ANALYTIC PROPERTIES OF THE FORWARD SCATTERING AMPLITUDE IN THE NONLOCAL THEORY

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The analyticity of the forward scattering amplitude is proved in the nonlocal theory under the assumption that the field commutator decreases sufficiently rapidly in space-like directions. Conditions for the appearance of complex singularities are indicated.

1. It is a generally accepted tenet (cf., e.g., [1,2]) that the analytic properties of the scattering amplitude follow from the condition of microcausality, which is essentially the requirement that the field operators be local. Only in this case has it been possible to represent the transition amplitude as the Fourier transform of a function which vanishes outside the light cone. However, it has been emphasized many times that the axiom of the locality of the fields is not so indisputable as, for example, the requirement of relativistic invariance or the unitarity of the S matrix.

Until now almost all results of quantum field theory (including the analytic properties of the amplitudes) have been obtained within the framework of a local theory. In this connection it is interesting to investigate to what extent these results are associated with the locality condition and what remains of them in a nonlocal theory. This applies, in particular, to various analytic properties, since the consequences of these properties can be checked by experiment.

In the present note we study the changes in the analytic properties of the forward scattering amplitude as one goes over to a nonlocal theory. It turns out that these properties are completely determined by the asymptotic behavior of the field commutator in space-like directions. For a sufficiently rapid decrease [for example, of the type (10)] the regions of analyticity of the local and nonlocal theories coincide, but there may be an essential singularity at infinity in the nonlocal theory. If the asymptotic form of the commutator is altered slightly [for example, choosing it to be of the type (11)], poles appear in the complex plane. The asymptotic form (12) leads to the appearance of cuts in the complex plane. For simplicity we consider a model of pseudoscalar neutral particles with mass m.

2. It has been shown in an earlier paper [3] (see also [4]) that the T product of the usual local theory (LSZ scattering formalism [4]) can also be used in a nonlocal theory (the same scattering formalism without the assumption of local fields). Despite the non-invariance of such a T product, the scattering amplitude will be an invariant quantity on the mass shell. This permits us to write the two-particle scattering amplitude, T, of the nonlocal theory, defined by

$$\langle p, k | S | p', k' \rangle = \langle p, k | p', k' \rangle + 2\pi i \delta (p + k - p' - k') T (p, k; p', k') \quad (1)$$

[here $\langle k | k' \rangle = 2k_0 \delta(\mathbf{k} - \mathbf{k}')$] in the usual form

$$T (p, k; p', k') = i \int dx e^{ikx} K \langle p | \theta (x_0) [A (x), j (0)] | p' \rangle . \quad (2)$$

Here $A(x)$ is the field operator of the particles under consideration, and $j(x)$ is the current operator, so that $KA(x) = j(x)$, where $K \equiv \Box - m^2$.

In the local theory the function under the integral sign is nonzero only inside the light cone ($x^2 = x_0^2 - \mathbf{x}^2 > 0$, $x_0 > 0$). For a proof of the analytic properties of the amplitude in the nonlocal theory we must require an exponential decrease of the matrix element of the commutator $\langle p | [A(x), j(0)] | p' \rangle$ as x goes to infinity in a space-like direction. Then we can make the replacement, in formula (2),

$$\theta (x_0) \to \theta (x_0) \theta ((x_0 + a)^2 - \mathbf{x}^2) \equiv \theta_a (x_0), \qquad a > 0,$$

without changing the value of the integral (this is easily shown by considering the difference of the new and old expressions and making an integration by parts in order to transfer the operator K to the function e^{ikx} in analogy to the procedure in [3]). The function θ_a cuts out the region $x_0 > 0$ inside the light cone with the vertex in the point $\{x_0 = -a, \mathbf{x} = 0\}$.

