SOVIET PHYSICS JETP VOLUME 18, NUMBER 3 MARCH, 1964

AMPLITUDE SINGULARITIES AT $l = -1$ IN THE BETHE-SALPETER EQUATIONS

V. G. VAKS and A. I. LARKIN

Submitted to JETP editor April 9, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) 45, 1087-1101 (October, 1963)

Singularities of the scattering amplitude for $l = -1$ are related to the singularity of the irreducible four-pole network in the Bethe-Salpeter equation. Under some reasonable assumptions concerning the properties of the network, it is shown that the contribution to the asymptotic amplitude due to accumulation of the poles is small, providing the poles are close to $l = -1$.

1. INTRODUCTION

It is known that the asymptotic value of the amplitude U(s, t) for the conversion of two particles into two others at high energies $\sqrt{s}$ and at fixed momentum transfer $\sqrt{-t}$ is determined by the singularities of the partial wave amplitudes $f_l(t)$ as functions of the angular momentum l in the channel where $\sqrt{t}$ is the energy[1,2]. Therefore the study of singularities of $f_l(t)$ in the l plane makes it possible to gain information on the behavior of the amplitude at high energies. Thus, if $f_l(t)$ has a singularity in the form of a Regge pole at $l = l(t)$, then U(s, t) contains a term proportional to $s^{l(t)}$. At the present time little is known about the position and character of the singularities of $f_l(t)$ in the l plane, and any information concerning these singularities is of considerable interest.

Gribov and Pomeranchuk[3] have pointed out that in relativistic theory $f_l(t)$ has an essential singularity at the point $l = -1$. They have shown that the jump in $f_l(t)$ across the left-hand cut in t is proportional to $(l + 1)^{-1}$. Inasmuch as a stationary, t-independent, pole in $f_l(t)$ is forbidden by the unitarity condition (since the amplitude of $f_l(t)$ has a limited modulus in the interval $4\mu^2 < t < 16\mu^2$ for real l) the contribution of this jump to the dispersion relation in t should be compensated by some sort of accumulation of the singularities at $l = -1$. This leads to a limitation on the rate at which different processes decrease: the invariant amplitude U(s, t) cannot decrease with decreasing energy more rapidly than s^{-1}.

Gribov and Pomeranchuk used only the most general properties of the amplitude, namely unitarity and analyticity in l and t, so that their results are quite general and at the same time yield no detailed indications on the position of the singularities $l_n(t)$ or on the values of the coefficients of the $s^{l_n(t)}$. In addition, their results are not applicable directly to cases of "anomalous" relations between the masses, that is, to processes in which weakly-bound systems participate. Yet all these problems have grown in interest in connection with a remark made by Azimov[4] that the presence of spin-possessing particles (nucleons, nuclei, and resonances) in intermediate states can cause the singularities in the l-plane to shift to the right, that is, it can lead to terms of order s^0, s^1, etc. in the amplitude.

In the present paper we study this singularity with the aid of the equations of field theory. In this approach, the mass ratios are generally speaking inessential, and the anomalous case can be considered together with the normal one. At the same time, we shall attempt to consider the general analytic properties of the equations, which do not depend on their specific form, particularly on the method of eliminating the divergences at large momenta.

Of importance to the analysis is the fact that the kernel of the integral equation has in the l representation a simple pole at $l = -1$. We shall demonstrate this in any order of perturbation theory for converging theories of the φ^3 type and for a model example, in which the asymptotic behavior of the Green's functions has at large momenta p a power-law form $p^{2\alpha-2}$. In theories of type φ^4, logarithmic divergences make it difficult to glean from perturbation theory any information on the singularities of the kernel at $l = -1$, but it is to be expected that the kernel has a singularity in the form of a simple pole, not connected with the large momenta, also in the real case.

Then, if the kernel of the equation is of the Fredholm type, that is, it decreases sufficiently well at large momenta, the scattering amplitude has an accumulation of poles near $l = -1$. We shall consider various properties of the poles and residues of the amplitude at these points. In the case when the poles are situated to the right of -1, the total contribution to the asymptotic value of the amplitude U(s, t) at large s is proportional to the square of the distance $\alpha_0(t)$ of the extreme right pole from -1:

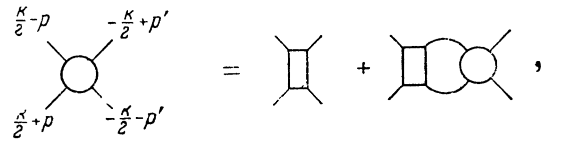
$$|U(s,\ t)| \lesssim \pi\alpha_0^2 s^{2\alpha_0-1}(\mu^2 - t/4)^{1-\alpha_0}, \quad (1)$$

and is small when α_0 is small.

If the kernel of the integral equation decreases slowly at large momenta, so that it is not of the Fredholm type, the accumulation of the poles apparently turns into a branch point, but it can be thought that an estimate of the type (1), in which α_0 has the meaning of the distance of the right end of the cut from -1, remains valid as before.

2. EQUATIONS FOR THE AMPLITUDE

The equations for the amplitude U(p, p′) are of the form[5]



$$U(p,\ p') = -iK(p, p')$$
$$-(2\pi)^{-4}\int K(p,\ q)\, D\left(\frac{k}{2}+q\right) D\left(\frac{k}{2}-q\right) U(q,\ p')\,dq^4. \quad (2)$$

We are considering identical spinless particles of mass μ; D is the exact (renormalized) Green's function:

$$D(p^2) = \int\limits_{\mu^2}^{\infty} \frac{\rho(\sigma)\,d\sigma}{p^2+\sigma}, \qquad D(p^2) \to \frac{1}{\mu^2+p^2}, \quad p^2 \to -\mu^2; \quad (3)$$

k is the 4-momentum of the center of inertia; K(p, q) is a 4-pole which is irreducible with respect to separation of the ends $k/2 \pm p$ from $-k/2 \pm p'$ by two lines. The quantities K(p, p′) and consequently U(p, p′) satisfy the symmetry relations

$$K(p,\ p') = K(p',\ p) = K(-p,\ p') = K(p,\ -p').$$

Equation (2) is analogous to nonrelativistic equation (5) of[6]. As in the latter case, the "initial" momentum p′ and the total 4-momentum k are parameters, while the quantity iK(p, q) plays the role of the potential V(**p**, **q**). The physical scattering amplitude $U(s, t) = U(-(p-p')^2, -k^2)$ is the value of U(p, p′) on the mass shell, that is, $p^2 = p'^2 = -\mu^2 - k^2/4$ when $pk = p'k = 0$, and is connected with the amplitude $f(\theta, t)$ of the phase shift theory of scattering by the relation

$$f(\theta,\ t) = (8\pi t^{1/2})^{-1}\, U(s,\ t).$$

In the present paper we are interested in the region $t = -k^2 < 0$. We can then change over in Equations (2) and (3) to a Euclidean metric, replacing the fourth components of all the vectors by imaginary values in accordance with $p_0 \to ip_0$[7,8], after which the quantities iK and U in (2) are obviously real. The values of U(p, p′) for timelike p and p′ will be obtained by analytic continuation.

