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CRITICAL CURRENT AND CRITICAL MAGNETIC FIELD IN HARD SUPERCONDUCTORS

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The distribution of the critical current over the cross section of a superconductor and its dependence on the density of filaments and on the external magnetic field are investigated within the framework of the filamentary model of a hard superconductor in which dislocation lines, appearing upon plastic deformation of the metal, or any other thin superconducting inclusions play the role of superconducting filaments. Various orientations of the magnetic field relative to a superconductor of rectangular cross section are considered.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

THE critical current and critical magnetic field for many superconducting alloys and intermetallic compounds considerably surpass the critical values of these parameters for pure metallic superconductors. Such alloys and compounds usually turn out to be physically or chemically inhomogeneous and mechanically harder. Therefore they are usually called hard superconductors.

We mention the basic differences between hard and soft superconductors.[1]

1. The critical magnetic field of hard superconductors (in the absence of current) does not change during plastic deformation of the sample and can reach a value ~ 100 kG or larger, whereas it does not exceed a value ~ 1 kG for soft superconductors.

2. The critical current of a hard superconductor can be increased more than an order of magnitude upon increase of the degree of plastic deformation of the sample and after homogenizing annealing can return to its original value. The magnitude of the critical current turns out to be approximately proportional to the transverse cross sectional area of the plastically deformed sample. The critical current in a tempered hard superconductor (the same as for a soft superconductor) is proportional to the perimeter of the transverse cross section. This means that in a plastically deformed hard superconductor the current flows through the total cross section, but in soft superconductors it only flows in a surface layer of thickness on the order of the penetration depth.

3. Hard superconductors can trap magnetic flux; if such a superconductor is first placed in an external near-critical magnetic field and then removed from this field, it is found that magnetic flux remains frozen in the sample. Thus magnetization of a hard superconductor is accompanied by hysteresis. The magnetization curve for hard superconductors shows that in large fields (but smaller than the critical field) the magnetic field penetrates almost completely into the sample, which still remains superconducting (absence of the Meissner effect). The Meissner effect is typical of soft superconductors.

4. Recent experiments [2] have shown that there is no latent heat associated with the transition from the superconducting state to the normal state for a hard superconductor located in an external magnetic field. This means that it is a second order phase transition, in contrast to soft superconductors which undergo a first order phase transition in an external magnetic field.

The magnetic properties of homogeneous hard superconductors are well explained by the theory of Abrikosov.[3] According to this theory, the special magnetic properties of such samples are related to the fact that the surface energy of the boundary between the normal and superconduction phases is negative. This theory also demonstrates that for such superconductors the transition in the presence of an external field will be a second order phase transition. However, an explanation of all the hysteresis effects was not obtained in this article. Nor was the flow of current in a superconductor considered in [3].

Another viewpoint concerning the reasons for the special properties of these superconductors has obtained wide prevalence. Mendelssohn [4] and Gorter [5] expressed the hypothesis that the main bulk of a hard superconductor, possessing the usual properties characteristic of soft superconductors, is interpenetrated by a random network of mutually intersecting thin filaments having a critical field much larger than the critical field of the bulk sample (the "sponge" model). It is easy to explain the high critical field of such a system, if the thickness of the filaments is assumed to be much less than the penetration depth.

Such a model of a hard superconductor enables us to explain many of its properties. Since the system of filaments forms a multiply connected system, a natural explanation then exists for the trapped magnetic flux. Further, the superconducting current flowing in the filaments which uniformly fill the entire volume of the superconductor will pass through the entire transverse cross section of the sample. Finally, according to the Ginzburg-Landau theory, [6] a superconductor with a characteristic dimension smaller than a certain critical size will undergo a second order phase transition in the presence of an external magnetic field. This explains the results of experiment [2].

Recently a number of authors [7-9] have suggested that the hypothetical superconducting filaments in hard superconductors may be dislocation lines in the plastically deformed metal. Such a conjecture enables one to qualitatively explain the effect of plastic deformation and heat treating of hard superconductors on the magnitude of their critical current.

Gorter [10] assumes that dislocations behave as thin superconducting filaments only for those superconductors whose surface tension at the boundary between normal and superconducting phases is negative (superconductors of the second kind). Thus the filamentary model of a hard superconductor and Abrikosov's model [3] do not mutually exclude one another.

The only attempt (known to us) to calculate the critical current for a cylindrical wire within the framework of the filamentary model of a hard superconductor was undertaken by Hauser and Buehler.[9] The equation defining the relation between the total critical current I in the wire and the applied external magnetic field H_a was written in the form

$$H_{cf\ av} = H_a + 0.2\ I/\rho\pi a^2 r.$$

Here $H_{cf\,av}$ is some average value of the critical field which destroys the superconductivity of a thin filament, ρ is the density of filaments, r is the radius of a filament, a is the radius of the wire. It is obvious that $(I/\rho\pi a^2)$ gives the expression for the current flowing in one filament, and the second term on the right hand side is the intensity of the magnetic field at the surface of the filament produced by the current flowing in this filament. Such a relation between H_a and I assumes that Silsbee's rule is valid for thin filaments, that the distribution of the current I in the filaments is uniform, and that the field produced by the remaining filaments can be neglected. The first assumption is not true for thin filaments, and the remaining assumptions are too crude for the present problem.