In the following it is easiest to use the method of Symanzik.[5] For forward scattering in a coordinate system where $\mathbf{p} = 0$, $p_0 = m$, formula (2) takes the form

$$T(\omega) = i\int dx e^{ikx} K \langle 1|\theta_a(x_0)[A(x),\ j(0)]|1\rangle, \quad (3)$$

where $\omega = k_0$ and $\langle 1|$ is a one-particle state with vanishing momentum. After integration over the angular variables Eq. (3) can be written as

$$T(\omega) = \int_0^\infty dr\, F_a(\omega, r), \quad (4)$$

$$F_a(\omega, r) \equiv 4\pi i r \frac{\sin kr}{k} \int_{-\infty}^{\infty} dt e^{i\omega t} K \langle 1|\theta_a(t)[A(t, \mathbf{x}),\ j(0)]|1\rangle, \quad (5)$$

where $k = (\omega^2 - m^2)^{1/2}$. The integration over t in (5) actually goes over the region $t \geq \max\{r - a, 0\}$, so that the function $F_a(\omega, r)$ will be an analytic function of ω in the upper half-plane and will behave like $\omega^n e^{-i\omega a'}$ for $\mathrm{Im}\,\omega \to \infty$.

Let us apply Cauchy's formula to the product of $F_a(\omega, r)$ and a function which is analytic in the upper half-plane and decreases sufficiently rapidly for $\mathrm{Im}\,\omega \to \infty$, for example, $F_b(\omega, r) = F_a(\omega, r)^1/\cos b\omega$, $b \geq a' \geq a$. (In order not to overburden the formulas, we shall assume that a further division by a polynomial is not necessary.) Then the integral along the half-circle in the upper plane will go to zero as the radius of the half-circle tends to infinity, and we obtain for $F_b(\omega, r)$ the formula

$$F_b(\omega, r) = \frac{1}{2\pi i} \int_{i\varepsilon-\infty}^{i\varepsilon+\infty} \frac{d\omega' F_b(\omega', r)}{\omega' - \omega}, \qquad \varepsilon > 0.$$

Letting $\epsilon \to 0$ and using the fact that $\mathrm{Re}\,F_b(\omega, r)$ and $\mathrm{Im}\,F_b(\omega, r)$ are even and odd functions, respectively, of ω, we easily obtain the following relation for real ω

$$\mathrm{Re}\,F_a(\omega, r) = 2\cos b\omega \sum_{n=0}^{\infty} \frac{\mathrm{Re}\,F_a(\omega_n, r)(-1)^{n+1}\omega_n}{(\omega^2 - \omega_n^2)\, b} + \frac{2\cos b\omega}{\pi} \mathrm{P} \int_0^\infty \frac{\omega'\, d\omega'\, \mathrm{Im}\,F_b(\omega', r)}{\omega'^2 - \omega^2}, \quad (6)$$

$\omega_n = (2n+1)\pi/2b$ are the zeros of $\cos b\omega$ and the symbol P denotes the principal value of the integral.

Let us fix $\omega > m$, $\pi/2b > m$ and integrate both sides of (6) over r. In the region $\omega' > m$ the integrations over ω' and r can be interchanged and we find as a result

$$\mathrm{Re}\ T(\omega) = 2\cos b\omega \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\omega_n\, \mathrm{Re}\,T(\omega_n)}{(\omega^2 - \omega_n^2)\, b} + \frac{2\cos b\omega}{\pi} \mathrm{P} \int_m^\infty \frac{\omega' d\omega'\, \mathrm{Im}\,T(\omega')}{(\omega'^2 - \omega^2)\cos b\omega'}$$
$$+ \frac{2\cos b\omega}{\pi} \int_0^\infty dr \int_0^m \frac{\omega'\, d\omega'\, 4\pi r}{(\omega'^2 - \omega^2)\cos b\omega'} \frac{\sin k'r}{k'} \times \int_{-\infty}^{\infty} dt e^{i\omega' t} K \langle 1|\bar{\theta}[A(t, \mathbf{x}),\ j(0)]|1\rangle, \quad (7)$$

where $\bar{\theta} \equiv \theta_a(t) + \theta_a(-t)$.