We change over, further, into the center of mass system (c.m.s.) with $k = 0$, and expand (2) in Legendre polynomials of the angle between p and p′. Introducing the notation

$$\varphi_l(p,\ p') = \frac{|\mathbf{p}||\mathbf{p}'|}{2}\int\limits_{-1}^{1} dz P_l(z)\, U(p,\ p'),$$

$$K_l(p,\ p') = \frac{|\mathbf{p}||\mathbf{p}'|}{2}\int\limits_{-1}^{1} dz P_l(z)(-i)K(p,\ p');$$

$$\int d^2q = \int\limits_0^{\infty} d|\mathbf{q}| \int\limits_{-\infty}^{\infty} dq_0, \quad D_\pm(q) = D\left(\frac{k}{2} \pm q\right),$$

we obtain

$$\varphi_l(p,\ p') = K_l(p,\ p')$$
$$+\frac{1}{4\pi^3}\int d^2q K_l(p,\ q) D_+(q) D_-(q)\, \varphi_l(q,\ p'). \quad (4)$$

Starting with Eq. (4), we shall designate by p the aggregate $|\mathbf{p}|$, p_0.

When (4) is analytically continued into the complex l plane it is necessary, taking into account the exchange character of the interaction, to continue separately the even and odd harmonics φ_l and K_l[1]:

$$\varphi_l = \tfrac{1}{2}(1+(-)^l)\,\varphi_l^+ + \tfrac{1}{2}(1-(-)^l)\,\varphi_l^-,$$
$$K_l = \tfrac{1}{2}[1+(-)^l]\,K_l^+ + \tfrac{1}{2}[1-(-)^l]\,K_l^-. \quad (5)$$

The $\varphi_l^\pm$ satisfy the equations

$$\varphi_l^\pm(p,\ p') = K_l^\pm(p,\ p')$$
$$+\frac{1}{4\pi^3}\int d^2q K_l^\pm(p,\ q)\, D\left(\frac{k}{2}+q\right) D\left(\frac{k}{2}-q\right) \varphi_l^\pm(q,p'), \quad (6)$$

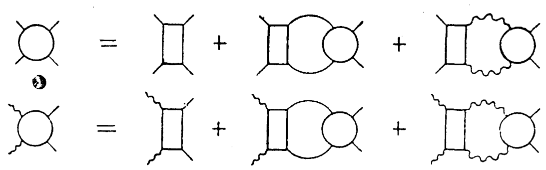
where $K_l^\pm$ stands for the analytic continuation of K_l from the even (odd) positive l; this continuation being regular and bounded in the right half plane $\mathrm{Re}\, l >$ const. On the mass shell $p_0 = p_0' = 0$ we have $\varphi_l^- = 0$, and we consider henceforth only φ_l^+.

If $K_l(p, q)$ decreases sufficiently well at large momenta, so that the integral

$$\int d^2p\, d^2q\, D_+(p)\, D_-(p)\, D_+(q)\, D_-(q)\, (K_l^+(p,\ q))^2 < \infty, \quad (7)$$

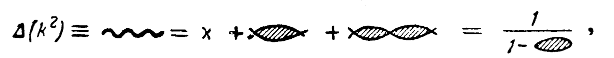
that is, it converges, then (6) is of the Fredholm type[9]. In this case, in the region of analyticity of the kernel K_l^+ in l, the only singularities of the amplitude in the l plane can be the poles $l = l_n$, which have no finite accumulation point [9]. All other singularities of φ_l^+ in the l plane should coincide with the singular points of the kernel K_l^+.

If there are several sorts of particles, then Equations (2) and (6) are best written in the form of a system of several equations, separating all the two-particle intermediate states. Thus, if there are two spinless fields φ and χ, this system is written graphically as



where the four-poles, which serve as kernels of the equations, are irreducible for the separation of the left ends from the right ones by any two lines.

In the case of interactions of the type $\lambda\varphi^4$, the wavy line corresponds to the chain



and the system of equations is written again in the form (8), where the left parts of the four-poles can not be separated from the right ones by either two lines or two chains.

Equation (8) can be rewritten in the form

$$U^{\alpha\beta}(p, p') = -iK^{\alpha\beta}(p, p')$$

$$-(2\pi)^{-4}\sum_{\gamma}\int d^4qK^{\alpha\gamma}(p, q)\,D_+^{\gamma}(q)\,D_-^{\gamma}(q)\,U^{\gamma\beta}(q, p'), \quad (2')$$

where the indices α, β, and γ correspond to solid or wavy lines. From (2′) follow all formulas (4)—(7), if the equations are interpreted as matrix equations. The properties of l-component integral equations (2) and (2′) considered below are perfectly identical, so that we shall discuss here Eq. (6).

Since we are interested in the region $l \sim -1 < -1/2$, then to reconstitute $U^{\pm}(z)$ in terms of $\varphi_l^{\pm}$ it is necessary to use the Regge formula in the Mandelstam form [10]:

$$|\mathbf{p}||\mathbf{p}'|\,U^{\pm}(p, p')$$

$$= \frac{i}{4\pi}\int_{L-i\infty}^{L+i\infty}\frac{dl\,(2l+1)}{\cos\pi l}\,\varphi_l^{\pm}\,[Q_{-l-1}(-z) \pm Q_{-l-1}(z)]$$

$$-\frac{1}{2\pi}\sum_{n=1}^{\infty}(-)^{n-1}2n\varphi_{n-1/2}^{\pm}\,(Q_{n-1/2}(z) \pm Q_{n-1/2}(-z))$$

$$-\sum_{\mathrm{Re}\, l_p > L}\frac{(2l_p+1)\,\mathrm{Res}\,\varphi_{l_p}^{\pm}}{2\cos\pi l_p}\,[Q_{-l_p-1}(-z) \pm Q_{-l_p-1}(z)], \quad (10)$$

where $-3/2 < L < -1$, l_p are the poles located to the right of the line $\mathrm{Re}\, l_p = L$, and $Q_l(z)$ is the Legendre function of the second kind.

