ON THE ANALYTIC PROPERTIES OF GREEN'S FUNCTIONS AND OF THE MASS OPERATOR

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A spectral representation for the mass operator of a system of interacting particles is obtained for finite temperature. The case when the mass operator has a pole is considered.

LUTTINGER[1] was the first to obtain a spectral representation for the mass operator of a system of fermions at absolute zero. Maleev[2] considered the case when the spectrum of the system has an energy gap and showed that in this case the mass operator may have one pole, and that in order to derive a spectral representation for the mass operator it is necessary to make one subtraction. Bonch-Bruevich[3] considered the analytic properties of the operators M^R and M^A for finite temperatures (M^R and M^A are the operators associated with the retarded and advanced Green's functions, respectively, in the same manner as the usual mass operator is associated with the causal Green's function).

In the present paper we obtain a spectral representation for the usual mass operator, at finite temperature, for a system of electrons in a lattice and for a homogeneous system of bosons. (We note that Abrikosov[7] has studied the Green's function for a system of electrons in a lattice in connection with other problems.) The Green's function and the mass operator are analytically continued from the real axis into the complex plane.

The Green's function for a system of interacting particles is given by

$$G(\mathbf{r}, \mathbf{r}', t-t') = -i\,\mathrm{Sp}\left\{\exp\left\{\frac{\Omega+\mu N-H}{T}\right\} T_t\,[\psi(\mathbf{r}, t)\,\psi^+(\mathbf{r}', t')]\right\}, \quad (1)$$

where T is the temperature of the system, T_t is the time ordering operator, Ω is the thermodynamic potential, μ is the chemical potential, N is the particle number operator, H is the Hamiltonian of the system, and $\psi(\mathbf{r}, t)$ and $\psi^+(\mathbf{r}', t')$ are the annihilation and creation operators of the particles.

Let us first consider a system of electrons in a crystal lattice.

We represent the Green's function as a Fourier integral in the time variable

$$G(\mathbf{r}, \mathbf{r}', t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\, G(\mathbf{r}, \mathbf{r}', \omega). \quad (2)$$

$G(\mathbf{r}, \mathbf{r}', \omega)$ is connected with the self-energy operator by means of the Dyson equation

$$G(\mathbf{r}, \mathbf{r}', \omega) = G_0(\mathbf{r}, \mathbf{r}', \omega) + \int d\mathbf{r}_1\, d\mathbf{r}_2\, G_0(\mathbf{r}, \mathbf{r}_1, \omega)\, M(\mathbf{r}_1, \mathbf{r}_2, \omega)\, G(\mathbf{r}_2, \mathbf{r}', \omega), \quad (3)$$

where $G_0(\mathbf{r}, \mathbf{r}', \omega)$ is the Green's function of the system of non-interacting particles. In order to find the analytic properties of the Green's function and of the mass operator it is necessary to diagonalize all the quantities which enter in Eq. (3). We show that if we choose as a base the solutions of the Schrödinger equation for non-interacting electrons in the metal, the function $G(\mathbf{r}, \mathbf{r}', \omega)$ and consequently also $M(\mathbf{r}, \mathbf{r}', \omega)$ will be diagonalized.

Let the functions $\psi(\rho, S_\rho, \mathbf{r})$ be a base for an irreducible representation ρ of the spatial group of the crystal, S_ρ the label of the base function of this representation, $\rho = (\mathbf{k}, \nu)$ with $\mathbf{k}$ the quasi-momentum, and ν the label of the representation. It is known that $\psi(\mathbf{k}, \mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r})\,u(\mathbf{k}, \mathbf{r})$ where $u(\mathbf{k}, \mathbf{r})$ is a periodic function of $\mathbf{r}$ with the period of the lattice.

