INFRARED SINGULARITIES OF MATRIX ELEMENTS IN SCALAR ELECTRODYNAMICS

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By summation of the perturbation-theory series by means of the renormalization group, expressions are obtained for the infrared (near-threshold) singularities of matrix elements in scalar electrodynamics. The matrix elements for photon-meson, electron-meson, and meson-meson scattering are considered. A comparison of the results of this work with the corresponding expressions for spinor electrodynamics[1] shows that the form of the main infrared singularities does not depend on the spin of the charged particles.

INTRODUCTION

THE infrared singularities of matrix elements in spinor electrodynamics have been treated in papers by one of the writers.[1] Here we extend this treatment to the case in which the charged particles have no spin.

As in [1], we shall mark with an index λ quantities calculated with a mass $\lambda^{1/2}$ introduced in the photon propagation function, and write the element of the S matrix for the elastic scattering of two particles in the form

$$\langle p_2k_2|S_\lambda - 1|p_1k_1\rangle = -i(2\pi)^4(16\,p_1^0p_2^0k_1^0k_2^0)^{-1/2}\,\delta(p_1+k_1-p_2-k_2)T_\lambda, \tag{1}$$

where p_1, k_1 and p_2, k_2 are the momenta of the particles before and after the scattering. We use the notations $s = (p_1 + k_2)^2$, $u = (p_1 - k_2)^2$, $t = (p_1 - p_2)^2$ for the squares of the total momenta in the direct, crossed, and third processes.

We take into account the infrared divergences in T_λ by means of the formula[2]

$$T_\lambda = e^{K_\lambda}T, \tag{2}$$

$$K_\lambda = -\sum_{i<j} z_ia_iz_ja_jF_\lambda((p_ia_i+p_ja_j)^2, p_i^2, p_j^2), \tag{3}$$

where the summation is taken over all charged particles before and after the reaction, z_i is the sign of the charge, $a_i = 1$ or -1 for an emerging or entering particle with the momentum p_i, and

$$F_\lambda((p_1-p_2)^2, p_1^2, p_2^2) = \frac{i\alpha}{8\pi^3}\int\frac{dk}{k^2-\lambda}\left(\frac{2p_1-k}{2p_1k-k^2}-\frac{2p_2-k}{2p_2k-k^2}\right)^2 \tag{4}$$

(α is the fine-structure constant, $\hbar = c = 1$, $ab = a^0b^0 - \mathbf{a}\cdot\mathbf{b}$). For F_λ (for $\lambda \to 0$) we have the following representation:[1)]

$$2F_\lambda(t, m^2, M^2) = \beta(x_t)\ln(mM/\lambda) - \varepsilon(x_t, \nu), \tag{5}$$

$$4mMx_t = t-(m-M)^2, \quad 4mM\nu = (m-M)^2, \tag{6}$$

$$\beta(x) = \frac{dx}{2\pi}\int_1^\infty\frac{(2z-1)\,dz}{\sqrt{z(z-1)}\,z(z-x-i\varepsilon)}, \tag{7}$$

$$\varepsilon(x,\nu) = \frac{\alpha x}{2\pi}\int_1^\infty\left[\frac{2z-1}{\sqrt{z(z-1)}}\ln\frac{z+\nu}{4z(z-1)}+\frac{\sqrt{z(z-1)}}{z+\nu}\right]\frac{dz}{z(z-x-i\varepsilon)}. \tag{8}$$

In what follows we shall examine the infrared singularities of T, i.e., the singularities which appear when s, u, or t approaches threshold values, for the processes of elastic photon-meson, electron-meson, and meson-meson scattering. It turns out that the form of the main singularities is the same as for the corresponding processes involving spinor charged particles, which have been treated earlier.[1]

In Section 4, fourth-order perturbation theory is used to study the analytic properties of the matrix element of meson-meson scattering which is obtained after the main infrared singularities are removed. In the infrared region this matrix element contains terms which go to infinity as $y^{-1/2}\ln y$, $y^{-1/2}\ln t$, $y^{-1/2}$, $t^{-1/2}$, $\ln t$, where $y = s - (m + M)^2$ or $u - (m + M)^2$; that is, it has integrable singularities. The Mandelstam representation holds for this matrix element in fourth-order perturbation theory.