The purpose of the present article is to investigate the current distribution and to find the dependence of the critical current in a "filamentary" superconductor on the magnitude of the external magnetic field. A plane-parallel slab of dielectric or nonsuperconducting metal, interpenetrated by a large number of thin superconducting filaments which are parallel to one another and also parallel to the surface of the slab, is considered as the model for a "filamentary" superconductor. The distribution of filaments is assumed to be random with a certain average density n (cm^{-2}). The filaments are electrically connected at the beginning and end of the slab and thus turn out to be connected in parallel.

Here the filaments represent not only dislocation lines which remain superconducting in the plastically deformed superconductor, but they also represent any thin superconducting inclusions which may arise in an inhomogeneous alloy or intermetallic compound.

We shall consider three cases: 1) The field is perpendicular to the filaments and parallel to the surface of the slab; 2) the field is parallel to the filaments; 3) the field is perpendicular to the filaments and to the surface of the slab.

2. FILAMENT WITH CURRENT IN A MAGNETIC FIELD

For the investigation of a filamentary superconductor, it is first necessary to determine the relation between the critical current of a single superconducting filament and the superimposed external magnetic field.

A superconducting cylinder in a transverse magnetic field and a superconducting cylinder with current in a longitudinal magnetic field were investigated by Silin.[11]

Let us consider a thin superconducting cylinder of radius r_0 with current I in a transverse, homogeneous at infinity, magnetic field H_0 and a longitudinal homogeneous magnetic field H. We shall solve the critical field problem for such a system within the framework of the Ginzburg-Landau [6] theory. Going over in the original equations of the theory to cylindrical coordinates (the oz axis coincides with the axis of the cylinder), we obtain the following equations:

$$-\frac{1}{\varkappa^2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)-\frac{1}{\varkappa^2 r^2}\frac{\partial^2\Psi}{\partial\varphi^2}+(A_z^2+A_\varphi^2)\Psi=\Psi-\Psi^3, \quad (1)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 A_z}{\partial\varphi^2}=\Psi^2 A_z, \quad (2)$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_\varphi}{dr}\right)-\frac{A_\varphi}{r^2}=\Psi^2 A_\varphi. \quad (3)$$

The boundary condition is: $(\partial\Psi/\partial r)_{r=r_0}=0$. In these equations A is the vector potential of the magnetic field, $|\Psi|^2$ defines the density of "superconducting electrons," $\kappa=(2\sqrt{2}\,e/\hbar c)\times H_{cm}\delta_0^2$ is the characteristic parameter of the Ginzburg-Landau theory, H_{cm} is the thermodynamic critical field, δ_0 is the penetration depth for a weak magnetic field. The system (1)—(3) is obtained under the assumption that Ψ is a real function. It is always possible to do this for a simply connected superconductor with the help of the appropriate gauge transformation. [12] All quantities are expressed in the relative system of units introduced in [6], i.e., δ_0 is taken to be the unit of length, the quantity $\sqrt{2}\,H_{cm}$ is taken as the unit of magnetic field intensity.

Since the cylinder is assumed to be thin ($\kappa r_0 \ll 1$), we shall look for [6,11] a solution Ψ of the form $\Psi(r,\varphi)=\Psi_0+u(r,\varphi)$, where $|u|\ll\Psi_0$, $\Psi_0=\text{const}$. Then, using the results of Silin, [11] we find the solution of Eqs. (2) and (3) for A in the zero-order approximation:

$$A_z=-\frac{2H_0}{\Psi_0}\frac{I_1(\Psi_0 r)}{I_0(\Psi_0 r_0)}\sin\varphi-\frac{H_I}{\Psi_0}\frac{I_0(\Psi_0 r)}{I_1(\Psi_0 r_0)},$$

$$A_\varphi=\frac{H}{\Psi_0}\frac{I_1(\Psi_0 r)}{I_0(\Psi_0 r_0)}; \quad (4)$$

$$I_m(x)=\sum_{\nu=0}^{\infty}\frac{(x/2)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}. \quad (5)$$

Here H_I is the intensity of the magnetic field at the surface of the cylinder produced by the total current I flowing in it.

For the derivation of the connection of Ψ_0 with H_I, H_0, H and r_0 we shall neglect the quantity u in comparison with Ψ_0 and, having substituted (4) into (1) we integrate Eq. (1) over a circle of radius r_0:

$$H_I^2=\tfrac{1}{4}(1-\Psi_0^2)\Psi_0^4 r_0^2-\tfrac{1}{16}\Psi_0^4 r_0^4\left(H_0^2+\tfrac{1}{2}H^2\right). \quad (6)$$

For $H_I=H=0$ this expression goes over into the formula $\Psi_0^2=1-(1/4)H_0^2 r_0^2$ which was obtained by Silin [11] for a thin cylinder ($\Psi_0 r_0 \ll 1$) in a transverse magnetic field H_0.