3. SINGULARITY OF THE KERNEL K_l^+ AT THE POINT $l = -1$

The kernel K_l^+ of Equation (6) is a sum of a series of diagrams which do not contain 2-particle cuts. We shall show that each of these diagrams, for which the Mandelstam spectral function $\rho(s, u) \neq 0$, has a pole at the point $l = -1$, and if the diagram does not contain logarithmic divergences at large momenta, the residue at this pole is finite, so that the pole is simple. Since this property is possessed by each term of the series, it can be thought that the sum of the series, that is, the complete kernel K_l^+, also has a simple pole at the point $l = -1$:

$$K_l^+(p, p') \xrightarrow[l\to -1]{} \Phi\,(p, p')/(l+1). \quad (11)$$

Let us consider first an interaction of the type $g\varphi^3$. Here K is represented by a sum of diagrams of the type of Fig. 1. For the Green's functions in Fig. 1 we shall use the Lehmann representation [3].

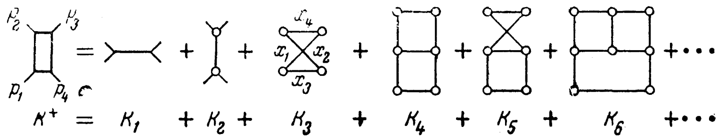


FIG. 1.

The term K_{1l} has vanishing analytic continuation in l and is of no interest to us. In the term K_{2l}, the members of lower order in Γ yield

$$K_{2l}^{(1)} = \frac{g^2}{2}\int_{\mu^2}^{\infty}\rho d\sigma\left[Q_l\left(\frac{p^2+p'^2-2p_0p_0'+\sigma}{2|\mathbf{p}||\mathbf{p}'|}\right)\right.$$

$$\left.-\,Q_l\left(-\frac{p^2+p'^2+2p_0p_0'+\sigma}{2|\mathbf{p}||\mathbf{p}'|}\right)\right]. \quad (12)$$

Hence

$$K_{2l}^{(1)+} = \frac{g^2}{2}\int_{\mu^2}^{\infty}\rho d\sigma\left[Q_l\left(\frac{p^2+p'^2-2p_0p_0'+\sigma}{2|\mathbf{p}||\mathbf{p}'|}\right)\right.$$

$$\left.+\,Q_l\left(\frac{p^2+p'^2+2p_0p_0'+\sigma}{2|\mathbf{p}||\mathbf{p}'|}\right)\right] \to \frac{g^2}{l+1}\int_{\mu^2}^{\infty}\rho d\sigma. \quad (13)$$

The integral in the right half of (13) converges in any order of perturbation theory, and for the time being we shall assume this property to be exact. Then (13) is the exact analog of the nonrelativistic formula (11) of [6] for a potential that behaves like r^{-1} for small r, and everything said previously with respect to Formulas (11) and (12) of [6] is applicable to (13), namely: this kernel is degenerate, so that account of only this one-meson interaction does not lead to an amplitude singularity at $l = -1$.

The remaining components of the kernel have the property, that if (prior to going over to the l-representation), we consider the asymptotic behavior of each diagram $K^{(n)}(s, t, (p \pm k/2)^2, (p' \pm k/2)^2)$ for large s and the other five arguments fixed (the "external masses" and t), then the diagram decreases more rapidly than s^{-1} (see Appendix 1).

In the nonrelativistic problem with non-exchange potential $V(p - p') = V(|p|, |p'|, z)$ it would follow from the fact that V(z) decreases more rapidly than z^{-1} at large z that $V_l(|p|, |p'|)$ is regular at $l = -1$. On the other hand, in the relativistic case the analytic continuation of the even and odd harmonics of K_l is given by different functions of l, namely K_l^+ and K_l^-, in which connection a formula of the type (10), which reconstitutes $K^+(z)$ through K_l^+, contains a factor $1 + \exp(i\pi l)$, which can cancel a simple pole in K_l^+. Therefore the indicated decrease of $K^{(n)}(z)$ still does not signify that there is no pole in $K_l^{(n)+}$, and the latter quantities must be investigated directly. A difference is then observed between the diagrams of type K_4 and K_6, for which only the spectral function $\rho(s, t)$ or $\rho(u, t)$ differs from zero, and diagrams of the type K_3 and K_5, for which $\rho(s, u) \neq 0$.

Any diagram can be represented, accurate to a positive factor, in the form of a Feynman integral

$$K^{(n)} = \int_{\mu^2}^{\infty} \rho(\sigma_1)\, d\sigma_1 \ldots \rho(\sigma_n)\, d\sigma_n \int_0^1 dx_1 \ldots dx_n$$

$$\times\, \delta(x_1 + \ldots + x_n - 1)$$

$$\times \left[\sum_{i=1}^{4} p_i^2 \varphi_i(x) - t\varphi_t(x) - 2pp'\varphi_s(x) + \sum_{k=1}^{n} \sigma_k x_k\right]^{-m},$$

$$p_{1,2}^2 = (k/2 \mp p)^2, \quad p_{3,4}^2 = (k/2 \mp p')^2, \quad -t = k^2, \quad m \geqslant 2. \tag{14}$$

Introducing for brevity the notation

$$\int \rho(\sigma_1)\, d\sigma_1 \ldots \rho(\sigma_n)\, d\sigma_n = \int d\sigma,$$

$$\int dx_1 \ldots dx_n \delta(x_1 + \ldots + x_n - 1) = \int dx,$$

$$\sum_{i=1}^{4} p_i^2 \varphi_i(x) - t\varphi_t(x) - 2p_0 p_0' \varphi_s(x) + \sum_{k=1}^{n} \sigma_k x_k = A, \tag{15}$$

we obtain for the l component

$$K_l^{(n)} = \frac{1}{2} \frac{(-)^m}{(m-1)!} \frac{d^{m-1}}{d\alpha^{m-1}}\bigg|_{\alpha=0} \int \frac{d\sigma\, dx}{\varphi_s(x)} Q_l\left(\frac{\alpha + A}{2|\mathbf{p}||\mathbf{p}'|\varphi_s(x)}\right). \tag{16}$$

A is a positive quantity, since it coincides with the denominator of (14) when $\mathbf{p} \cdot \mathbf{p}' = 0$ and cannot vanish because (14) is real for the Euclidean p, p', k being considered.

If only $\rho(s, t)$ differs from zero in the diagram, then $\varphi_s(x) \geq 0$. Indeed, were we to have, for example, for some x $\varphi_s(x) < 0$, then by taking the quantity $-2pp' = (u - s)/2$ sufficiently large, we would make the denominator of (14) go through zero, that is, (14) would have an imaginary part at $s, t, \mu_i^2 < 0$, and this contradicts the known analytic properties of this diagram. For this diagram the argument of $Q_l(z)$ is positive and, as in the nonrelativistic non-exchange case, $K_l^{(n)+} = K_l^{(n)-} = K_l$, so that the singularity at $l = -1$ is excluded by the already indicated fact that $K^{(n)}(z)$ falls off rapidly with z 1).