Let g be an element of the spatial group of the crystal. The action of this element on the function space is by definition

$$T(g)\,\psi(\mathbf{r}) = \psi(g^{-1}\mathbf{r}).$$

The Green's function must be invariant with respect to all elements of the spatial group, i.e.,

$$T(g)\,G(\mathbf{r}, \mathbf{r}', \omega) = G(g^{-1}\mathbf{r}, g^{-1}\mathbf{r}', \omega) = G(\mathbf{r}, \mathbf{r}', \omega). \quad (4)$$

In general, a phase factor could have appeared, however in the absence of an external electromagnetic field the phase cannot depend on the coordinates, since in that case $G(\mathbf{r}, \mathbf{r}', \omega)$ would satisfy another equation. The coordinate independent phase factor must vanish, since $G(\mathbf{r}, \mathbf{r}, t)_{t=-0}$ is the particle density.

We expand the Green's function with respect to the system of functions $\psi(\rho, S_\rho, \mathbf{r})$

$$G(\mathbf{r}, \mathbf{r}', \omega) = \sum_{\rho\rho' S_\rho S'_{\rho'}} \langle \rho, S_\rho | G(\omega) | \rho', S'_{\rho'} \rangle \psi(\rho, S_\rho, \mathbf{r}) \psi^*(\rho', S'_{\rho'}, \mathbf{r}'). \quad (5)$$

Let us show that

$$\langle \rho, S_\rho | G(\omega) | \rho', S'_{\rho'} \rangle = G_\rho(\omega) \delta_{\rho\rho'} \delta_{SS'}. \quad (6)$$

Applying the operator $T(g)$ to both sides of Eq. (5), we have

$$T(g) G(\mathbf{r}, \mathbf{r}', \omega) = \sum \langle \rho, S_\rho | G(\omega) | \rho', S'_{\rho'} \rangle \psi(\rho, S_\rho; g^{-1}\mathbf{r}) \times \psi^*(\rho', S'_{\rho'}; g^{-1}\mathbf{r}). \quad (7)$$

Since the functions $\psi(\rho, S_\rho, \mathbf{r})$ form a base for an irreducible representation of the spatial group

$$\psi(\rho, S_\rho, g^{-1}\mathbf{r}) = \sum_{q_\rho} \langle q_q | T(g) | S_\rho \rangle \psi(\rho, q_\rho, \mathbf{r}), \quad (8)$$

where $\langle q_\rho | T(g) | S_\rho \rangle$ is the matrix which realizes the representation of the group element g in the representation ρ.

Substituting (8) into (7) and taking into account the fact that the representation of the spatial group can always be chosen unitary, we obtain

$$\sum_{S_\rho S'_{\rho'}} \langle q_\rho | T(g) | S_\rho \rangle \langle \rho, S_\rho | G(\omega) | \rho', S'_{\rho'} \rangle \times \langle S'_{\rho'} | T^{-1}(g) q'_{\rho'} \rangle = \langle \rho, q_\rho | G(\omega) | \rho', q'_{\rho'} \rangle \quad (9)$$

for all ρ and ρ'. The equality (9) holds for all elements of the spatial group. According to Schur's lemma[4] for $\rho \neq \rho'$ the matrix element $\langle \rho, S_\rho | G(\omega) | \rho', S'_{\rho'} \rangle$ must vanish and for $\rho = \rho'$, according to a group-theoretical lemma, the matrix $\langle \rho, S_\rho | G(\omega) | \rho, S'_\rho \rangle$ is a multiple of the unit matrix, i.e., (6) is proved.

