is necessary to introduce supplementary conditions in one form or another,[3] which greatly complicates the entire calculation.

We shall show that the effect of relativistic precession of the axis of a gyroscope can be obtained much more simply if we start from the Lagrangian function, which is not hard to construct. This method is analogous to the solution of the problem of Lense and Thirring[11] which is given in a book by Landau and Lifshitz.[12] It is well known[12] that since the emission of gravitational waves is an effect of the order $(v/c)^5$ a system of gravitating bodies can be described by means of a Lagrangian to accuracy $(v/c)^4$, which is quite sufficient for the calculation of the effect. Moreover, the generally covariant method for obtaining the equations of motion of a spinning particle is a specific method of the general theory of relativity and cannot be directly applied to the linear theories. For the purpose of comparing the results of the various theories we believe it is desirable to conduct the calculations by a single method.

The Lagrangian for a point mass dm which is in the gravitational field g_{ik} is of the form

$$dL = -dmc\frac{ds}{dt} = -dmc\sqrt{g_{ik}\frac{dx^i}{dt}\frac{dx^k}{dt}}. \quad (1)$$

Substituting in Eq. (1) the values $g_{ik} = \delta_{ik} + h_{ik}$ for the metric tensor, where h_{ik} are small deviations from the Galilean metric, we get to accuracy $(v/c)^4$, [12]

$$dL = dmc^2 \left[\frac{h_{00}}{2} + \frac{v^2}{2c^2} + h_{0\alpha}\frac{v^\alpha}{c} + \frac{1}{2}h_{\alpha\beta}\frac{v_\alpha v_\beta}{c^2} + \frac{h_{00}}{4c^2}v^2 + \frac{v^4}{8c^4}\right], \quad (2)$$

where **v** is the three-dimensional velocity of the mass dm, and the components of the tensor h_{ik}, with effects of the rotation of the central body included are given by [12]

$$h_{ik} = 2\varphi\delta_{ik}, \qquad h_{0\alpha} = \frac{4\varkappa M l^2}{5c^3R^3}[\mathbf{\Omega}_0\mathbf{R}]_\alpha,$$
$$\varphi \equiv \frac{\varkappa M}{c^2 R}, \qquad \delta_{ij} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad (3)^*$$

with i, j, k = 1, 2, 3, 0; α, β, γ = 1, 2, 3, where $\kappa = 6.67 \cdot 10^{-8}\ \mathrm{g}^{-1}\ \mathrm{cm}^3\ \mathrm{sec}^{-2}$ is the gravitational constant, M and l are the mass and radius of the central body, $\mathbf{\Omega}_0$ is the angular velocity of the rotation of the central body around its axis, and **R** is the radius vector of the mass dm.

To obtain the Lagrangian for a gyroscope located in the given field g_{ik}, it is obviously necessary to integrate the expression (2) over the entire volume of the gyroscope; that is, [1)]

$$L = \int_{(V)} dL. \quad (4)$$

Let $\mathbf{R}_0$ be the radius vector of the center of mass of the gyroscope; then $\mathbf{R} = \mathbf{R}_0 + \mathbf{r}$, where r is the coordinate of the mass dm in the reference system with its center at the center of mass of the gyroscope, so that

$$\int_{(V)} r_\alpha dm = 0. \quad (5)$$

The velocity of the mass dm is obviously $\mathbf{v} = \mathbf{v}_0 + \omega \times \mathbf{r}$, where $\mathbf{v}_0$ is the orbital velocity of the satellite and ω is the angular velocity of the rotation of the gyroscope around its axis. Substituting the expressions (2) and (3) in Eq. (4) and using the condition (5) and the definition of the inertia tensor [13]

$$I_{\alpha\beta} = \int_{(V)} (r_\gamma^2\delta_{\alpha\beta} - r_\alpha r_\beta)dm, \qquad I_{\alpha\beta} = \begin{pmatrix} I' & & 0 \\ & I' & \\ 0 & & I \end{pmatrix}, \quad (6)$$

we get after an easy integration

$$L = \frac{mv_0^2}{2} + \frac{S^2}{2I} + \frac{3}{2}\frac{\varkappa M}{c^2R_0^3}(\mathbf{S}[\mathbf{v}_0\mathbf{R}_0])$$
$$+ \frac{\varkappa}{c^2R_0^3}(\mathbf{S}, \mathbf{M} - 3\mathbf{n}(\mathbf{nM})). \quad (7)$$

Here $\mathbf{S} = I\omega$ is the angular momentum of the motion of the gyroscope, $\mathbf{M} = (2/5)Ml^2\mathbf{\Omega}_0$ is the angular momentum of the earth, and $\mathbf{n} = \mathbf{R}_0/R_0$ is a unit vector. In deriving Eq. (7) we have used the facts that

$$|r/R_0| \ll 1, \qquad |\varkappa M/c^2R_0| \ll 1.$$

The resulting expression (7) for the Lagrangian has a clear physical meaning: the first two terms are the energies of translational and rotational motion of the gyroscope, the third term is the energy of the "spin-orbit" interaction, and the last term is the energy of the "spin-spin" interaction.