Let us consider now terms of the type K_3 and K_5, for which $\rho(s, u)$ differs from zero. The quantity $\varphi_s(x)$ now, generally speaking, passes through zero. In particular, if the diagram is symmetrical under the substitutions $s \to u$ and $pp' \to -pp'$, as in the case of the simplest diagram in K_3, then the region of variation of $\varphi_s(x)$ is symmetrical, and $\varphi_s(x)$ must have a zero inside the integration region. Therefore

$$K_l^{(n)+} = \frac{1}{2} \frac{(-)^{m-1}}{(m-1)!} \frac{d^{m-1}}{d\alpha^{m-1}}\bigg|_{\alpha=0} \int d\sigma \left[\int_{\varphi_s > 0} \frac{dx}{\varphi_s(x)} Q_l\left(\frac{\alpha + A}{2|\mathbf{p}||\mathbf{p}'|\varphi_s}\right)\right.$$

$$\left. + \int_{\varphi_s < 0} \frac{dx}{|\varphi_s(x)|} Q_l\left(\frac{\alpha + A}{2|\mathbf{p}||\mathbf{p}'|\varphi_s}\right)\right]. \tag{17}$$

If we attempt to expand Q_l in (17) near $l = -1$, then the integral will diverge at small φ_s. We therefore use for $Q_l(z)$ the limiting expression for large z:

$$K_l^{(n)+} = \frac{1}{2} \frac{(-)^{m-1}}{(m-1)!} \frac{d^{m-1}}{d\alpha^{m-1}}\bigg|_{\alpha=0} \int \frac{d\sigma}{l+1} \int_{\varphi_{s\,min}}^{\varphi_{s\,max}} \frac{dx}{|\varphi_s(x)|} \left(\frac{|\varphi_s(x)|}{A + \alpha}\right)^{l+1}$$

$$\approx \frac{1}{l+1} \int \frac{d\sigma\, dx\, \delta(\varphi_s(x))}{A^{m-1}}. \tag{18}$$

The quantity $K_l^{(n)-}$ is regular when $l = -1$. The integral in (18) is already convergent, for otherwise $K_l^{(n)+}$ would have a pole to the right of $l = -1$, or would have at $l = -1$ a singularity which is stronger than a simple pole; in either case the asymptotic expression for $K^{(n)}(z)$ would contain z^{l_p} with $l_p \geq -1$.

1)If we expand $Q_l(z)$ near $l = -1$ and differentiate with respect to α in (16), then the pole term Q_l obviously drops out; but in order for this operation to be valid, it is necessary to verify that the resultant integral is convergent at small $\varphi_s(x)$; thus, for example for ladder diagrams of the type of Fig. 2e (see below) this integral diverges, and a singularity exists at $l = -1$.

Returning to the notation of (14), we have

$$K_l^{(n)+} \underset{l\to-1}{=} \frac{1}{l+1}\int d\sigma\, dx\, \delta(\varphi_s(x)) \times \left[\sum_{i=1}^{4} p_i^2\varphi_i(x) - t\varphi_t(x) + \sum_{k=1}^{n} \sigma_k x_k\right]^{-(m-1)}. \quad (19)$$

Thus, the singularity of the kernel K_l^+ is a simple pole and is given by a sum of a series of positive terms of the form (19).

By way of an example, which will also be used in what follows, let us consider the simplest diagram of the type K_3. Using the well-known expression for a square diagram [2] and introducing in place of the Feynman parameters x_i indicated in Fig. 1 the variables ξ and η using the formulas $\xi + 1 = 2(x_2 + x_3)$ and $\eta + 1 = 2(x_1 + x_3)$, we obtain after eliminating the δ functions

$$K_{3l}^{(1)+} \underset{l\to-1}{=} \left(\frac{g}{4\pi}\right)^2 \frac{1}{l+1} \int_{-1}^{1}\int_{-1}^{1} \frac{d\xi\, d\eta}{(p+k\eta/2)^2(1-\xi^2) + (p'+k\xi/2)^2(1-\eta^2) + {}^1/_4 k^2(1-\xi^2)(1-\eta^2) + 4\mu^2}. \quad (20)$$

All the foregoing pertained to a type φ^3 interaction and can be applied without modification to the case of a coupling of two fields of the type $\varphi^2\chi$. To discuss the possible generalizations of the result (11), let us consider the case when the asymptotic value of $\rho(\sigma)$ in (3) has at large σ a power-law form

$$\rho(\sigma) \underset{\sigma\to\infty}{\to} \text{const}\, \sigma^{\alpha-1}, \quad 0 < \alpha < 1, \quad (21)$$

which corresponds to an asymptotic form $D(p^2) \sim p^{2\alpha-2}$ as $p^2 \to \infty$. We do not discuss here such questions as whether such an asymptotic form is possible in theories of the φ^3 or $\varphi^2\chi$ type, or whether it is sensible to use the exact asymptotic form of D when using perturbation theory for the vertex part Γ, inasmuch as example (21) is in character of an illustration.

For the "one-meson" term $K_{2l}^{(1)+}$ the first equation of (13) is valid as before, but the second becomes meaningless because of the divergence of the integral. Separating in $Q_l(z)$ the part that is singular at $l = -1$, we obtain from (13)

$$K_{2l}^{(1)+} = g^2 \frac{\Gamma(l+1)\sqrt{\pi}}{\Gamma(l+{}^3/_2)} (|\mathbf{p}||\mathbf{p}'|)^{l+1} \int_{\mu^2}^{\infty} \rho d\sigma \frac{1}{2}[(A_+ + \sigma)^{-l-1} + (A_- + \sigma)^{-l-1}];$$

$$A_\pm = p^2 + p'^2 \pm 2p_0p_0'. \quad (22)$$

It is convenient to rewrite the integral over σ in the form

$$J(l) = \int_{\mu^2}^{\infty} \rho d\sigma\left[\frac{1}{2}(A_+ + \sigma)^{-l-1} + \frac{1}{2}(A_- + \sigma)^{-l-1} - \sigma^{-l-1}\right] + \int_{\mu^2}^{\infty} \rho d\sigma \sigma^{-l-1}. \quad (23)$$