Thus

$$G(\mathbf{r}, \mathbf{r}', \omega) = \sum_\rho G_\rho(\omega) \sum_{S_\rho} \psi(\rho, S_\rho, \mathbf{r}) \psi^*(\rho, S_\rho, \mathbf{r}'). \quad (10)$$

The Dyson equation takes the form

$$G_\rho^{-1}(\omega) = G_{0\rho}^{-1}(\omega) - M_\rho(\omega). \quad (11)$$

For $G_\rho(\omega)$ there exists a Lehmann representation (cf. e.g. [5,6])

$$G_\rho(\omega) = \int_{-\infty}^{\infty} d\omega' \frac{A_\rho(\omega') n(\omega')}{\omega' - \omega + i\delta} + \int_{-\infty}^{\infty} d\omega' \frac{A_\rho(\omega')(1 - n(\omega'))}{\omega' - \omega - i\delta}, \quad (12)$$

$$A_\rho(\omega) = -\sum_{nm} \exp\left\{\frac{\Omega + \mu N_n - E_n}{T}\right\} |(a_\rho)_{nm}|^2 \times (1 + e^{-\omega_{mn} T}) \delta(\omega - \omega_{mn}) \leqslant 0, \quad (13)$$

where a_ρ is the expansion coefficient of the function $\psi(\mathbf{r})$ with respect to the functions $\psi(\rho, S_\rho, \mathbf{r})$, and

$$\int_{-\infty}^{\infty} A_\rho(\omega) d\omega = -1, \qquad n(\omega) = (1 + e^{\omega/T})^{-1}. \quad (14)$$

The Green's function for the system of non-interacting particles has the form[5]

$$G_{0\rho}(\omega) = \frac{n(E_\rho)}{\omega - E_\rho + \mu - i\delta} + \frac{1 - n(E_\rho)}{\omega - E_\rho + \mu + i\delta}, \quad (15)$$

where $E_\rho = E_\nu(\mathbf{k})$ is the energy of the electron with momentum $\mathbf{k}$ in the zone ν. It can be seen from (12), (14) and (15), that

$$\lim_{|\omega| \to \infty} \omega G_\rho(\omega) = -1, \qquad \lim_{|\omega| \to \infty} \omega^{-1} M_\rho(\omega) = 0. \quad (16)$$

We define the function $G_\rho(\omega)$ for complex ω through the formula (12). We note that for complex ω, (12) does not define an analytic continuation of the Green's function, but yields the functions G^R and G^A in the upper and lower half-planes, respectively. This is clear from the fact that (12) is a Cauchy integral along two straight lines, above and beneath the real axis, and consequently (12) defines three analytic functions in the three different domains: the real axis and the upper and lower half planes. The mass operator must have similar properties. The spectral representation for $M_\rho(\omega)$ to be obtained below defines the operators $M_\rho^R(\omega)$ and $M_\rho^A(\omega)$ in the upper and lower half planes, respectively, and the mass operator on the real axis.

Let us now consider the problem of poles of the function $M_\rho(\omega)$. The poles of the mass operator correspond to zeros of the Green's function. For complex ω it follows from (12) that $G_\rho^*(\omega) = G_\rho(\omega^*)$, and that $(-G_\rho(\omega))$ is an R-function (i.e., a function for which the sign of the imaginary part coincides with the sign of the imaginary part of the argument). It is easy to see that an R-function can only have a zero of first order with a positive coefficient, i.e., $G_\rho(\omega) \sim -a^2(\omega - \omega_0)$. It is easy to show that the imaginary part of $G_\rho(\omega)$ does not vanish in the complex plane (cf., e.g., [1]) and consequently ω_0 is real. In [2] it has been shown that $G_\rho(\omega)$ can have a zero only if the spectrum of the one-fermion excitations has a gap, and that moreover there can be only one zero. Consequently, $M_\rho(\omega)$ can only have a single pole of first order with a positive residue situated in the region of the gap.