1)We note that the expression (8) corresponds to the transverse gauge. If the Coulomb gauge is used, the second term in the parentheses in the integrand must be omitted.

2. PHOTON-MESON SCATTERING

The kinematics of photon-meson scattering has been treated in [4]. Let p_1, p_2, and m be the momenta and mass of the meson, and let k_1, k_2 and l_1, l_2 be the momenta and polarization vectors of the photon. Then the matrix element can be written in the form

$$T = A\,(s, u, t)\,H_A + B\,(s, u, t)\,H_B, \qquad (9)$$

where the structure expressions H_A, H_B are gauge invariant and are given by

$$H_A = (e_1e_2) - (e_1k_2)\,(e_2k_1)/k_1k_2, \qquad (10)$$

$$H_B = (e_1q)\,(e_2q) - (e_1k_2)\,(e_2q)\,(k_1q)/k_1k_2 - (e_1q)\,(e_2k_1)\,(k_2q)/k_1k_2 + (e_1k_2)\,(e_2k_1)\,(k_1q)\,(k_2q)/(k_1k_2)^2; \quad q = p_1 + p_2. \qquad (11)$$

We note that for $t = -2k_1k_2 = 0$ the momenta k_1 and k_2 are equal and $k_2e_1 = k_1e_2 = 0$. Therefore H_A and H_B are finite at $t = 0$. In the center-of-mass system of the direct process

$$H_A = -\mathbf{e}_1\mathbf{e}_2 + 2t^{-1}\,(\mathbf{e}_1\mathbf{p}_2)\,(\mathbf{e}_2\mathbf{p}_1), \qquad (12)$$

$$H_B = 4t^{-2}\,(s - m^2)^2\,(\mathbf{e}_1\mathbf{p}_2)\,(\mathbf{e}_2\mathbf{p}_1). \qquad (13)$$

Let us examine the analytic properties of A and B in the lower orders of perturbation theory. In second order (Fig. 1)

$$A^{(2)} = -2e^2, \quad B^{(2)} = e^2\left(\frac{1}{s - m^2} + \frac{1}{u - m^2}\right). \qquad (14)$$

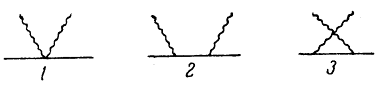

FIG. 1

In fourth order the diagrams 1—9 of Fig. 2 contain infrared divergences (for $\lambda \to 0$). After renormalization of the meson wave functions and removal of the infrared divergences by Eq. (2), where for the present case $K_\lambda = F_\lambda(t, m^2, m^2)$, we find that in the infrared region, i.e., for $s \to m^2$ or $u \to m^2$, the main singularities in $T^{(4)}$ come, as in spinor electrodynamics,[1] from diagrams 8—15 of Fig. 2.

The main singularities are of the form

$$T^{(4)} = [\beta\,(t/4m^2)\,\ln\,[(m^2 - s)/m^2] + \gamma\,(t/4m^2)]\,T_s^{(2)} + [\beta\,(t/4m^2)\,\ln\,[(m^2 - u)/m^2] + \gamma\,(t/4m^2)]\,T_u^{(2)} + \ldots, \qquad (15)$$

$$\gamma\,(x) = -\frac{\alpha x}{4\pi}\int_1^\infty \left[\frac{(2z-1)\ln 4z}{\sqrt{z(z-1)}} - \sqrt{\frac{z-1}{z}}\,\right]\frac{dz}{z\,(z - x - i\varepsilon)}, \qquad (16)$$

where the dots denote terms less singular than a pole at $s \to m^2$ or $u \to m^2$; $\beta\,(x)$ is given by Eq. (7); $T_s^{(2)}$, $T_u^{(2)}$ are the respective contributions of diagrams 2 and 3 in Fig. 1 [concerning Eq. (16) see footnote[1)]].