The equations of motion of the gyroscope (the Euler-Lagrange equations) are

$$\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}} - \frac{\partial L}{\partial \varphi} = 0, \quad (8)$$

where φ is the angle. Substituting the expression (7) in Eq. (8), we get

$$d\mathbf{S}/dt = [\mathbf{S}\mathbf{\Omega}], \quad (9)$$

where the angular velocity of precession Ω is given by

$$\mathbf{\Omega} = \frac{3}{2}\frac{\varkappa M}{c^2R_0^3}[\mathbf{v}_0\mathbf{R}_0] + \frac{\varkappa}{c^2R_0^3}(\mathbf{M} - 3\mathbf{n}(\mathbf{nM})). \quad (10)$$

This result agrees with the final formula of Schiff's paper [3] if in that formula we drop the term corresponding to the Thomas precession. [14]

The Thomas precession [14] is an effect of the special theory of relativity; it is the same in all theories and can be obtained without difficulty by the same method, if we use the fact that $\partial L/\partial \mathbf{r} = \mathbf{f}$, the external force of nongravitational origin. Then instead of Eq. (10) we get

$$\mathbf{\Omega} = \frac{1}{2c^2}[\mathbf{v}_0\mathbf{f}] + \frac{3}{2}\frac{\varkappa M}{c^2R_0^3}[\mathbf{v}_0\mathbf{R}_0]$$
$$+ \frac{\varkappa}{c^2R_0}(\mathbf{M} - 3\mathbf{n}(\mathbf{nM})). \quad (10')$$

This expression describes the precession of the gyroscope in the presence of the additional force **f**. For a satellite **f** = 0, and if the satellite travels close to the earth the main term in Eq. (10) gives the approximate value

$$\Omega = 7 \cdot 10^{-9}\ \text{rad/revolution} \approx 3 \cdot 10^{-7}\ ''/\text{sec}.$$

We note that the result can be obtained not only from the Lagrangian but also directly from the equations of motion,

*$[\mathbf{\Omega}_0\mathbf{R}] = \mathbf{\Omega}_0 \times \mathbf{R}$.

1)Strictly speaking, the Lagrangian of a system of gravitating particles is not equal to the sum of the Lagrangians of the individual particles, but for the case we are considering the correction is entirely negligible.

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = 0. \quad (11)$$

To do this one must take the vector product of Eq. (11) by the vector **r** from the left and integrate over the volume of the gyroscope, which at once gives Eq. (9).

2. GYROSCOPE MOTION IN LINEAR THEORIES OF GRAVITATION

We shall not go into a theoretical analysis of the linear theories, since this is not our problem, but shall consider only those consequences of the theories that may possibly be checked by experiment. We merely emphasize that the linear theories involve serious theoretical difficulties, the main one being that the energy density of the gravitational field is not positive definite.[15,16]

Let us first consider the effect in the theory of Birkhoff.[6,7] In this theory the field equations and the equations of motion are

$$\Box h_{ik} = -\frac{8\pi\varkappa}{c^4}T_{ik}, \quad (12a)$$

$$\frac{d^2x^i}{ds^2} = g_0^{ik}\left(\frac{\partial h_{lm}}{\partial x^k} - \frac{\partial h_{lk}}{\partial x^m}\right)\frac{dx^l}{ds}\frac{dx^m}{ds}, \quad (12b)$$

where $ds^2 = g_0^{ik}dx_i dx_k$ is the line element of the special theory of relativity ($g_0^{\alpha\beta} = -\delta_{\alpha\beta}$, $g_0^{00} = 1$), and T_{ik} is the energy-momentum tensor (without the gravitational field).[2)]

The solution of Eq. (12a) that corresponds to the field of a rotating sphere is of the form[7,8]

$$h_{ik} = \varphi\delta_{ik},\ h_{0\alpha} = \frac{2}{5}\frac{\varkappa M l^2}{c^3R^3}[\Omega\mathbf{R}]_\alpha. \quad (13)$$