Taking into account all that was said above, and applying the Cauchy formula, we obtain (cf. [2,3])

$$M_\rho(\omega) = M_\rho(\infty) + \int_{-\infty}^{\infty} \frac{\alpha_\rho(\omega')\, d\omega'}{\omega' - \omega + i\delta} + \int_{-\infty}^{\infty} \frac{\beta_\rho(\omega')\, d\omega'}{\omega' - \omega - i\delta} \tag{17}$$

in the absence of an energy gap and if $\lim_{|\omega|\to\infty} M_\rho(\omega) = \text{const}$. The reasoning above proves (17) only in the complex plane. On the real axis Eq. (17) implies that

$$M_\rho^R(\omega) - M_\rho(\omega) = 2i\pi\alpha_\rho(\omega),$$

$$M_\rho(\omega) - M_\rho^A(\omega) = 2i\pi\beta_\rho(\omega).$$

It is in this sense that one is supposed to interpret Eq. (17). Taking into account the fact that $\alpha_\rho + \beta_\rho = \gamma_\rho = \pi^{-1}\,\mathrm{Im}\, M_\rho^R(\omega)$ we obtain the interpretation of the functions $\alpha_\rho(\omega)$ and $\beta_\rho(\omega)$. Since

$$\mathrm{Im}\, M_\rho^R(\omega) = \mathrm{Im}\, G_\rho^R(\omega)\, [(\mathrm{Im}\, G_\rho^R(\omega))^2 + (\mathrm{Re}\, G_\rho^R(\omega))^2]^{-1},$$

$\gamma_\rho(\omega) \le 0$. Taking into account the fact that $M_\rho(\infty)$ is a real quantity, we obtain that $(-M_\rho(\omega))$ is an R-function, $\alpha_\rho(\omega)$ and $\beta_\rho(\omega)$ will in general be complex functions.

M^R has the form which has been obtained by Bonch-Bruevich[3]

$$M_\rho^R(\omega) = M_\rho(\infty) + \int_{-\infty}^{\infty} \frac{\gamma_\rho(\omega')}{\omega' - \omega - i\delta}\, d\omega'.$$

At temperature zero the spectral representation has the form which has been obtained by Maleev[2].

If $M_\rho(\omega)$ increases for $|\omega| \to \infty$ it is necessary to carry out a subtraction

$$M_\rho(\omega) = M_\rho(0) + \omega \int_{-\infty}^{\infty} \frac{\alpha_\rho(\omega')\, d\omega'}{\omega'(\omega' - \omega + i\delta)} + \omega \int_{-\infty}^{\infty} \frac{\beta_\rho(\omega')\, d\omega'}{\omega'(\omega' - \omega - i\delta)}. \tag{18}$$

In the presence of a gap (the notation and assumptions regarding the gap are the same as in [2]), if $G_\rho(\omega)$ has a zero in the point ω_ρ:

$$M_\rho(\omega) = M_\rho(\infty) + \frac{\delta_\rho^2}{\omega - \omega_\rho} + \int_{-\infty}^{-\Delta_\rho^-} \frac{\alpha_\rho(\omega')\, d\omega'}{\omega' - \omega + i\delta} + \int_{\Delta_\rho^+}^{\infty} \frac{\alpha_\rho(\omega')\, d\omega'}{\omega' - \omega + i\delta} + \int_{-\infty}^{-\Delta_\rho^-} \frac{\beta_\rho(\omega')\, d\omega'}{\omega' - \omega' + i\delta} + \int_{\Delta_\rho^+}^{\infty} \frac{\beta_\rho(\omega')\, d\omega'}{\omega' - \omega - i\delta}. \tag{19}$$

The generalization to the case of an increasing $M_\rho(\omega)$ is obvious. If in (19) we can retain only the pole term, $G_\rho(\omega)$ has the form (cf. [2]) on the real axis:

$$G_\rho(\omega) = u_\rho^+ \left\{\frac{1 - n_\rho^+}{\omega - \varepsilon_\rho^+ + i\delta} + \frac{n_\rho^+}{\omega - \varepsilon_\rho^+ - i\delta}\right\} + u_\rho^- \left\{\frac{1 - n_\rho^-}{\omega - \varepsilon_\rho^- + i\delta} + \frac{n_\rho^-}{\omega - \varepsilon_\rho^- - i\delta}\right\}, \tag{20}$$