We determine the value of Ψ_{0c}^2 corresponding to the transition from the superconducting to the normal state in the same way that this was done in [6] for a thin film. The dependence of H_I^2 on Ψ_0 is represented by a positive function having a maximum in the interval $0 \le \Psi_0 \le 1$. Let us denote the abscissa of this maximum by Ψ_{0c}. It is obvious that the states with $\Psi_0 < \Psi_{0c}$ are unstable, since a decrease of Ψ_0 corresponds here to a decrease of H_I. Consequently Ψ_{0c} is the minimum stable value of Ψ_0 and upon being substituted into Eq. (6) it gives the critical value of H_I for given external fields, H_0 and H. Solving the equation $d(H_I)^2/d\Psi_0=0$, we find

$$H_{I_c}^2=\tfrac{1}{27}r_0^2-\tfrac{1}{36}r_0^4\left(H_0^2+\tfrac{1}{2}H^2\right). \quad (7)$$

3. DISTRIBUTION OF A WEAK CURRENT IN A LAMINATED SUPERCONDUCTING SLAB

Before going on to the filamentary superconductor, it is convenient for methodological reasons to investigate the current distribution in a composite plane superconductor which consists of a collection of thin parallel superconducting plane layers, separated from one another by dielectric. Let the thickness of the entire slab $2L \gg \delta_0$, but the thickness of each layer $2d \ll \delta_0$. We denote the distance between adjacent layers by a.

All calculations in this and in subsequent Sections will be carried out in the absolute Gaussian system of units, in contrast to the previous. All layers are parallel to the xz plane. The origin of coordinates is located at the center of the slab.

Let us consider one individual layer located a distance y from the center of the slab. We denote the total current flowing in this layer (in the direction ox) by I(y). The layer under consideration is located in an external magnetic field $H_0(y)$ which is parallel to its surface and perpendicular to the direction of the current. The field H_0 is produced by the current flowing in all the other layers located in the interval $(-y, +y)$.

We investigate the selected layer in the London approximation. The free energy per unit area of such a layer will be

$$F(y)=\int_{-d}^{d}\left[\frac{1}{8\pi}H^2(\xi)+\frac{1}{2}\Lambda j^2(\xi)\right]d\xi,$$

where ξ is the distance along the oy axis from the center of the layer, $\Lambda=4\pi\delta_0^2/c^2$, H is the intensity of the magnetic field inside the layer, j is the current density inside the layer. Using the well-known solutions of the London equations for a layer with current in a magnetic field and taking the inequality $|\xi| \leq d \leq \delta_0$ into account, we obtain

$$F(y) = \frac{1}{4\pi}[H_0^2(y)\, d + H_I^2(y)\, \delta_0^2/d], \tag{8}$$

where $H_I = 2\pi I/c$ is the field at the surface of the layer produced by the total current I flowing in the layer. The free energy per unit area of the dielectric region between layers is obviously $F_a(y) = H_0^2(y)\, a/8\pi$.

We shall now assume that a physically infinitesimal volume contains many such superconducting layers, and we consider some average picture of the distribution of current $j(y)$ and field $H_0(y)$ inside our superconducting slab. The average free energy density will be $\mathcal{F}(y) = (F + F_a)/(2d + a)$; the average current density $j(y) = I(y)/(2d + a)$ is related to the field $H_0(y)$ by the equation $dH_0/dy = 4\pi j(y)/c$. Considering this relation and the formula for H_I, we obtain the following expression for the average free energy density:

$$\mathcal{F}(y) = \frac{\delta_0^2}{16\pi}\frac{2d+a}{d}\left(\frac{dH_0}{dy}\right)^2 + \frac{H_0^2}{8\pi}.$$

The desired function $H_0(y)$ must minimize the total free energy of the laminated superconductor. The variational problem which arises reduces to the solution of the Euler equation

$$d^2H_0/dy^2 - \delta_1^{-2}H_0 = 0,$$

where $\delta_1 = \delta_0[(2d + a)/2d]^{1/2}$. The solution of this equation for the boundary conditions $H_0(\pm L) = \pm H_J$ is*

$$H_0 = H_J\frac{\text{sh}(y/\delta_1)}{\text{sh}(L/\delta_1)},$$

where $H_J = 2\pi J/c$ (J is the total current in the slab).

The current distribution in the slab will be

$$j(y) = \frac{J}{2\delta_1}\frac{\text{ch}(y/\delta_1)}{\text{sh}(L/\delta_1)}. \tag{9}\dagger$$

Thus the current in a laminated slab will flow in a surface layer of thickness $\sim\delta_1$ which does not exceed by much the penetration depth δ_0 of a bulk sample. Actually, even if the distance between layers, a, is an order of magnitude larger than the thickness of the layers, 2d, the value of $\delta_1 \sim 3\delta_0$.

