THE SHAPE OF THE MENISCUS OF ROTATING He II

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It is shown that in the presence of a free surface rotation of He II cannot be uniform, owing to the bending of the quantized vortex lines as they move out onto a nonplanar surface, and that this circumstance in turn is the cause of additional deepening of the meniscus near the axis of rotation.

It is well known that a central macroscopic vortex is formed in rotating He II; this was first observed by Andronikashvili.[1] Recently the experiments of Tsakadze[2] have shown that such a vortex is formed only in He I. So far as He II is concerned, the central macroscopic vortex in it represents a metastable formation, which quickly disappears after the He I—He II transition. Nevertheless, as was also first noted by Andronikashvili, the meniscus of the rotating He II, in contrast to the meniscus of rotating classical liquids, differs somewhat from parabolic—it is characterized by a conical deepening in the form of a small crater on the axis of rotation.

Our first attempt to explain this phenomenon was unsuccessful. It was based on the assumption that in He II rotating in a cylinder of infinite length (or, what amounts to the same thing, in a vessel filled to a plane lid) the formation of an internal region of irrotational rotation with a central vortex is energetically feasible; the circulation of this vortex slightly exceeds the unit quantum $2\pi\hbar/m$ (m is the mass of the helium atom). However, it was shown that minimization of the free energy led to values of the circulation of the central vortex of the order of one quantum, while the radius of the irrotational region, even if it exceeds the usual distance between vortices, is not so large as to explain the experimentally observed width of the conical deepening of the meniscus.[3]

It thus became clear that the deepening of the meniscus is not a simple discharge on the free surface of a formation which is characteristic for the fundamental mass of He II. One must assume that it is the appearance of those changes which the very presence of the free surface brings about in the character of the rotation. The basis for such an assumption is the circumstance that the vortices should be perpendicular to the free surface of the liquid. However, being perpendicular to the curved surface, they cannot guarantee a uniform (in the mean) rotation of the superfluid component with an average curl $\boldsymbol{\omega} \equiv \text{curl}\, \mathbf{v}_s = 2\omega_0$, where ω_0 is the angular velocity of rotation of the vessel.

In this connection, we return to the equations of hydrodynamics of rotating He II with the purpose of clarifying the character of their stationary solution and the shape of the mechanics for boundary conditions corresponding to rotation of a cylindrical vessel with a flat bottom which is not filled up to its brim. The results, obtained at the present time, refer to the case of small departures from homogeneous rotation. It was assumed that the normal component rotates as a whole:

$$\mathbf{v}_n = [\boldsymbol{\omega}_0 \mathbf{r}], \qquad (1)^*$$

while the velocity of the superfluid component $\mathbf{v}_s$, and the pressure $\mathbf{p}$ are expressed by the formulas

$$\mathbf{v}_s = [\boldsymbol{\omega}_0 \mathbf{r}] + \mathbf{u}, \qquad (2)$$

$$p = -\rho g z + \tfrac{1}{2}\omega_0^2 r^2 + \chi, \qquad (3)$$

where r is the distance to the axis of the cylinder. Here $u = u(r, z)$ and $\chi = \chi(r, z)$ are the contributions to the first terms of Eqs. (2) and (3), which correspond to homogeneous rotation, while the value u is assumed to be small ($u \ll v_s$).

Linearization of the system of hydrodynamic equations of rotating He II,[4,5] carried out in correspondence with the assumption of the smallness of u, show that, first, the force of mutual friction $\mathbf{F}_{sn} = 0$ and, second, the equations for the φ component of $\mathbf{u}$ and χ are separated from the equations for the r and z components of u. Inasmuch

*$[\omega_0 \mathbf{r}] = \omega_0 \times \mathbf{r}$.

as for the determination of the shape of the meniscus it suffices to know u_φ and χ, we limit ourselves to consideration only to the first of these two systems of equations. It has the form

$$\frac{\partial^2 u_\varphi}{\partial z^2} - k_0^2 u_\varphi = 0,$$

$$\frac{\partial \chi}{\partial z} = -\rho_s \nu_s \frac{\partial}{\partial z}\frac{1}{r}\frac{\partial}{\partial r} r u_\varphi, \qquad \frac{\partial \chi}{\partial r} = -\rho_s \nu_s \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r u_\varphi, \quad (4)$$

where $k_0^2 = 2\omega_0/\nu_S$, $\nu_S = \epsilon/\rho_S\Gamma$, ϵ is the energy of the vortex filament referred to a unit length of the latter, $\Gamma = 2\pi\hbar/m$ is the circulation. The general solution of this set is the following:

$$u_\varphi = A(r)\exp(k_0 z) + B(r)\exp(-k_0 z),$$

$$\chi = -\rho_s \nu_s \frac{1}{r}\frac{\partial}{\partial r} r\,[A(r)\exp(k_0 z) + B(r)\exp(-k_0 z)] + C. \quad (5)$$