with

$$\varepsilon_\rho^\pm = \tfrac{1}{2}\{\omega_\rho + \Sigma_\rho \pm [(\omega_\rho - \Sigma_\rho)^2 + 4\delta_\rho^2]^{1/2}\},$$

$$\Sigma_\rho = E_{0\rho} - \mu + M_\rho(\infty), \qquad n_\rho^\pm = n(\varepsilon_\rho^\pm),$$

$$u_\rho^\pm = \pm(\varepsilon_\rho^\pm - \omega_\rho)/(\varepsilon_\rho^+ - \varepsilon_\rho^-), \qquad u_\rho^+ + u_\rho^- = 1.$$

We consider now the Green's function in the complex plane. For this it is necessary to continue Eq. (12) analytically off the real axis. On the real axis we have[6]

$$G_\rho^{R,A} = \int_{-\infty}^{\infty} d\omega' \frac{A_\rho(\omega')}{\omega' - \omega \mp i\delta}. \tag{21}$$

Comparing (21) and (12), we obtain on the real axis

$$G_\rho(\omega) = (1 - n(\omega))\, G_\rho^R(\omega) + n(\omega)\, G_\rho^A(\omega). \tag{22}$$

We continue (22) into the complex plane. G_ρ^R is analytic in the upper half-plane, G_ρ^A is analytic in the lower half-plane, and $G_\rho^R(\omega) = G_\rho^{A*}(\omega^*)$. This relation will also be true for G^R in the lower half plane and G^A in the upper half-plane. Equation (22) defines in the complex plane a function which is the analytic continuation of the Green's function off the real axis. It can be seen from (22) that $G_\rho(\omega)$ has the same singularities as $G_\rho^R(\omega)$ and $G_\rho^A(\omega)$, and besides poles in the points $i\omega_n = i(2n+1)\pi T$ (n is an integer). The residues are $[G_\rho^R(i\omega_n) - G_\rho^A(i\omega_n)]T$. In general, G_ρ^A and G_ρ^R are the same function on different sheets, but we regard them as different functions on the same sheet.

From (17), it follows that the mass operator is

$$M_\rho(\omega) = \frac{\beta_\rho(\omega)}{\gamma_\rho(\omega)} M_\rho^R(\omega) + \frac{\alpha_\rho(\omega)}{\gamma_\rho(\omega)} M_\rho^A(\omega). \tag{23}$$

All quantities entering Eq. (23) are analytic continuations of the corresponding functions off the real axis.

It was proved above that in the upper half-plane G_ρ^R has no zeros, and in the lower half-plane G_ρ^A has no zeros, i.e. both functions do not vanish simultaneously except possibly on the real axis. The case $G_\rho^R = G_\rho^A = 0$ on the real axis has been considered above. Consequently, outside the real axis $G_\rho = 0$ only if $\exp(\omega/T)\, G_\rho^R = -G^A$.

We note that on the real axis this cannot happen, since there $\mathrm{Re}\, G_\rho^R = \mathrm{Re}\, G_\rho^A$, i.e., in general $M_\rho(\omega)$ can have poles in the complex plane, although, by all appearances this is improbable.

Let us now consider a system of bosons, regarding the system as spatially homogeneous. We show that also in the case of bosons the mass operator may have a pole, this pole being situated on the real axis. For a system of bosons this case is realized below the temperature of Bose-condensation.