The current distribution (9) lends itself to a clear physical interpretation. A laminated superconductor, with the layers electrically connected in parallel, is a multiply connected superconducting region. If prior to the flow of current this multiply connected region did not trap any magnetic flux, then the switching on of the current J induces in this region currents that tend to counteract the magnetic field penetrating into the region. This leads to the consequence that the resulting current J will flow on the surface of our multiply connected superconductor.

It is clear that qualitatively the same result is obtained for the distribution of current in a composite "filamentary" superconductor.

*sh = sinh.

†ch = cosh.

4. DISTRIBUTION OF THE CRITICAL CURRENT IN A FILAMENTARY SUPERCONDUCTOR

At first glance it appears that the result obtained is in complete contradiction with the experimental results. It was stated in Sec. 1 that the critical current in plastically deformed hard superconductors is proportional to the transverse cross sectional area of the conductor, from which the conclusion was drawn that the current in such superconductors flows through the entire transverse cross section. Now the result is obtained that in a "filamentary" superconductor the current flows in a surface layer of depth δ_1.

However, this is only an apparent contradiction. The point is that the result obtained just now is only valid for a very weak total current J. Actually the current distribution (9) will only be realized as long as the current in the last layer has not reached its critical value. As soon as this occurs, a further increase of the total current J does not change the values of the current in the last layers (filaments). The critical current is preserved there. An increase of the total current will occur at the expense of increasing the current flown through the deeper lying layers or filaments. This process will continue until all reserves are exhausted, then the total current through the filamentary superconductor has achieved its maximum possible value. This occurs when all the filaments are simultaneously in the critical regime. Let us investigate this state.

The filamentary model of a superconductor is represented by a plane dielectric slab of thickness 2L, interpenetrated by fine superconducting filaments of radius $r_0 \ll \delta_0/\kappa$ which are parallel to the ox axis. Let the x, z plane be parallel to the sides of the slab, and the origin of coordinates be located in the middle of the slab. Let us select a layer of thickness dy located a distance y from the center of the slab. We shall assume that it contains a sufficiently large number of filaments so that for an average density of current it can be regarded as a physically infinitesimal volume.

Let the critical current $I_c(y)$ determined from condition (7) flow in each of the filaments of this layer. In absolute Gaussian units (7) is written in the form

$$H_{I_c}^2 = \frac{2}{27} H_{cm}^2 \frac{r_0^2}{\delta_0^2} - \frac{1}{36}\frac{r_0^4}{\delta_0^4}\left(H_0^2 + \frac{1}{2}H^2\right), \qquad (10)$$

where $H_{I_c} = 2I_c/cr_0$ is the intensity of the magnetic field at the surface of the filament produced by the current I_c flowing in the filament, H_{cm} is the thermodynamic critical field of the filament material, H_0 is the field produced by the current which flows in those filaments whose coordinates lie in the interval $(-y, +y)$. In the case under consideration $H = 0$. Expressing H_{I_c} in Eq. (10) in terms of I_c and using the equation

$$dH_0/dy = 4\pi c^{-1} I(y)n, \qquad (11)$$

where n is the number of filaments passing through a unit transverse cross sectional area of our superconductor, we obtain the equation

$$\frac{dH_0}{dy} = \frac{4\pi}{c} n\sqrt{a^2 - b^2H_0^2};$$

$$a = \frac{c}{3\sqrt{6}} H_{cm}\frac{r_0^2}{\delta_0}, \qquad b = \frac{c}{12}\frac{r_0^3}{\delta_0^2}. \qquad (12)$$

The solution of Eq. (12) obviously gives that distribution of the field H_0, and correspondingly of the current $j(y)$, for which all filaments are simultaneously in the critical state:

$$H_0 = (a/b)\sin(4\pi c^{-1}nby), \qquad (13)$$

$$j = an\cos(4\pi c^{-1}nby). \qquad (14)$$

The integration constant is set equal to zero from symmetry considerations.

It is necessary to consider two cases: $n \leq n_0$ and $n > n_0$, where

$$n_0 = 3\delta_0^2/2Lr_0^3. \qquad (15)$$

For $n = n_0$, the argument of the cosine in (14) becomes $\pi/2$ for $y = L$. For $n \leq n_0$ the distribution of the current over the cross section of the slab is described by Eq. (14) (see Fig. 1). The total critical current will be

$$J_c = \int_{-L}^{L} j\,dy = \frac{2}{\pi\sqrt{6}} cH_{cm}\frac{\delta_0}{r_0}\sin\left(\frac{\pi n}{2n_0}\right). \qquad (16)$$

Let us consider the case $n > n_0$. Now $j(y)$ reaches the value zero inside the interval $(-L, +L)$. This happens at the points $\pm y_0$ where $y_0 = 3\delta_0^2/2nr_0^3$. Then the current distribution inside the slab is determined by the formula

$$j = \begin{cases} an\cos(4\pi c^{-1}nby), & |y| \leqslant y_0 \\ 0, & |y| > y_0 \end{cases}. \qquad (17)$$

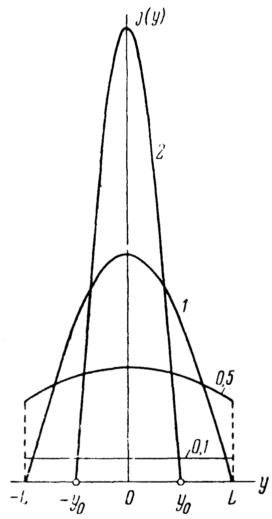

FIG. 1. The distribution of the current density inside a filamentary superconductor for various filament densities; the numbers on the curves indicate the value of n/n_0.