The other fourth-order diagrams are finite for $\lambda \to 0$, and for $s \to m^2$ ($u \to m^2$) they have only integrable singularities (i.e., weaker than a pole).

Up to terms which are less singular for $s \to m^2$ or $u \to m^2$, the expression (15) can be expanded in terms of the structures (10), (11). The result we get is that in the fourth order

$$A^4 = A_a^4, \qquad (17)$$

$$B^4 = e^2\,\{(s - m^2)^{-1}\,[\beta\,(t/4m^2)\,\ln\,[(m^2 - s)/m^2] + \gamma\,(t/4m^2)] + (u - m^2)^{-1}\,[\beta\,(t/4m^2)\,\ln\,[(m^2 - u)/m^2] + \gamma\,(t/4m^2)]\} + B_a^{(4)}, \qquad (18)$$

where in the infrared region $A_a^{(4)}$ and $B_a^{(4)}$ have singularities weaker than a pole.

The equations of the renormalization group[5] enable us to sum the main singularities in B. Consider, for example, the region $s \to m^2$. Representing B in the form $B = e^2(s - m^2)^{-1}M$ and considering M for various normalizations with fixed factors for the external lines, we can write for M the differential equations of the group. We choose the normalization momentum k_0 so that $\mathrm{Re}\,d(1, m^2/k_0^2, e^2) = 1$, where $d(k^2/k_0^2, m^2/k_0^2, e^2)$ is the transverse Green's function of the photon. Letting k_0^2 go to m^2 for $s \to m^2$, we get a connection between $M[(m^2 - s)/m^2]$ and the expression $M[(m^2 - s)/(m^2 - k_0^2)]$, for which the perturbation-theory series converges well for all t in the domain of definition of the function $\beta\,(t/4m^2)$. The resulting expression for M is $M = \psi\,(t)\exp\{e^2\beta\,(t/4m^2)\ln[(m^2 - s)/m^2]\}$, where e^2 is the observed charge. For $\psi\,(t)$ we can again write the equation of the renormalization group, which gives for all finite t (including $t \to 4m^2$) the result $\psi\,(t) = e^{\gamma(t/4m^2)}$.

When we use an analogous procedure for the region $u \to m^2$, we get for B the following representation:

$$B = -e^2e^{\gamma(t/4m^2)}[(m^2 - s)^{\beta(t)} + (m^2 - u)^{\beta(t)}] + B_a, \qquad (19)$$

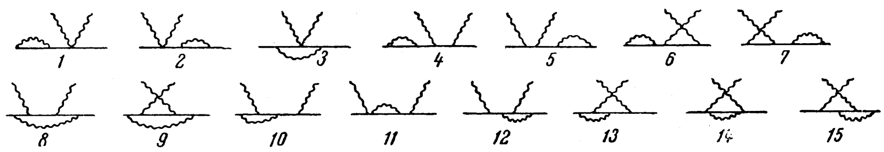

FIG. 2

$$\delta(t) = -1 + \beta(t/4m^2), \tag{20}$$

where β and γ are in general series in α, whose first terms are given in Eqs. (7) and (16), and B_a does not contain the main infrared singularities. The physical interpretation of the function $\delta(t)$ in Eq. (19) has been considered by a number of authors.[1,6-8]

3. ELECTRON-MESON SCATTERING

Let p_1, p_2, and m be the momenta and mass of the electron, and k_1, k_2, and M those of a positive meson. In second order (diagram 1 of Fig. 3) the matrix element

$$T^{(2)} = e^2\bar{u}(p_2)\gamma^n u(p_1)(k_1 + k_2)_n t^{-1}, \tag{21}$$

($\bar{u}u = 2m$) has a pole at $t = 0$. This case is unlike the preceding one in that in fourth order the main infrared singularities occur for $t \to 0$. To examine these singularities, we first separate out the infrared divergences.