We represent the Green's function in the form of a Fourier integral with respect to time and coordinates

$$G(\mathbf{r}, t) = \frac{1}{(2\pi)^4}\int d\mathbf{p} \int_{-\infty}^{\infty} d\omega\, e^{i\mathbf{pr}-i\omega t}\, G(\mathbf{p}, \omega). \quad (24)$$

For $G(\mathbf{p}, \omega)$ there exists the Lehmann representation

$$G(\mathbf{p}, \omega) = \int_{-\infty}^{\infty} d\omega' \left\{\frac{(n(\omega')+1)\, A(\mathbf{p}, \omega')}{\omega' - \omega - i\delta} - \frac{n(\omega')\, A(\mathbf{p}, \omega')}{\omega' - \omega + i\delta}\right\}. \quad (25)$$

$$A(\mathbf{p}, \omega) = -(2\pi)^3 \sum_{nm} \exp\left\{\frac{\Omega + \mu N_n - E_n}{T}\right\} \times |a_{nm}(\mathbf{p})|^2 \delta(\omega - \omega_{mn})(1 - e^{-\omega_{mn}/T}), \quad (26)$$

where $a(\mathbf{p})$ is the expansion coefficient for the expansion of $\psi(\mathbf{r})$ in plane waves,

$$\int_{-\infty}^{\infty} A(\mathbf{p}, \omega)\, d\omega = -1, \qquad n(\omega) = (e^{\omega/T} - 1)^{-1}. \quad (27)$$

The Green's function for non-interacting particles has the form

$$G_0(\mathbf{p}, \omega) = \frac{n(\mathbf{p}) + 1}{\omega - \varepsilon_0(\mathbf{p}) + i\delta} - \frac{n(\mathbf{p})}{\omega - \varepsilon_0(\mathbf{p}) - i\delta},$$

$$\varepsilon_0(\mathbf{p}) = \frac{p^2}{2m}, \qquad n(\mathbf{p}) = n(\varepsilon(\mathbf{p})). \quad (28)$$

As in the case of the fermion system we define $G(\mathbf{p}, \omega)$ in the complex plane by Eq. (25). For complex ω we obtain

$$G(\mathbf{p}, \omega) = \int_{-\infty}^{\infty} \frac{A(\mathbf{p}, \omega')}{\omega' - \omega}\, d\omega', \quad G(\mathbf{p}, \omega^*) = G^*(\mathbf{p}, \omega). \quad (29)$$

For a fermion system $A(\mathbf{p}, \omega)$ does not change sign and therefore the Green's function is an R-function. For a boson system, as can be seen from Eq. (26), $A(\mathbf{p}, \omega)$ changes sign in the origin.

We prove that the Green's function does not have zeros in the complex plane. From (29) it follows ($\omega = \epsilon + i\gamma$)

$$G(\mathbf{p}, \omega) = G_1(\mathbf{p}, \omega) + iG_2(\mathbf{p}, \omega),$$

$$G_1(\mathbf{p}, \omega) = \int_{-\infty}^{\infty} \frac{A(\mathbf{p}, \omega')(\omega' - \varepsilon)}{(\omega' - \varepsilon)^2 + \gamma^2}\, d\omega',$$

$$G_2(\mathbf{p}, \omega) = \gamma \int_{-\infty}^{\infty} \frac{A(\mathbf{p}, \omega')}{(\omega' - \varepsilon)^2 + \gamma^2}\, d\omega'. \quad (30)$$

In order for $G(\mathbf{p}, \omega)$ to vanish it is necessary that $G_1 = 0$ and $G_2 = 0$. Let $G_2 = 0$, then (30) implies

$$G_1(\mathbf{p}, \omega) = \int_{-\infty}^{\infty} \frac{\omega' A(\mathbf{p}, \omega')}{(\omega' - \varepsilon)^2 + \gamma^2}\, d\omega'.$$

Since $\omega' A(\mathbf{p}, \omega')$ does not change sign, it follows that $G_1 = 0$.

We now consider G_1 and G_2 as γ goes to zero. On the real axis $G_2 \sim A(\mathbf{p}, \omega)$. Since $A(\mathbf{p}, \omega)$ changes sign, the derivative $\partial G/\partial\epsilon$ does not have a constant sign on the real axis, and consequently, in distinction from the case of a fermion system, the Green's function can have several zeros.