The boundary condition at the bottom of the vessel ($\partial u_\varphi/\partial z = 0$ for $z = 0$) gives

$$u_\varphi = 2A(r)\,\mathrm{ch}\,k_0 z, \qquad \chi = -2\rho_s\nu_s\,\mathrm{ch}\,k_0 z \frac{1}{r}\frac{d}{dr} rA(r) + C. \quad (6)^*$$

The boundary conditions on the free surface $z = \zeta(r)$ have the form[3,4]

$$\left[(p + \eta_s\omega)\,\delta_{ik} - \eta_s\frac{\omega_i\omega_k}{\omega}\right] N_k = 0, \quad (7)$$

where $\mathbf{N}$ is the unit normal vector, and $\eta_S = \rho_S\nu_S$. It is then not difficult to get for $z = \zeta(r)$:

$$p = 0, \qquad \omega_\varphi = 0, \qquad \omega_z N_r - \omega_r N_z = 0 \quad (8)$$

(i.e., the pressure on the free surface is equal to 0 and the vortices are perpendicular to this surface).

Making use of Eqs. (2) and (3), and also the equalities

$$N_r = -\frac{d\zeta}{dr}\Big/\sqrt{1 + \left(\frac{d\zeta}{dr}\right)^2}, \qquad N_z = 1\Big/\sqrt{1 + \left(\frac{d\zeta}{dr}\right)^2}, \quad (9)$$

we get from the first and third conditions of (8) two equations for the determination of the functions $A(r)$ and $\zeta(r)$:

$$-\rho g\zeta + \frac{1}{2}\rho\omega_0^2 r^2 - 2\eta_s\,\mathrm{ch}\,k_0\zeta\,\frac{1}{r}\frac{d}{dr} rA(r) + C = 0,$$

$$\left[\omega_0 + \mathrm{ch}\,k_0\zeta\,\frac{1}{r}\frac{d}{dr} rA(r)\right]\frac{d\zeta}{dr} - k_0 A(r)\,\mathrm{ch}\,k_0\zeta = 0. \quad (10)$$

For solution of these equations, use is again made of the assumption of the smallness of u [A(r) is considered small] and, moreover, it is also assumed that the meniscus is slightly curved. Then

$$\zeta = H - \frac{\omega_0^2}{2g}(R^2 - r^2) + C_1\,[J_0(ar) - J_0(aR)] + C_2\,[Y_0(ar) - Y_0(aR)], \quad (11)$$

$$u_\varphi = \frac{\sqrt{2\omega_0\nu_s}}{\mathrm{sh}\,k_0 H}\,\mathrm{ch}\,k_0 z\left[\frac{\omega_0^2 r}{g} - aC_1 J_1(ar) - aC_2 Y_1(ar)\right], \quad (12)$$

$$\omega - 2\omega_0 = \frac{\sqrt{2\omega_0\nu_s}}{\mathrm{sh}\,k_0 H}\,\mathrm{ch}\,k_0 z\left[\frac{r\omega_0^2}{g} - a^2 C_1 J_0(ar) - a^2 C_2 Y_0(ar)\right], \quad (13)$$

where $a^2 = (\rho g k_0 \tanh k_0 H)/r\eta_S\omega_0$. In writing these formulas, the condition $\zeta = H$ for $r = R$ is employed.

If we now set $\zeta = h$ for $r = 0$ (in order to stay within the limits of these stated approximations, one should assume that $H - h \ll H$), then C_2 would be equal to 0. However, it is evident that such a requirement is not necessary. Apparently Eqs. (11)—(13) are valid only in a weakly curved part of the meniscus, and one must consider as proof of this the fact that in the approach to the axis of the rotating container an increase in the angular velocity of rotation takes place, accompanied by a deepening of the meniscus. It is not excluded that $u_\varphi \to \infty$ and $\omega \to \infty$ for $r \to 0$. Then the deepening should reach the bottom of the vessel and the axis of rotation is hollow (a thin central vortex which extends to the meniscus). Our solution cannot give a quantitative description of these phenomena in the form of the assumptions made on the weak inhomogeneity of the rotation and the small bending of the meniscus. At the present time, calculations are being carried out that are free from limitations which prevent the investigation of the region of strong curvature of the meniscus and also which take into account the role of surface tension.

[1] E. L. Andronikashvili and I. P. Kaverkin, JETP **28**, 126 (1955), Soviet Phys. JETP **1**, 174 (1955).

[2] Dzh. S. Tsakadze, JETP **44**, 105 (1963), Soviet Phys. JETP **17**, 72 (1963).

[3] M. P. Kemoklidze and Yu. G. Mamaladze, JETP **46**, 165 (1964), Soviet Phys. JETP **19**, 118 (1964).

[4] I. L. Bekaresvich and I. M. Khalatnikov, JETP **40**, 920 (1961), Soviet Phys. JETP **13**, 643 (1961).

[5] Andronikashvili, Mamaladze, Matinyan, and Tsakadze, Usp. Fiz. Nauk **73**, 2 (1961), Soviet Phys. Uspekhi **4**, 1 (1961).

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*ch = cosh.