In fourth order the divergences are given by diagrams of the same type as in the preceding case, but since each charged particle contributed to the divergences, the infrared factor K_λ in Eq. (2) is in this case given by

$$K_\lambda = 2F_\lambda(s, m^2, M^2) - 2F_\lambda(u, m^2, M^2) + F_\lambda(t, m^2, m^2) + F_\lambda(t, M^2, M^2). \tag{22}$$

We have here diagrams unlike those of the preceding case, diagrams 2 and 3 of Fig. 3, which contain two virtual photons and give infrared divergences and singularities when the momentum of each goes to zero. Consider, for example, diagram 2. Putting its contribution in the form

$$T^{(4)}_{\lambda 2} = \frac{ie^4}{(2\pi)^4}\int \frac{dk}{(k^2 - 2p_1k)(k^2 + 2k_1k)} \left\{\left[\frac{N(0)}{k^2 - \lambda} + \frac{N(q)}{(k-q)^2 - \lambda}\right]\frac{1}{q^2} + \left[\frac{N(0)}{k^2-\lambda}\left(\frac{1}{q^2 - 2qk} - \frac{1}{q^2}\right) - \frac{N(q)}{(k-q)^2 - \lambda}\left(\frac{1}{q^2 - 2qk} + \frac{1}{q^2}\right)\right] + \left[\frac{N(k) - N(0)}{k^2 - \lambda} - \frac{N(k) - N(q)}{(k-q)^2 - \lambda}\right]\frac{1}{q^2 - 2qk}\right\}, \tag{23}$$

where $q = p_1 - p_2$ and

$$N(k) = \bar{u}(p_2)\gamma^m d_{mn}(k_1 + k_2 + k)^n(\hat{p}_1 - \hat{k} + m) \times (2k_1 + k)^r d_{rs}\gamma^s u(p_1) \tag{24}$$

(d_{ij} is the factor for the photon Green's function, which is equal to g_{ij} for the Coulomb gauge or to $g_{ij} - k_ik_j/k^2$ for the transverse gauge; in the latter case in $N(0)$ and $N(q)$ the momentum k in d_{ij} is not fixed), we see that for $\lambda \to 0$ only the first term in the curly brackets is divergent. It gives a contribution to $2F_\lambda(s, m^2, M^2)$ in Eq. (22). In the second and third terms in Eq. (23) we can set $\lambda = 0$. Then the second term has at least a pole for $t \to 0$. It can be shown that the singularities of the third term are weaker than a pole. Thus the main infrared singularities in $T_2^{(4)}$ are contained in the second term in Eq. (23).

Diagram 3 of Fig. 3 is treated in a similar way. Diagrams 2 and 3 of Fig. 3 give the following contribution to the main singularities of the matrix element for $t \to 0$:

$$T^{(4)}_{2,3} = \Phi T^{(2)} + \ldots, \tag{25}$$

$$\Phi = [\beta(x_s) - \beta(x_u)]\ln(-t/mM) + \varepsilon(x_s, \nu) - \varepsilon(x_u, \nu), \tag{26}$$

where the functions β and ϵ are given by Eqs. (6) – (8).

Besides those from diagrams 2 and 3, pole singularities in T come from diagram 4 of Fig. 3, which contains an electron-photon vertex function. Apart from the factor $F_\lambda(t, m^2, m^2)$ in Eq. (22), it gives a contribution to the infrared singularities which depends on the supplementary magnetic moment μ' of the electron,

$$T^{(4)}_4 = \bar{u}(p_2)\mu'\cdot\tfrac{1}{2}[\hat{q}, \gamma^n]_- u(p_1)e(k_1 + k_2)_n t^{-1} + \ldots \tag{27}$$

It is not hard to verify that in the sixth order (diagrams 5—8 of Fig. 3) the part of the electron-photon vertex function that depends on μ' leads to singularities of the form (25), where instead of $T^{(2)}$ one must insert the expression (27). Thus in the lowest orders perturbation theory gives for the main infrared singularities of the matrix element T the sum of the expressions (21) and (25), where the matrices γ^n are to be replaced by $\gamma^n + (\mu'/2e)[\hat{q}, \gamma^n]$.