Let ω_0 denote one of the zeros of the Green's function. Obviously, $A(\mathbf{p}, \omega_0) = 0$. From (10) it can be seen that G_2 is an odd function of γ, i.e., in the neighborhood of the zero the Green's function is either an R-function or a (−R)-function and consequently[2] the mass operator has in this point a simple pole with real residue, but the sign of the residue is unknown, in distinction from the fermion case.

This yields directly the spectral representation for the mass operator

$$M(\mathbf{p}, \omega) = M(\mathbf{p}) + \omega \int_{-\infty}^{\infty} \frac{\alpha(\mathbf{p}, \omega')\, d\omega'}{\omega'(\omega' - \omega - i\delta)} + \omega \int_{-\infty}^{\infty} \frac{\beta(\mathbf{p}, \omega')\, d\omega'}{\omega'(\omega' - \omega + i\delta)} + \sum_i \frac{a_i(\mathbf{p})}{\omega - \omega_i(\mathbf{p})}, \quad (31a)$$

M^R and M^A are respectively equal to ($\gamma = \alpha + \beta$)

$$M^{R,A}(\mathbf{p}, \omega) = M(\mathbf{p}) + \omega \int_{-\infty}^{\infty} \frac{\gamma(\mathbf{p}, \omega')\, d\omega'}{\omega'(\omega' - \omega \mp i\delta)} + \sum_i \frac{a_i(\mathbf{p})}{\omega - \omega_i(\mathbf{p})}. \quad (31b)$$

It is clear from (31) that

$$\gamma(\mathbf{p}, \omega) = \pi^{-1}\, \mathrm{Im}\, M^R(\omega),$$

$$M^R(\mathbf{p}, \omega) - M(\mathbf{p}, \omega) = -2i\pi\alpha(\mathbf{p}, \omega),$$

$$M(\mathbf{p}, \omega) - M^A(\mathbf{p}, \omega) = 2i\pi\beta(\mathbf{p}, \omega).$$

It is easy to show that $\gamma(\mathbf{p}, \omega) \leq 0$ for $\omega > 0$ and $\gamma(\mathbf{p}, \omega) \geq 0$ for $\omega < 0$.

Let there exist one pole. Then if one neglects the integral in the vicinity of the pole, one obtains, as in the case of fermions

$$G(\mathbf{p}, \omega) = u^+(\mathbf{p}) \left\{ \frac{n^+(\mathbf{p}) + 1}{\omega - \varepsilon^+(\mathbf{p}) + i\delta} - \frac{n^+(\mathbf{p})}{\omega - \varepsilon^+(\mathbf{p}) - i\delta} \right\} + u^-(\mathbf{p}) \left\{ \frac{n^-(\mathbf{p}) + 1}{\omega - \varepsilon^-(\mathbf{p}) + i\delta} - \frac{n^-(\mathbf{p})}{\omega - \varepsilon^-(\mathbf{p}) - i\delta} \right\}; \quad (32)$$

$$\varepsilon^\pm(\mathbf{p}) = \tfrac{1}{2}\{\omega_0(\mathbf{p}) + \Sigma(\mathbf{p}) \pm [(\omega_0(\mathbf{p}) - \Sigma(\mathbf{p}))^2 + 4a_0(\mathbf{p})]^{1/2}\},$$

$$\Sigma(\mathbf{p}) = \varepsilon_0(\mathbf{p}) - \mu + M(\mathbf{p}), \quad n^\pm(\mathbf{p}) = n(\varepsilon^\pm(\mathbf{p})),$$

$$u^\pm(\mathbf{p}) = \pm[\varepsilon^\pm(\mathbf{p}) - \omega_0(\mathbf{p})]/[\varepsilon^+(\mathbf{p}) - \varepsilon^-(\mathbf{p})],$$

$$u^+(\mathbf{p}) + u^-(\mathbf{p}) = 1,$$

where $\omega_0(\mathbf{p})$ is the pole of the mass operator and $a_0(\mathbf{p})$ is the residue in that pole.