As $l \to -1$ the first integral in (23) converges and tends to zero. The second term has a pole at $l = \alpha - 1$, and to the left of this point the integral diverges, so that its value near $l = -1$ is obtained by analytic continuation in l. For example, when

$$\rho = \delta(\sigma - \mu^2) + \theta(\sigma - 4\mu^2)(\sigma/4\mu^2)^{\alpha-1} a/4\mu^2$$

we have [2)]

$$\int_{\mu^2}^{\infty} \rho d\sigma \sigma^{-l-1} = (\mu^2)^{-l-1} + \frac{a}{l+1-\alpha} \underset{l\to-1}{\longrightarrow} 1 - \frac{a}{\alpha}. \quad (24)$$

We note now that the answers (22) and (13) can be written in an identical manner with the aid of the formula

$$K_{2l}^{(1)} \underset{l\to-1}{\longrightarrow} \frac{1}{l+1}\left(\int_{\mu^2}^{\infty} \rho d\sigma \sigma^{-l-1}\right)_{l=-1}, \quad (25)$$

where it is necessary to take, in the case when the integral diverges, the analytic continuation of the function in the brackets to the point $l = -1$. As can be seen from the example (24), the residue at $l = -1$ is now not necessarily positive, but in all other respects the formulas retain the same form, including the degeneracy of the kernel of (25), and consequently, including the result that the amplitude φ_l^+ in (8) has no singularity at the point $l = -1$ in the case of the interaction represented by diagrams K_2.

Generalizing Formula (19) by the same method, we obtain

$$K_l^{(n)+} \underset{l\to-1}{=} \frac{1}{l+1}\left(\int d\sigma\, dx\, \delta(\varphi_s(x)) \times \left[\sum_{i=1}^{4} p_i^2\varphi_i - t\varphi_t(x) + \sum_{r=1}^{n} \sigma_r x_r\right]^{-l-k}\right)_{l=-1}. \quad (26)$$

As in (25), in the case of divergence at large σ (which is possible when $\alpha > 1/4$), the integral is

2)The resultant degenerate pole of the kernel (24) at $l = \alpha - 1$ is precisely analogous to the pole in the nonrelativistic potential $-V_l(|\mathbf{p}|, |\mathbf{p}'|)$ in Formula (11) of [6], inasmuch as V(r), which corresponds to the D-function (14), behaves like $r^{-1-2\alpha}$ for small r.

meant in the sense of analytic continuation from the region of large l, where it converges, and is not necessarily positive. As indicated in Appendix 1, the powers of s in the individual terms of the kernel $K^{(n)}(s)$ are generally speaking different from -1, from which it follows, as in the case of (20), that the residue in (25) is regular at $l = -1$.

Thus, although on going from the nonsingular case $\int \rho d\sigma < \infty$ to (21) the properties connected with the behavior at large momenta and small distances change, including for example the appearance of degenerate poles to the right of $l = -1$ in the "potentials" $K_l^{(n)+}$ [see (24)], the singularity at $l = -1$ keeps the form of a simple pole (25), (26). As can be seen from the derivation of (14)—(19), this singularity is connected not with the behavior at large momenta, but with the symmetry properties of the two-particle states, the need for introducing two functions φ_l^+ and φ_l^- for the single-valued and regular continuation in the l-plane, and therefore does not change in form on going over to the more singular interaction.

We have considered cases when the diagrams of perturbation theory were convergent and it was possible to investigate directly their analytic properties near $l = -1$. On going over to the type φ^4 case, the situation becomes more complicated by the fact that the diagrams of perturbation theory, both for the functions $D(p^2)$ and $\Delta(p^2)$ (9) and for each term of the series K_n, diverge logarithmically. If we calculate these diagrams in the usual manner, cutting off the integrals at large momenta and then subtracting the divergent constants, then there appear in K_n for large s expressions of the type $s^{-1}\ln^m s$, corresponding to $(l+1)^{-m-1}$ in K_{ln}. This accumulation of logarithms occurs in the same manner both in diagrams with $\rho(s, u) \neq 0$ and with $\rho(s, u) = 0$, and the resultant singularity of the kernel K_l^+ is then not clear. We note also that Equation (8) ceases to be of the Fredholm type and the integrals (7) calculated by perturbation theory diverge logarithmically.

It can be assumed that the method described for obtaining information on the singularity of K_l^+ from perturbation theory is not adequate in this case. Thus, if the increasing powers of the logarithm in $D(p^2)$ or $\Delta(p^2)$ were to add up to a finite power: $D, \Delta \sim (p^2)^\alpha$ as $p^2 \to \infty$, we would encounter a situation of the type (21). It is even more natural to expect that the adding up of the logarithms into a power of the "Regge" type occurs in those internal vertex parts of the K_n diagrams, which are four-point diagrams with virtual momenta. In any such case when the K_n diagrams exhibit a non-logarithmic behavior at large virtual momenta, the residue at $(l+1)^{-1}$ in formulas of the type (26) is finite, and we again arrive at the result (11).

We note, finally that even if (11) does not correspond to a real case and the singularity of K_l^+ has a different form at $l = -1$, Eq. (6) can apparently be investigated by methods analogous to those employed below.

4. ACCUMULATION OF POLES AT $l = -1$ AND THEIR CONTRIBUTION TO THE ASYMPTOTIC VALUE OF U(s, t) AT LARGE s

We shall consider essentially case (7). Then Equation (6) has near $l = -1$, in accordance with (11), the form of a Fredholm equation of the second kind, in which the role of the parameter λ is played by $(l+1)^{-1}$:

$$\varphi_l^+(p, p') = \frac{\Phi(p, p')}{l+1} + \frac{1}{l+1}\frac{1}{4\pi^3}\int d^2q\,\Phi(p, q) D_+(q) D_-(q) \varphi^+(q, p'). \quad (27)$$

The symmetrical kernel $\Phi(p, q)$ can be represented as a series in the eigenfunctions $\varphi_n(p)$ and eigenvalues $l_n + 1 = \alpha_n(t)$ of the homogeneous equation (27) [9]:

$$\Phi(p, q) = \sum_n \alpha_n(t) \varphi_n(p) \varphi_n(q),$$

$$\frac{1}{4\pi^3}\int \varphi_n(q)\varphi_m(q) D_+(q) D_-(q)\, d^2q = \delta_{mn}, \quad |\alpha_{n+1}| < |\alpha_n|. \quad (28)$$

If the kernel $\Phi(p, q)$ is nondegenerate, as in our cases (20) and (26), then the series (28) contains an infinite number of terms, and the eigenvalues α_n decrease without limit in absolute value with increasing n (so that the series $\Sigma\alpha_n^2$ converges).