Let us consider the last integral. Here it is not possible to interchange the order of integrations, since $k' = (\omega'^2 - m^2)^{1/2}$ becomes pure imaginary for $\omega' < m$ and $\sin k'r$ becomes an exponentially increasing function of r. In contrast to the local theory, this integral contains the commutator $[A(x), j(0)]$ multiplied by the weight function $\bar{\theta}$ depending on t. If the function $\bar{\theta}$ is replaced by unity, the integral agrees with the corresponding term in the local theory and will vanish in the case of scattering of pseudoscalar particles. Indeed, without the function $\bar{\theta}$ the integral over t in (7) can be written, after expanding the commutator in a complete set of functions,

$$\int_{-\infty}^{\infty} dt\, e^{i\omega' t} \sum_{k,n} [e^{i(m-k_{n0})t} - e^{-i(m-k_{n0})t}]\, e^{i\mathbf{k}_n \mathbf{x}} |\langle 1|j(0)|k, n\rangle|^2. \quad (8)$$

The integration over t gives rise to δ functions in the square brackets, $[\delta(m - k_{n0} + \omega') - \delta(k_{n0} - m + \omega')]$, and since the summation over n begins only with $n = 2$ because of the condition $\langle 1|j(0)|1'\rangle$, the integral (8) vanishes.

Thus the replacement $\bar{\theta} \to \bar{\theta} - 1$ in formula (7) is justified. This means that the last integral in (7) is completely determined by the behavior of the commutator in the space-like region, since the function $\bar{\theta} - 1$ cuts out the region outside the light cones with the vertices $\{\mathbf{x} = 0,\ t = \pm a\}$. After interchanging the integrations over ω' and t this term takes the following form (a factor $2\cos b\omega/\pi$ is omitted):

$$R(\omega) = 4\pi \int_0^\infty dr\, r^2 \int_{-\infty}^{\infty} dt\, f_\omega(r, t)\, K \langle 1|(\bar{\theta} - 1)[A(x),\ j(0)]|1\rangle,$$

where

$$f_\omega(r, t) = \frac{1}{2} \int_{-m}^{m} d\omega'\, e^{i\omega' t} \frac{\sin k'r}{r} \frac{\omega'}{(\omega'^2 - \omega^2)\, k' \cos b\omega'}$$

represents the odd solution of $Kf_\omega = 0$ with respect to t. We consider only the function f which is odd in t, since the function $(\bar{\theta} - 1) \times \langle 1[A(x), j(0)]|1\rangle$ is odd in t.

By integration by parts we can transfer the operator K to the function f_ω which, by virtue of the equality $Kf_\omega = 0$, leads to the following expression for $R(\omega)$:

$$R(\omega) = \lim \int_l (Pdr - Qdt), \tag{9}$$

i.e., $R(\omega)$ is given by an integral over the boundary of the region alone. We have introduced the following notation:

$$P = \frac{\partial F}{\partial t}\Phi - F\frac{\partial \Phi}{\partial t}, \qquad Q = \frac{\partial F}{\partial r}\Phi - F\frac{\partial \Phi}{\partial r},$$

$$F = 4\pi r f_\omega(r, t), \qquad \Phi = r(\bar{\theta} - 1)\langle 1|[A(x), j(0)]|1\rangle.$$

The integration contour l should, in the limit, enclose the whole region $r > 0$ of the (r, t) plane. Actually, the integral is taken only over that part of the contour which lies inside the region cut out by the function $\bar{\theta} - 1$. In this way all has been reduced to the investigation of the limit (9). In order to find this limit we must know the asymptotic behavior of the integrand for $r \to \infty$.

The asymptotic form of the function f_ω is found easily. One readily verifies by the method of the stationary phase that

$$f_\omega(r, t) \sim r^{-3/2}\exp(m\sqrt{r^2 - t^2})\left[\frac{\pi m\gamma^3}{2\alpha^3}\right]^{1/2}\varphi(im\gamma),$$

$$\varphi(\omega') = i\omega'/(\omega'^2 - \omega^2)\, k' \cos b\omega',$$

$$\alpha = r/t, \qquad \gamma(\alpha) = \alpha/\sqrt{1 - \alpha^2},$$

i.e., f_ω increases exponentially for $r \to \infty$, $\alpha = \text{const} < 1$.