Applying the renormalization group, we find that

$$T = e^2\bar{u}(p_2)(\gamma^n + (\mu'/2e)[\hat{q}, \gamma^n]_-)u(p_1)(k_1 + k_2)_n t^{-1}e^{\Phi} + T_a, \tag{28}$$

where Φ is in general a series in α whose first term is given by Eq. (26), and for $t \to 0$ the singularities of the quantity T_a are weaker than a pole.

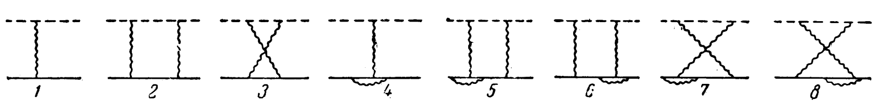

FIG. 3

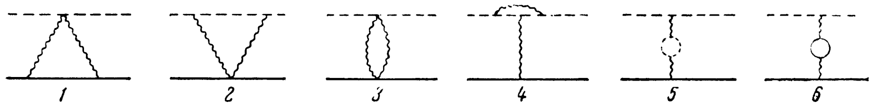

FIG. 4

If we express T in terms of invariant functions in the usual way

$$T = \bar{u}(p_2)\left(A(s, u, t) + (\hat{k}_1 + \hat{k}_2) B(s, u, t)\right) u(p_1), \quad (29)$$

then

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} e\mu'(s-u) \\ e^2 - 2me\mu' \end{pmatrix} \frac{e^{\Phi}}{t} + \begin{pmatrix} A_a \\ B_a \end{pmatrix}. \quad (30)$$

4. MESON-MESON SCATTERING

Let us consider the matrix element for the scattering of two oppositely charged mesons with the masses m and M. When, as in the preceding case, we separate out the infrared divergences and the main infrared singularities, we get the following expression for the matrix element

$$T = e^2 (s-u) e^{\Phi} t^{-1} + T_a, \quad (31)$$

where Φ is given by Eq. (26) and the singularities of T_a are weaker than a pole.

Consider the singularities of T_a in a low order (the fourth) of perturbation theory. The strongest of them are given by diagrams 2 and 3 of Fig. 3, and also by diagrams 1—3 of Fig. 4, and are of the form

$$T_a^{(4)} = -4\alpha^2 \Big\{ (s - m^2 - M^2) \int_{(m+M)^2}^{\infty} \frac{1}{\sqrt{k(s')}} \ln \frac{-ts'}{k(s')} \frac{ds'}{s' - s - i\varepsilon} + (s \to u) + \pi^2 (m+M)/\sqrt{-t} + \ln(-t/mM) \Big\} + \ldots,$$

$$k(s) = [s - (m+M)^2]\,[s - (m-M)^2], \quad (32)$$

where $(s \to u)$ means the preceding terms with s replaced by u, and the dots stand for finite terms. We see that for $t \to 0$, $s \to (m+M)^2$, or $u \to (m+M)^2$, $T_a^{(4)}$ goes to infinity, but these singularities are integrable.

An examination of the fourth-order diagrams (2—4 of Fig. 3 and 1—6 of Fig. 4) shows that the Mandelstam representation[3] holds for $T_a^{(4)}$ in the following form:

$$T_a^{(4)} = (s - s_0) \int_{(M+m)^2}^{\infty} \frac{b_1(s')\, ds'}{(s' - s_0)(s' - s)} + (s - s_0)(t - t_0) \int_{(M+m)^2}^{\infty} ds' \int_0^{\infty} dt' \times \frac{b_2(s')}{(s' - s_0)(s' - s)(t' - t)(t' - t_0)}$$
$$+ (s \to u) + (t - t_0) \int_0^{\infty} \frac{b_3(t')\, dt'}{(t' - t_0)(t' - t)} + (s - u) \left(\int_{4m^2}^{\infty} \frac{b_4(t')\, dt'}{t' - t} + \int_{4M^2}^{\infty} \frac{b_5(t')\, dt'}{t' - t} \right). \quad (33)$$

We note that the spectral function $b_2(s')$ does not depend on t'.

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