Substituting Eq. (13) in Eq. (12b), we get the equations of motion of a point mass in the field (13):

$$\frac{d^2R^\alpha}{dt^2} = -\frac{\varkappa M R^\alpha}{R^3} - \frac{\varkappa M}{c^2R^3}v^2R^\alpha + \frac{2\varkappa M}{c^2R^3}v^\alpha(\mathbf{Rv}) + 2cv^\gamma\frac{\partial h_{0\gamma}}{\partial R^\alpha} - cv^\gamma\frac{\partial h_{0\alpha}}{\partial R^\gamma}. \quad (14)$$

It is easy to see that all of the terms of Eq. (14) except the rotation terms can be obtained from the Lagrangian

$$dL = dmc\left\{\frac{v^2}{2c^2} + \frac{\varkappa M}{c^2R}\left(1 + \frac{v^2}{c^2}\right)\right\}. \quad (15)$$

The further procedure for calculating the angular velocity of precession caused by the "spin-orbit" interaction does not differ in any way from the case of the general theory of relativity, which has been treated above. We can obtain the result at once by comparing the coefficients of the corresponding terms in the functions (15) and (2):

$$\Omega = \frac{\varkappa M}{c^2R_0^3}[\mathbf{v}_0\mathbf{R}_0]. \quad (16)$$

As for the rotation terms, we must use a different procedure for them, since we do not know a Lagrangian for these terms. If we multiply Eq. (14) from the left by **r** (vector product) and integrate over the volume of the gyroscope, we get for a circular orbit

$$\frac{d\mathbf{S}}{dt} = \frac{\varkappa M}{c^2R_0^3}[\mathbf{S}[\mathbf{v}_0\mathbf{R}_0]] - \frac{3}{2}\frac{\varkappa}{c^2R_0^3}[\mathbf{SM}] + \frac{3}{2}\frac{\varkappa\mathbf{n}}{c^2R_0^3}(\mathbf{n}[\mathbf{SM}]) - \frac{3\varkappa}{c^2R_0^3}[\mathbf{Mn}](\mathbf{Sn}). \quad (17)$$

The first term naturally corresponds to the result (16), and the other three terms give the precession of the gyroscope caused by the rotation of the central body. Thus the first main term of the precession is only two thirds as large as in the general theory of relativity, and the precession caused by the rotation differs even in its structure from the results of the general theory of relativity.

Let us now consider the effect in the theory of Belinfante and Swihart.[6] Although in this theory a Lagrangian exists and both the field equations and the equations of motion can be obtained from a variation principle, we cannot make direct use of the Lagrangian method to calculate the effect. This is because the authors of[6] use a Lagrangian of the second kind,[18] in which the momentum, the coordinates, and certain specific combinations of these variables are all varied independently. The field equations and equations of motion obtained in this way are[6]

$$\frac{c^4}{8\pi\varkappa}[\alpha\Box h_{ik} + f\delta_{ik}\Box h_l^l] = -\frac{1}{2}T_{ik} - K\delta_{ik}\Sigma mbc^2\delta(\mathbf{x} - \mathbf{x}(t)), \quad (18)$$

$$\frac{dp_i}{ds} = \frac{1}{2}a^l p^m\frac{\partial h_{lm}}{\partial x^i} + Kmc^2b\frac{\partial h_l^l}{dx^i},$$

$$\frac{dx^i}{\partial s} = c\left[a^i - \frac{1}{2}h^{ik}a_k\right],$$

$$mca_i[1 - Kh_l^l] = b\left[p_i - \frac{1}{2}h_i^l p_l\right]. \quad (19)$$

Here α, f, K are numerical constants, which are chosen in the theory so that all three of the "critical" effects take the known values; p_i are the momentum components; and $b \equiv (-a_ia^i)^{1/2}$, where a^i is the velocity, whose definition is contained in the system (19).