Actually our task from the very beginning was to find that distribution of field and current in the filaments which would guarantee the critical state for all filaments at once. The filaments located outside the interval $(-y_0, y_0)$ are in the magnetic field produced by the current flowing in the remaining filaments, which turns out to be critical for these exterior filaments. It is clear that current can no longer flow in them.

As is evident from Fig. 1, the current flow for the case $n > n_0$ is concentrated in the middle of the slab. In this case the total critical current turns out to be independent of n and equal to $J_{c\,max} = 2cH_{cm}\delta_0/\pi\sqrt{6}\,r_0$.

Thus, the dependence of the critical current in the slab on the density of filaments will be expressed by the formula

$$J_c = \begin{cases} J_{c\,max}\sin(\pi n/2n_0), & n \leqslant n_0 \\ J_{c\,max}, & n > n_0 \end{cases} \qquad (18)$$

5. DEPENDENCE OF THE CRITICAL CURRENT OF A FILAMENTARY SUPERCONDUCTOR ON THE EXTERNAL MAGNETIC FIELD

We now consider the case when the filamentary superconductor is located in an external homogeneous magnetic field H, and we shall determine the dependence of its total critical current J_c on this field.

1. Let the magnetic field H be directed along the oz axis, i.e., parallel to the surface of the slab and perpendicular to the filaments.

Then instead of Eq. (12) the following equation expresses the condition that all filaments are simultaneously in the critical regime:

$$dH_0/dy = 4\pi c^{-1}n\,[a^2 - b^2(H_0 + H)^2]^{1/2}, \qquad (19)$$

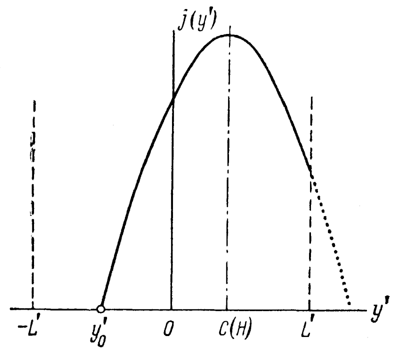

FIG. 2. The distribution of current over the cross section of a composite filamentary superconducting slab of thickness 2L', located in a transverse external magnetic field $H_z = H$, for $n \geq n_0$.

since now both the field H_0 produced by the current flowing in the filaments as well as the external field H act on the layer of thickness dy under consideration.

The solution of Eq. (19) is

$$H_0 + H = (a/b)\sin(4\pi c^{-1} nby - C(H)),$$

from which

$$j_c(y) = I_c n = an\cos(4\pi c^{-1} nby - C(H)). \qquad (20)$$

Here C(H) is an integration constant which depends on the parameter H. Let us determine this dependence.

First we consider the case $n \geq n_0$. In this case the flow of current in a filamentary superconductor in the absence of an external magnetic field is concentrated in the central part of the superconductor cross section. According to Eq. (20), the presence of an external magnetic field leads to a displacement of this concentration as a whole by an amount $cC(H)/4\pi nb$.

We introduce for convenience a new unit for measuring distance along the oy axis, so that $4\pi nby/c = y'$, $4\pi nbL/c = L'$. Then the current distribution over the cross section is written, on the basis of Eq. (20), in the form

$$j_c = \begin{cases} an\cos(y' - C(H)), & y' > y_0' \\ 0, & y' \leqslant y_0' \end{cases},$$

and displacement of the current concentration along the oy' axis is exactly equal to C(H) (see Fig. 2). The maximum critical current flows in those filaments located a distance C(H) from the center of the slab. This means that these filaments are located in zero external magnetic field. This condition enables us to establish the dependence of C on H. The total current flowing in the filaments located to the left of the point y' = C(H) will be

$$J_{\text{left}} = \frac{c}{4\pi}\frac{a}{b}\int_{-\pi/2}^{0} \cos y'\, dy' = \frac{c}{4\pi}\frac{a}{b}.$$

And for the filaments located to the right of the point y' = C(H), it amounts to

$$J_{\text{right}} = \frac{c}{4\pi}\frac{a}{b}\int_{0}^{L'-C(H)} \cos y' dy' = \frac{c}{4\pi}\frac{a}{b}\sin(L' - C(H)).$$

We can determine the dependence of C on H from the condition that the filaments located at the point y' = C(H) are in zero external magnetic field:

$$J_{\text{right}} + cH/2\pi - J_{\text{left}} = 0, \qquad (21)$$

from which

$$H = (a/2b)[1 - \sin(L' - C)]. \qquad (22)$$

On the other hand, the total critical current in a filamentary superconductor in the presence of an external magnetic field H will evidently be

$$J_c = J_{\text{left}} + J_{\text{right}} = \frac{c}{4\pi}\frac{a}{b}[1 + \sin(L' - C)]. \qquad (23)$$

From (22) and (23) we find

$$J_c = J_{c\,max} - cH/2\pi, \qquad (24)$$

where $J_{c\,max} = ca/2\pi b = 2cH_{cm}\delta_0/\pi\sqrt{6}\, r_0$ is the maximum critical current of a filamentary superconductor in the absence of an external magnetic field, for $n \geq n_0$.