Substituting (28) in (27) we obtain

$$\varphi_l^+(p, p') = \sum_n \frac{\alpha_n(t)\varphi_n(p)\varphi_n(p')}{l+1-\alpha_n(t)}. \quad (29)$$

From (29) we see that near $l = -1$ the amplitude has an infinite number of poles [3,1], which are real for the values $t < 0$ being considered. Inasmuch as the functions $\varphi_n(p)$ are orthonormal in the sense of (28) and bounded, the residues at the poles are proportional to their distance from $l = -1$, and tend to zero as $l_n \to -1$.

Let us substitute (29) in (10) and go over to large $s = -(p - p')^2$. Assuming that $\alpha_n(t) \ll 1$, which is the condition for the applicability of our formulas, we obtain for the contribution made to the total amplitude U by the poles located near $l = -1$:

$$U(p, p', k) = \pi i \sum_n \alpha_n^2(t) \varphi_n(p) \varphi_n(p') \left(\frac{s}{|\mathbf{p}||\mathbf{p}'|}\right)^{\alpha_n(t)} s^{-1}. \quad (30)$$

To change over to the physical amplitude U(s, t) it is necessary to continue (30) analytically from Euclidean values $p^2, p'^2 > 0$ to values on the mass shell $p^2 = p'^2 = -\mu^2 - k^2/4$, $pk = p'k = 0$. It is no longer possible to state here that $\varphi_n(p)$ do not increase with increasing n. To the contrary, we can see, for example, that the series (29) will diverge starting with $t \leq -8\mu^2$, because in this case an imaginary part appears in the amplitude $\varphi_l^+(p, p')$ [more accurately, in its lowest term $K_{3l}^{(1)}$ [(27)], and the functions $\varphi_n(p)$, as can be seen from the homogeneous equation (27), are real for all stable $-(p \pm k/2)^2 < 4\mu^2$. Therefore the series (29) is a sum of real terms, so that the imaginary part can appear in the function represented by it only at the point where the series is divergent.

But the series (38) converges on the mass shell. Indeed, if we replace all the s^{α_n} in (30) by the maximum value s^{α_0}, then, accurate to a factor, (30) will go over into the iterated kernel of (27):

$$\sum_n \alpha_n^2 \varphi_n(p)\varphi_n(p') = \frac{1}{4\pi^3}\int d^2q\,\Phi(p, q) D_+(q) D_-(q)\Phi(q, p') = \Phi^{(2)}(p, p'). \tag{31}$$

Therefore expression (39) divided by $(l + 1)^2$ is the term that is principal in $(l + 1)^{-1}$ in the aggregate of diagrams comprising the first iteration of the kernel $K_l^+(p, p')$. The divergence of the series (31) for certain $p^2, p'^2 < 0$ would indicate the appearance of a singularity and consequently of an imaginary part in the function represented by it. This would lead to the presence of a discontinuity in this expression at $t < 0$, proportional to $(l + 1)^{-2}$, in contradiction to the result of Gribov and Pomeranchuk [3], we have shown that when $t < 0$ the discontinuity in the expression, which satisfies the dispersion relation in s, is proportional to $(l + 1)^{-1}$.

It is seen from the foregoing, incidentally, that the discontinuity in the total amplitude near $l = -1$ coincides with the discontinuity of the kernel $K_l(p, p')$, which is equal to $(l + 1)^{-1}\Phi(p, p')$. The foregoing is well illustrated by the example (20): by putting $(k/2 \pm p)^2 = -m^2$, we obtain in the denominator the expression $-m^2(1 - \xi^2) + (q - k\xi/2)(1 - \eta^2) + 4\mu^2$, and inasmuch as integration is over Euclidean q, in the case of stable $m^2 < 4\mu^2$ the denominator does not go through zero and there is no singularity.

Thus, the contribution of the considered aggregate of poles to the asymptotic value of U(s, t) for large s has the form

$$U(s, t) = \frac{\pi i}{s}\sum_n \alpha_n^2(t)\varphi_n^2(t)\left(\frac{s}{-\mu^2 + t/4}\right)^{\alpha_n(t)}, \quad \varphi_n(t) = \varphi_n(p)\big|_{p_0=0,\ p^2=-\mu^2+t/4}. \tag{32}$$

The presence of the term (32) in the amplitude leads to a limitation on the rate of decrease of the different processes: even when at large $-t$ all the singularities of the amplitude in the l-plane go over to the left, so that, in particular, for the extreme right pole $\alpha_0(t)$ the condition $\alpha_0 \ll 1$ begins to be satisfied in (29), still the amplitude always contains the term (32), that is, terms of order s^{-1}. However, on the basis of Formulas (31) and (32) it can be thought that the total contribution of (32) also becomes small under these conditions, and satisfies the estimate

$$|U(s, t)| \leqslant \pi\alpha_0^2 (s)^{\alpha_0 - 1}(\mu^2 + k^2/4)^{1-\alpha_0}, \quad 0 < \alpha_0 \leqslant 1. \tag{33}$$

Let us assume first that the eigenvalue with the largest absolute value in (29), α_1, is positive, that is, $\alpha_0 = \alpha_1$; this is satisfied, for example, for all the well-converging theories (see below). Then it follows from the variational properties of the positive kernel $\Phi^{(2)}(p, p')$ that $\alpha_0^2(t)$ is the maximum value of the integral of $\Phi^{(2)}(p, q)$ with arbitrary (normalized in accord with (28)) function $\psi(p)$:

$$\frac{1}{(4\pi^3)^2}\int \psi(p)\Phi^{(2)}(p, q)\psi(q) D_+(p) D_-(p) D_+(q) \times D_-(q)\, d^2p\, d^2q \leqslant \alpha_0^2(t). \tag{34}$$

If the function $\Phi^{(2)}(p, q)$ has a smooth dependence on all the arguments, then it follows from the variational inequality (42) that $\Phi^{(2)}(p, q)$ is generally speaking small, of order $\alpha_0^2(t)$ in the entire region of Euclidean $p^2, q^2 > 0$. If furthermore $\Phi^{(2)}$ does not have an essential singularity at infinity with respect to p^2 and q^2, so that its limits as p^2, $q^2 \to \pm\infty$ coincide, then it can be thought that this smallness remains in force also for non-Euclidian $p^2, q^2 < 0$. Such smoothness properties are possessed, for example, by all perturbation-theory diagrams, particularly expression (20). Strengthening the inequality by making the substitution $s^{\alpha_n} \to s^{\alpha_0}$, we arrive at (33). Thus, if $\Phi(p, q)$ is given by (20), then the first eigenvalue α_1 is small only if the parameter $(g^2/4\pi)^2(\mu^2 + k^2/4)^{-2} \sim \alpha_1(t) \ll 1$ is small. Going over from (20) to $\Phi^{(2)}(p, q)$ in accordance with (31) and putting $p^2 = p'^2 = -\mu^2 + t/4$ and $p_0 = p_0' = 0$, we see that the estimate (33) is correct.