In linear approximation in the gravitational po-

[2)]The Birkhoff theory is not a linear approximation to the general theory of relativity; moreover, as Barajas[17] has shown, the equations (12b) are not geodesic equations in any kind of Riemannian space.

tentials h_{ik} the system (19) takes the form

$$\frac{d^2x^\mu}{dt^2} = \frac{1}{2}\frac{dx^i}{dt}\frac{dx^k}{dt}\frac{\partial h_{ik}}{\partial x^\mu} + Kc^2b^2\frac{\partial h_l^l}{\partial x^\mu}. \quad (20)$$

The solutions of the field equations (18) for the case of a spherically symmetrical central body are of the form[6,8]

$$h_{\alpha\beta} = c_1\varphi\delta_{\alpha\beta}, \quad h_{00} = c_0\varphi, \quad h_l^l = -c_T\varphi, \qquad h_{0\alpha} = \frac{\varkappa M l^2}{5\alpha c^2 R^3}[\Omega_0 \mathbf{R}]_\alpha. \quad (21)$$

Multiplying Eq. (20) from the left by r and integrating over the volume of the gyroscope, after some simple calculations we get

$$\frac{d\mathbf{S}}{dt} = -\left(\frac{1}{2}c_1 + Kc_T\right)\frac{\varkappa M}{c^2R_0^3}\{[\mathbf{S}[\mathbf{v}_0\mathbf{R}_0]] + \mathbf{R}_0(\mathbf{S}\mathbf{v}_0)\} + \frac{3\varkappa}{4\alpha c^2R_0^3}[\mathbf{R}_0\mathbf{M}](\mathbf{R}_0\mathbf{S}) - \frac{\varkappa}{4\alpha c^2R_0^3}[\mathbf{S}\mathbf{M}]. \quad (22)$$

The expression (22) shows that independently of the values of the constants c_1, c_T, K the motion of a gyroscope in the Belinfante-Swihart theory is of an entirely different character from the motion in the general theory of relativity. For example, according to Eq. (22) the angular velocity vector of the top's rotation will not only precess but also change in magnitude. Thus, as can be seen from a comparison of Eqs. (17), (22), and (10), pure precession will occur only in the general theory of relativity.

Both on a satellite and under laboratory conditions there are great difficulties in observing the precession of the axis of a gyroscope. Nevertheless, the observation of this effect under laboratory conditions or with a satellite is undoubtedly a simpler problem than the generation and detection of gravitational waves,[19] since terms of order $(v/c)^5$ in the Lagrangian are responsible for the emission of gravitational waves, and the terms responsible for the precession of a gyroscope are of order $(v/c)^4$.

In conclusion we express our deep gratitude to Professor V. L. Ginzburg for suggesting this topic and directing the work.

[1] V. L. Ginzburg, "Experimental Verification of the General Theory of Relativity," in: Recent Developments in General Relativity, Pergamon Press—PWN, 1962.

[2] V. L. Ginzburg, JETP **30**, 213 (1956), Soviet Phys. JETP **3**, 136 (1956), UFN **59**, 11 (1956), **63**, 119 (1957); Fortschritte der Physik **5**, 16 (1957).

[3] L. J. Schiff, Proc. Natl. Acad. Sci. U.S.A. **46**, 871 (1960).

[4] L. J. Schiff, Phys. Rev. Letters **4**, 215 (1960).

[5] G. D. Birkhoff, Proc. Natl. Acad. Sci. U.S.A. **29**, 231 (1943), **30**, 324 (1944).

[6] F. J. Belinfante and J. C. Swihart, Ann. Phys. **1**, 168, 196 (1957), **2**, 81 (1957).

[7] M. Moshinsky, Phys. Rev. **80**, 514 (1950).

[8] B. I. Pustovoĭt, JETP **37**, 870, (1959), Soviet Phys. JETP **10**, 619 (1960); Izv. VUZov Fizika **3**, 63 (1960).

[9] A. Papapetrou, Proc. Roy. Soc. **A209**, 248 (1951).

[10] V. A. Fock, J. Phys. USSR **1**, 61 (1939).

[11] J. Lense and H. Thirring, Physikal. Z. **19**, 156 (1918).

[12] L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), Fizmatgiz, 1960.

[13] L. D. Landau and E. M. Lifshitz, Mekhanika (Mechanics), Fizmatgiz, 1957.

[14] L. H. Thomas, Nature **117**, 514 (1926).

[15] H. Weyl. Amer. J. Math. **66**, 591 (1944).

[16] F. J. Belinfante, Phys. Rev. **98**, 1175 (1955). F. J. Belinfante and J. C. Swihart, Phys. Rev. **95**, 630 (1954).

[17] A. Barajas, Proc. Natl. Acad. Sci. U.S.A. **30**, 54 (1944).

[18] F. J. Belinfante, Physica **7**, 449 (1940).

[19] V. B. Braginskiĭ, JETP **44**, 1562 (1963), Soviet Phys. JETP **17**, 1050 (1963).

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