Now let us consider the case $n < n_0$. In this case the distribution of current over the cross section of a filamentary superconductor in the presence of the field H is shown in Fig. 3. The function C(H) is found in the same way as for the previous case:

$$J_{\text{left}} = \frac{c}{4\pi}\frac{a}{b}\int_{-L'-C}^{0} \cos y' dy' = \frac{c}{4\pi}\frac{a}{b}\sin(L' + C),$$

$$J_{\text{right}} = \frac{c}{4\pi}\frac{a}{b}\int_{0}^{L'-C} \cos y' dy' = \frac{c}{4\pi}\frac{a}{b}\sin(L' - C),$$

from this and from condition (21) we obtain $H = (a/b)\cos L' \sin C$. Considering that the total critical current $J_c = J_{\text{right}} + J_{\text{left}}$, we finally obtain

$$J_c = J_{c\,max}\sin\left(\frac{\pi n}{2n_0}\right)\left[1 - \frac{(H/H_{max})^2}{\cos^2(\pi n/2n_0)}\right]^{1/2}, \qquad (25)$$

where $H_{max} = a/b = 4H_{cm}\delta_0/\sqrt{6}\, r_0$ is the critical transverse magnetic field of a filamentary superconductor, destroying the superconductivity of such a superconductor without current flowing in it. This quantity is exactly equal to the critical field of a thin circular cylinder of radius r_0 with-

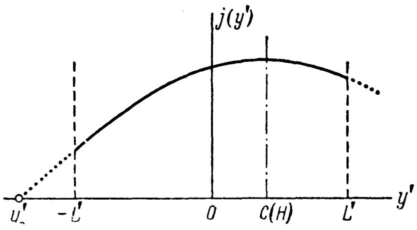

FIG. 3. The distribution of current over the cross section of a composite filamentary superconducting slab of thickness 2L', located in a transverse external magnetic field $H_z = H$, for $n < n_o$.

out current, if the field is perpendicular to the axis of the cylinder (10).

The dependence of J_c on H given by (25) will be realized only as long as $y_0' < -L$ (see Fig. 3), which corresponds to the condition

$$H < H^*,\ H^* = H_{max} \cos^2 (\pi n / 2n_0).$$

The curve $J_c(H)$ given by (25) is tangent to the straight line (24) at the point $H = H^*$. For $H > H^*$ the current distribution over the cross section of a filamentary superconductor will in principle be that shown in Fig. 2, and consequently the dependence of J_c on H will be determined by Eq. (24). The function $J_c(H)$ is shown in Fig. 4 for various values of n.

Thus, finally, for $n \leq n_0$

$$J_c = \begin{cases} J_{c\,max} \sin\left(\dfrac{\pi n}{2n_0}\right)\left[1 - \dfrac{(H/H_{max})^2}{\cos^2(\pi n/2n_0)}\right]^{1/2}, & H \leqslant H^* \\ J_{c\,max}(1 - H/H_{max}), & H > H^* \end{cases} \quad (26)$$

Up to now the case when the external magnetic field is directed along the oz axis has been considered. If the external homogeneous magnetic field H is directed along the ox axis, i.e., if it is parallel to the filaments, then according to Eq. (10) it is necessary to write the equation

$$dH_0/dy = 4\pi c^{-1} n\, [a^2 - b^2 (H_0^2 + H^2/2)]^{1/2}, \quad (27)$$

instead of Eq. (19). The solution of this equation is

$$H_0 = \frac{1}{b} \sqrt{a^2 - b^2 H^2/2}\ \sin\left(\frac{4\pi}{c} nby\right),$$

and hence the current density distribution over the cross section of a filamentary superconductor has the form

$$j_c = n \sqrt{a^2 - b^2 H^2/2}\ \cos(4\pi c^{-1} nby); \quad (28)$$

the integration constant is set equal to zero according to symmetry considerations.