In the case $\alpha_0 = \alpha_1$ under consideration, the sum of the series (32) is of the order of the first term, and $\varphi_0^2(t)(\mu^2 + k^2/4)^{-1} \sim 1$. Both properties are natural, since the series converges and the function represented by it is regarded as smooth. It is reasonable to assume that (33) is valid also in the case when several poles lie far to the left of -1, without making any contribution to the asymptotic value—it is to be expected that the series is here, too, of the order of the first term. The case when all poles lie to the left of -1 is little probable on the basis of the examples presented below; it is apparently similar in its properties to the previously considered [6] example of a non-Fredholm equation with a continuous eigenvalue spectrum $-\alpha_1 < l + 1 < 0$ situated to the left of -1. In this case the total contribution of (32) is of the order of the smaller of the numbers $s^{-1}\ln^{-2} s$ or $\alpha_1^2 s^{-1}$.

If the integrals (13) and (19) as well as the series of terms (19) converge, which, as explained above, occurs for convergent theories with not too singular an interaction, then the kernel in (27) is the sum of positive terms and $\Phi(p, q) > 0$. In this case the eigenvalue $\alpha_1(t)$ with the largest absolute value is positive [11], that is, the pole farthest from $l = -1$ in (29) is on the right for all values of t. For the simplest kernel (20) it is shown in Appendix 2 that at $k = 0$ and $k^2 \gg 16\mu^2$ all the poles are situated to the right of $l = -1$. The clarification of the problem necessitates in the general case an investigation of the quasi-classical case $\alpha_n \to 0$ of integral equations of type (27), but some idea of the position of the poles can be obtained from physical considerations. As was already explained (see [6]) the accumulation of the poles at large $(l + 1)^{-1}$ has the simple meaning of the existence of a large number of levels in the problem with some nonlocal (and retarded) potential $\Phi(p, p')$ when its "charge" $(l + 1)^{-1} \to \infty$. The constant-sign expressions of the type (19) and (20) apparently correspond to the case when this "potential" corresponds to attraction in all of space, so that the accumulation of the poles occurs only in the case of positive "charges" $(l + 1)^{-1}$.

So far we have considered only the term (11), which is of highest order in $(l + 1)^{-1}$ in the kernel K_l^+. If for some reason $\Phi(p, q)$ in (11) is small compared with the other terms K_l^+, which are nonsingular at $l = -1$, [3)] then it is necessary to retain also the next higher term in the expansion near -1:

$$K_l^+(p, q) = (l + 1)^{-1}\Phi(p, q) + \Phi_1(p, q)$$
$$= (l + 1)^{-1}(\Phi(p, q) + (l + 1)\Phi_1(p, q)). \quad (35)$$

From (35) we obtain a formula of the type (29), in which the quantities $\alpha_n(t)$ and $\varphi_n(t)$ will now depend on $l + 1$. The poles are determined by the zeroes of the denominators, that is, by the equations $l_n + 1 = \alpha_n(t, l_n + 1)$. In Formula (32) α_n^2 is replaced by $\alpha_n^2(l_n + 1)(1 - \alpha_n'(l_n + 1))^{-1}$, but in all other respects the formulas retain the same form and, expanding the function $\alpha_n'(l_n + 1)$ for small $l_n + 1$, we again arrive at the estimates (31) and (33).

We note that for sufficiently large eigenvalues $n \gg 1$ the correction to α_n is apparently small for arbitrary Φ_1. Indeed, the equation for the eigenvalues α_n is in the form of an equation that is linear in $l + 1$ with a "perturbation" $(l + 1)\Phi_1(p, q)$:

$$(l+1)\psi_n(p) = \frac{1}{4\pi^3}\int [\Phi(p, q)$$
$$+ (l+1)\Phi_1(p, q)]\psi_n(q)D^+(q)D^-(q)\,d^2q. \quad (36)$$

If we assume for simplicity that the system of eigenfunctions $\varphi_n(p)$ of the operator $\Phi(p, q)$ is complete, then we can apply to (36) the usual methods of perturbation theory, and we obtain in first order

$$l_n + 1 = l_n^{(0)} + 1 + (l_n + 1)\frac{1}{(4\pi^3)^2}\varphi_n(p)\Phi_1(p, q)\varphi_n(q)$$
$$\times D^+(p)D^-(p)D^+(q)D^-(q)\,dp^2 dq^2. \quad (37)$$

At large $n \gg 1$ the functions $\varphi_n(p)$ in the integral (37) oscillate rapidly, so that the correction to $l_n^{(0)} + 1$ is, generally speaking, small. We note also that the form of equation (27) which is linear in $l + 1$ makes it possible (at least when $n \gg 1$) to obtain for the "levels" $l_n + 1$ also other "quantum-mechanical" results, for example, the theorem on the non-crossing of the terms—poles—as t is varied.

We have considered above the Fredholm case (7). The non-Fredholm behavior of the kernels corresponds to singular potentials $|V(r)| \geq r^{-2}$ as $r \to 0$ in the nonrelativistic problem, and to logarithmic and stronger divergences in the relativistic problem. If, in spite of the uncertainty discussed in section 3, we assume for K_l^+ a behavior (12) as before, then the accumulation of the poles will apparently be replaced by branching, which stretches to the right in the case of "effective attraction" and to the left in the case of "repulsion." We have already considered [6] some of the simplest equations of this type. As mentioned above, the general form of Formulas (29)—(33) remains the same if the sums over n are replaced by integrals over the region of variation of α_n. It can be thought that an estimate of the type (33) will be valid also in the general case.

3)This case has been considered in greater detail (see [6]) in connection with the problem of the passage of poles through a region of l close to -1.

The authors are grateful to V. N. Gribov, I. Ya. Pomeranchuk, and L. B. Okun' for a discussion of the results.