It is known[6] that $\epsilon^\pm(0) = 0$. From this we obtain a condition for the determination of the chemical potential μ. If the spectrum is symmetric, then obviously $\omega_0(\mathbf{p}) = -\Sigma(\mathbf{p})$ and all expressions take on a simpler form

$$\varepsilon^+(\mathbf{p}) = -\varepsilon^-(\mathbf{p}) = \varepsilon(\mathbf{p}) = \sqrt{\Sigma^2(\mathbf{p}) + a_0(\mathbf{p})},$$
$$u^\pm(\mathbf{p}) = [\varepsilon(\mathbf{p}) \pm \Sigma(\mathbf{p})]/2\varepsilon(\mathbf{p}).$$

Using the fact that $\epsilon(0) = 0$, we represent $\epsilon(\mathbf{p})$ in the form

$$\varepsilon(\mathbf{p}) = [\Sigma^2(\mathbf{p}) - \Sigma^2(0) + b(\mathbf{p})]^{1/2},$$

where $b(0) = 0$. Assuming $b(\mathbf{p}) = 0$ and $M(\mathbf{p}) = \text{const}$, we obtain the usual spectrum of the theory of superfluidity:

$$\varepsilon(\mathbf{p}) = \{[\varepsilon_0(\mathbf{p}) + \Sigma(0)]^2 - \Sigma^2(0)\}^{1/2}. \quad (33)$$

We now consider the causal Green's function in the complex plane. We continue (25) into the complex plane. Comparing (25) with (21), we obtain

$$G(\mathbf{p}, \omega) = (n(\omega) + 1)\, G^R(\mathbf{p}, \omega) - n(\omega)\, G^A(\mathbf{p}, \omega). \quad (34)$$

We continue (34) into the complex plane. From (34) it can be seen that $G(\mathbf{p}, \omega)$ has the same singularities as $G^{R,A}(\mathbf{p}, \omega)$ and besides, poles in the points $i\omega_n = 2i\pi nT$, where $n = \pm 1, \pm 2, \ldots$ As can be seen from (21), (26), (34) there is no pole for $n = 0$. As in the case of the fermion system the residues are

$$[G^R(\mathbf{p}, i\omega_n) - G^A(\mathbf{p}, i\omega_n)]\, T.$$

For the mass operator we obtain

$$M(\mathbf{p}, \omega) = \frac{\beta(\mathbf{p}, \omega)}{\gamma(\mathbf{p}, \omega)} M^R(\mathbf{p}, \omega) + \frac{\alpha(\mathbf{p}, \omega)}{\gamma(\mathbf{p}, \omega)} M^A(\mathbf{p}, \omega). \quad (35)$$

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1 J. M. Luttinger, Phys. Rev. **121**, 942 (1961).

2 B. M. Maleev, JETP **41**, 1675 (1961), Soviet Phys. JETP **14**, 1191 (1962).

3 V. L. Bonch-Bruevich, Phys. Lett. **2**, 146 (1962).

4 G. Ya. Lyubarskiĭ, Teoriya grupp i ee primenenie v fizika (Group Theory and Its Application in Physics) Gostekhizdat, 1958.

5 V. L. Bonch-Bruevich and S. V. Tyablikov, Metod funktsiĭ Grina v statisticheskoĭ mekhanike (The Green's Function Method in Statistical Mechanics), Fizmatgiz, 1961.

6 Abrikosov, Gor'kov, and Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Quantum Field-theoretical Methods in Statistical Physics), Fizmatgiz, 1962.

7 A. A. Abrikosov, JETP **41**, 569 (1961) and **43**, 1083 (1962), Soviet Phys. JETP **14**, 408 (1962) and **16**, 765 (1963).

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