Carrying out an investigation analogous to that carried out in Sec 4, we obtain the dependence of the critical current in a filamentary superconductor on the external longitudinal magnetic field H (see Fig. 4):

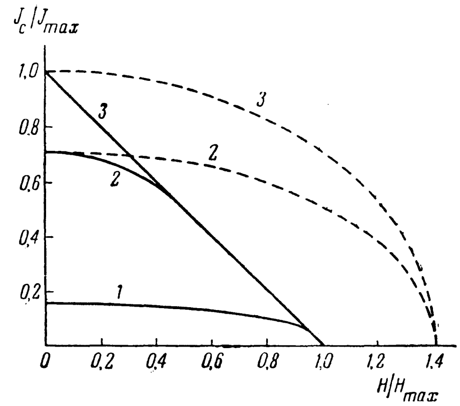

FIG. 4. The dependence of the critical current on the intensity of the external magnetic field for a composite filamentary superconducting slab. The solid curves are for the field parallel to the surface of the slab and perpendicular to the filaments; the dashed curves are for the field parallel to the filaments: curve 1 is for $n/n_o = 0.1$, curve 2 for $n/n_o = 0.5$, curve 3 for $n/n_o \geq 1$.

$$J_c = \begin{cases} J_{c\,max} \sin(\pi n/2n_0) \sqrt{1 - H^2/2H_{max}^2}, & n \leqslant n_0 \\ J_{c\,max} \sqrt{1 - H^2/2H_{max}^2}, & n > n_0 \end{cases} \quad (29)$$

Finally, if the external homogeneous magnetic field H is directed along the oy axis, i.e., if it is perpendicular to the filaments and to the surface of the slab, then instead of Eq. (19) the initial equation will obviously be

$$dH_0/dy = 4\pi c^{-1} n \sqrt{a^2 - b^2 (H_0^2 + H^2)}.$$

We solve this equation and set the integration constant equal to zero:

$$H_0 = b^{-1} \sqrt{a^2 - b^2 H^2}\ \sin(4\pi c^{-1} nby),$$

hence

$$j_c = n \sqrt{a^2 - b^2 H^2}\ \cos(4\pi c^{-1} nby), \quad (30)$$

$$J_c = \begin{cases} J_{c\,max} \sin(\pi n/2n_0) \sqrt{1 - H^2/H_{max}^2}, & n \leqslant n_0 \\ I_{c\,max} \sqrt{1 - H^2/H_{max}^2}, & n > n_0 \end{cases}.$$

6. DISCUSSION OF THE RESULTS

In the present article, the dependence of the critical current in a composite filamentary superconductor on the magnitude of the external magnetic field and on the density of filaments n has been obtained. In this connection, it was assumed that such a filamentary superconductor resembles a plastically deformed hard superconductor, and the density of filaments increases with plastic

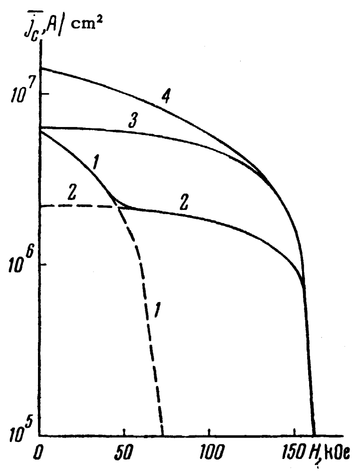

FIG. 5. Theoretically calculated dependence of the critical current on the external magnetic field for a Nb_3Sn band with different filament densities: curve 1 is for a bulk sample; curve 2 is for $n/n_o = 0.1$; curve 3 is for $n/n_o = 0.3$; curve 4 is for $n/n_o \geq 1$.

strain. We once again emphasize that the superconducting filaments of our model represent not only dislocation lines along which superconductivity is preserved, but also any other thin superconducting inclusions.

If no external magnetic field is present, then the distribution of the critical current over the cross section of our slab will be determined by Eqs. (14) or (17). For a filament density much less than a certain characteristic density n_0, the critical current is distributed practically uniformly over the cross section. On the contrary, for $n \gg n_0$ the critical current is concentrated near the middle of the slab (Fig. 1).

The dependence of the total critical current on n [see Eq. (18)] is characterized by the fact that for $n \geq n_0$ the critical current reaches its maximum value and saturation occurs.

Let us estimate the value of n_0. If dislocations play the role of superconducting filaments in our model, then it is natural to take $r_0 \sim 10^{-7}$ cm as an estimate for the radius of a filament. Since the penetration depth for a weak magnetic field is $\delta_0 \sim 10^{-5}$ to 10^{-6} cm, we take $\delta_0/r_0 \sim 10$. Let the transverse dimension of the superconductor be $L \sim 10^{-2}$ cm—ordinarily encountered in experimental situations. Then, according to (15), $n_0 \sim 10^{11}$ cm^{-2}. On the other hand, it is known that the density of dislocations in metals is within the limits 10^8 to 10^{12} cm^{-2}, depending on the degree of plastic deformation. Thus, it is possible to state that the estimate obtained for n_0 agrees in order of magnitude with the density of dislocations inside a plastically deformed metal.

If $n \geq n_0$, then the critical current flow is not uniform over the entire cross section of the slab, and one can only talk about a certain average density of the critical current, $\bar{j}_{c\,max} = J_{c\,max}/2L$. Having assumed $H_{cm} \sim 1$ to 10 kOe and $\delta_0/r_0 \sim 10$ for the filament material, from (18) we obtain $\bar{j}_{c\,max} \sim 10^6$ to 10^7 A/cm^2.