APPENDIX I

The asymptotic behavior of the diagrams at large $s = -(p - p')^2$ and fixed $(p \pm k/2)^2$, $(p' \pm k/2)^2$, k^2 can be investigated by a method which is close to that used by Sudakov in electrodynamics[12]. In each diagram we introduce, in integrating over the virtual 4-momenta f, the variables x, y, z, and $f_\perp$ in accordance with the formulas

$$f = px + p'y + kz/2 + f_\perp, \quad pf_\perp = p'f_\perp = kf_\perp = 0. \quad \text{(A.1)}$$

Typical denominators of the diagrams have the form

$$f^2 + \sigma_i = p^2x^2 + p'^2y^2 + k^2z^2/4 + z(pkx + p'ky)$$
$$+ f_\perp^2 + \sigma_i + 2pp'xy = B^i_{xyz} + 2pp'xy \approx B^i_{xyz} + sxy;$$
$$(f \pm k/2)^2 + \sigma_i = B^i_{xyz\pm1} + sxy,$$
$$(f - p)^2 + \sigma_i = B^i_{x-1yz} + s(x-1)y,$$
$$(f - p')^2 + \sigma_i = B^i_{xy-1z} + sx(y-1). \quad \text{(A.2)}$$

The Jacobian of the transformation from f to $xyzf_\perp$ is equal to

$$D(f)/D(xyzf_\perp) = \tfrac{1}{2}[p^2(p'k)^2 + p'^2(pk)^2 + k^2(pp')^2$$
$$- 2(kp)(p'k)(pp') - p^2p'^2k^2]^{1/2} \approx s\sqrt{k^2}/4. \quad \text{(A.3)}$$

The variables (A.1) enable us to estimate simply the essential region in the integrals and its contribution in the case of large s; from (A.2) we see that in the case of good convergence, generally speaking small x or y are important at large momenta.

The diagram of Fig. 2a was considered by Sudakov; the greatest contribution $\sim s^{-1}\ln^2 s$ is made by the region of small x and y. Inasmuch as the terms K_2 in Fig. 1 contain Γ multiplied by $D \sim s^{-1}$, the diagrams of K_2 with insertions Γ are of the order of $s^{-2}\ln^2 s$. The diagrams of lower order in Γ in K_3 and K_4 (see Fig. 2b, c) are given by the expressions

$$K_3^{(1)} = \int df_\perp\, dz\, d\sigma$$
$$\times \int \frac{dx\, dy\, s\sqrt{-t}/4}{(B^1_{xyz+1} + sxy)(B^2_{x-1yz} + sy(x+1))(B^3_{xy-1z} + sx(y-1))(B^4_{x-1,\,y-1,\,z-1} + s(x-1)(y-1))}, \quad \text{(A.4)}$$

$$K_4^{(1)} = \int \frac{df_\perp\, df'_\perp\, dz\, dz'\, d\sigma\, dx\, dx'\, dy\, dy'\,(s\sqrt{-t}/4)^2}{(B^1_{xyz+1} + sxy)(B^2_{x-1yz} + sy(x-1))(B^3_{xy-1z} + sx(y-1))(B^4_{x-x'y-y'z-z'} + s(x-x')(y-y'))}$$
$$\times \frac{1}{(B^5_{x'y'z'-1} + sx'y')(B^6_{x'-1y'z'} + sy'(x'-1))(B^7_{x'y'-1z'} + sx'(y'-1))}. \quad \text{(A.5)}$$

We see that in (A.4) there is no region of variables that would make a contribution $\sim s^{-1}$, so that $K_3 \sim s^{-2}$, apart from the logarithms. In (A.5) the essential contribution is made by small x, x' or y, y'; in Fig. 2c, the first region corresponds to the absence of an extinction factor s^{-1} in the right-hand and middle lines, the second corresponds to absence of such a factor in the left-hand and middle lines, but in both cases there remain two "uncompensated" lines each, with contribution s^{-2}. The insertions Γ do not give increasing powers of s and do not change the estimates. In similar fashion the maximum contribution in the more complicated diagrams, for example Fig. 2d, is the one in the region of small x, x', y, y' of the "joining" lines, but the number of these lines is not less than three, which leads to the estimate $K_n \lesssim s^{-2}$. We note that the statement concerning a fall-off faster than s^{-1} is valid only for diagrams without 2-particle cuts. Ladder diagrams of the type of Fig. 2e are pro-

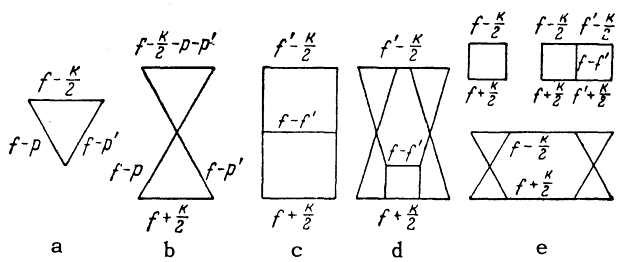


FIG. 2.

portional to $s^{-1}\ln s$, $s^{-1}\ln^2 s$ and $1/s$, respectively, as can be seen from a more thorough examination of the contribution of the region of small x, y, or more simply from the l-representation of these diagrams. Further, the foregoing is valid only for each $K^{(n)}$ diagram; if the powers of $\ln^n s$ at s^{-2}, which we did not estimate, were to add up, for example, into a Regge-type power s^α, then the behavior of K(s) at large momenta would have the form $s^{\alpha-2}$, but this power is, generally speaking, also different from -1.

Let us discuss the case (21). For $\alpha < 1/4$ each term of K_n with $n > 2$ is as before $< s^{-1}$, although for the simplest diagrams with a small, α-dependent, number of virtual lines the essential region is now $f_i^2 \sim s$, and not $f_i^2 \sim \mu_i^2, -t$. When $1/4 < \alpha < 1/3$ the simple diagrams decrease more slowly than s^{-1} (for example, $K_3^{(1)} \sim s^{4\alpha-2}$), but the powers obtained are linear in α and are, generally speaking, different from -1. For $\alpha > 1/3$, each new complication of the diagram leads to increasing powers of s (at that for $\alpha \geq 1/3$ Eq. (8) ceases to be of the Fredholm type), and when $\alpha > 1/2$ the integrals diverge, so that some subtraction procedure is necessary, but the powers of s in the individual terms are as before, generally speaking, different from -1.

APPENDIX II

All the eigenvalues of the kernel K(p, q) are positive if for each real $\psi(p)$ we have [9]

$$\int d^2p \int d^2q \psi(p) K(p, q) \psi(q) \geqslant 0. \quad (A.6)$$

The condition (A.6) is always satisfied for kernels of the type

$$K(p, q) = \int \frac{d\xi\, d\eta\, g(\xi, p)\, g(\eta, q)}{[\varphi(p, \xi) + \varphi(q, \eta) + C]^n}$$
$$= \int_0^\infty \frac{x^{n-1}dx}{\Gamma(n)} \int d\xi\, d\eta\, g(\xi, p)\, g(\eta, q) \exp\{-x[\varphi(p, \xi) + \varphi(q, \eta) + C]\}, \quad (A.7)$$

if ξ, η, g, φ are real and $\varphi(p, \xi) + \varphi(q, \eta) + C \geq 0$. In cases $k = 0$ and $k^2 \gg 16\mu^2$, the kernel (20) assumes the form (A.7), so that all its eigenvalues are positive.

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Translated by A. M. Bincer and J. G. Adashko
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