Finally we note that the magnitude of the maximum critical field H_{max} of a filamentary superconductor does not depend on n, which follows from the expression for H_{max}. If $H_{cm} \sim 1$ to 10 kOe and $\delta_0/r_0 \sim 10$, then $H_{max} \sim 10$ to 100 kOe.

Thus it is evident that a filamentary superconductor represents a plastically deformed hard superconductor sufficiently well and leads to reasonable numerical estimates.

The relations obtained in the present article enable us to carry out a specific calculation. Let us take, for example, a sample of Nb_3Sn. Without dealing with the question of the nature of the superconducting filaments appearing in this sample, we assume for them the value extrapolated by Kunzler, $H_{max} = 170$ kOe, and $r_0 \sim 10^{-7}$ cm (according to Kunzler, $r_0 \ll 10^{-6}$ cm). The treatment carried out in [15] of the results of Bozorth et al. [14] gave $\kappa = 3.5$, $H_{cm} = 15$ kOe, $H_{c2} = 72$ kOe (H_{c2} is the second critical field at which electrical resistance is reestablished) for this intermetallic compound. Finally we assume that the thickness of the sample $2L = 0.02$ cm. Let the filament density be $n/n_0 = 0.1$. The indicated parameters are sufficient for complete calculation of the function $\bar{j}_c(H)$. If there were no filaments present in general, then the relation between the critical current J_c and the external magnetic field H would be determined by the equation $H_{c2} = (2\pi J_c/c) + H$.

The dependence of the average current density $\bar{j}_c = J_c/2L$ on H is shown in Fig. 5 (curve 1) for this case. This curve will be realized until, at a given value of the external field the critical current of the bulk sample is larger than the critical current in the filamentary superconductor ($n/n_0 = 0.1$, curve 2). For values of H and $\bar{j}_c$ corresponding to the point of intersection of curves 1 and 2, the total volume sample goes over into the normal state and the current begins to flow in the filaments. The dependence of the critical current density on the external magnetic field will be determined by curve 2. The function $\bar{j}_c(H)$ is also shown in Fig. 5 (curves 3 and 4) for the cases $n/n_0 = 0.3$ and $n/n_0 \geq 1$.

For comparison, one can point out that a critical current density equal to 1.4×10^6 A/cm^2 was obtained in [13] for a Nb_3Sn wire of diameter 0.015 cm.

The curves obtained are in good agreement (at least qualitatively) with the experimental curves, for which a sudden drop of $\bar{j}_c$ is frequently observed in weak fields, changing into a plateau for average values of the field, and ending with an almost vertical drop near H_{max}. In addition, the experimental curves are characterized by a strong dependence of the middle portion of a characteristic on the extent of plastic deformation, with the critical current changing by more than an order of magnitude, whereas no dependence of H_{max} on the degree of plastic deformation was observed in the experiment.

The dependence of $\overline{j}_c$ on H for the case of a longitudinal magnetic field [see Eq. (29)] is also qualitatively in agreement with experiment.

It is necessary to dwell separately on the results of the calculation for the case when the field is perpendicular to the direction of the current and to the surface of the band. Its comparison with Eq. (25), giving $J_c(H)$ for the case of a field that is parallel to the surface of the band and perpendicular to the current, shows that the curve corresponding to Eq. (30) will lie above the curve corresponding to Eq. (25) for all points. At this point a sharp qualitative discrepancy between theory and experiment appears, since the experiments [16,17] carried out on rolled tapes of Nb-Zr alloy gave exactly the opposite mutual position of the curves for these two cases.

It is probably necessary to seek the explanation for this discrepancy in the fact that for rolled tapes the cross sections of the superconducting regions will not be circular. Such superconducting regions will consist of thin superconducting layers whose surfaces are parallel to the surface of the rolled tape. It is, of course, impossible to apply the calculation carried out in Sec 5 to this case, since the critical current of such a layer will depend significantly on the orientation of the transverse field relative to the surface of the layer.

In the present article, it has been assumed everywhere that the density of superconducting filaments is constant. An additional investigation was carried out with the purpose of explaining how the function $J_c(H)$ is changed if it is assumed that $n = n(y)$. The field H was assumed to be parallel to the surface of the slab and perpendicular to the filaments. In order to simplify the calculations it was assumed that $n = \beta\,|y|$ where $\beta > 0$. It was found that in this case $J_c(H)$ will be determined by Eq. (25) as before, provided that the average value $\overline{n} = L^{-1}\int_0^L n\,dy$ is substituted in place of n. This gives a basis to conjecture that the results obtained in the present article do not depend on the assumption concerning the constancy of n.

In conclusion, it is necessary to remark that one must be careful in comparing the results obtained here with experimental results, since our calculation applies to superconductors with markedly elongated rectangular cross sections whereas the majority of experiments is carried out on wires of circular cross section. In addition, the calculations strictly speaking were carried out for an infinitely high slab, and therefore the theory does not take any account of the edge effect which arises for a superconductor of rectangular cross section.

The collection of results obtained in this article gives a good qualitative description of the behavior of plastically deformed hard superconductors with current, located in an external magnetic field.

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