The first derivatives of f_ω with respect to t and r have the same type of growth. We must therefore require for the existence of the limit (9) that the function Φ (together with its first derivatives with respect to r and t) vanish exponentially for $r \to \infty$. If we postulate the asymptotic form

$$\Phi \sim cr^{-p}\exp(-m\sqrt{r^2 - t^2}), \qquad p > 1/2 \tag{10}$$

(or an even more rapid decrease), then $R(\omega)$ vanishes and there are no singularities over and above those of the local theory.[1)]

If we assume that the commutator (together with its derivatives) has the asymptotic form

$$r^{-1}\Phi \sim r^{-1} c e^{i(rz - tz_0)} + (z \to -z^*), \qquad \operatorname{Im} z > m, \tag{11}$$

where $z_0 = \sqrt{z^2 + m^2}$, then poles appear in the complex plane.[2)] Indeed, going over to the variables $\rho = (r^2 - t^2)^{1/2}$, $\alpha = t/r$ and choosing the curve $\rho = \text{const}$ as integration contour in (9), we find the following expression for $R(\omega)$:

$$R(\omega) = \lim_{\rho\to\infty} \rho^{1/2} \int_{-\tau}^{\tau} d\alpha\, e^{\chi(\alpha)\rho}\psi_\omega(\alpha) + (z \to -z^*),$$

where

$$\tau = (\rho^2 - a^2)/(\rho^2 + a^2), \quad \chi(\alpha) = m + i(z - \alpha z_0)/\sqrt{1 - \alpha^2},$$

$$\psi_\omega(\alpha) = \nu(\alpha)/[(m\gamma(\alpha))^2 + \omega^2],$$

and $\nu(\alpha)$ is some analytic function whose singularities are located on the straight lines $\operatorname{Im}\alpha = 0$, $|\operatorname{Re}\alpha| \geq 1$ [$\nu(\alpha)$ is independent of ω and ρ]. Again using the method of the stationary phase, we find (omitting obvious intermediate steps)

$$R(\omega) = c(z^2)/(z_0^2 - \omega^2) + (z \to -z^*),$$

i.e., the forward scattering amplitude has poles in the complex plane in a theory with an asymptotic behavior of the commutator of the type (11).

As is easily seen, the asymptotic form

$$\Phi \sim \int dz\, c(z)\, e^{i(rz - tz_0)} \tag{12}$$

would imply the presence of complex cuts in the amplitude.

3. It is seen from (7) that in the nonlocal theory (as compared to the local theory) the dispersion formula contains the additional term $R(\omega)$. Under certain conditions [if we have the asymptotic form (10)] $R(\omega) = 0$. In this case the regions of analyticity of the scattering amplitudes in the local and nonlocal theories coincide, but the amplitudes may differ in the character of the singularity at infinity. Indeed, if it turns out that (7) is true only for $b \geq b_{min} > 0$, this may be connected with the presence of an essential singularity of the scattering amplitude at infinity.[3)]

The case $b_{min} = 0$ is not excluded. Here the dispersion relations for the forward scattering amplitude are the same in both theories.[4)] If $R(\omega) \neq 0$, the simplest reasonable assumption

[1)]Here and in the following we must, of course, assume that it is possible to integrate over the asymptotic expansion in a parameter. The terms with the δ functions arising from the differentiation of the function $\bar{\theta}$ in (9) always vanish as $r \to \infty$.

[2)]In virtue of the symmetrization $(z \to -z^*)$ the function $R(\omega)$ is real on the real axis. Here we must in general take into account that Φ is a pure imaginary function. Condition (11) can easily be given in relativistic form. This is seen most easily by considering the asymptote in lowest order perturbation theory.

[3)]We note that the choice of the function $1/\cos b\omega$ (which has an infinite number of poles) as weight function is not mandatory. Without making any fundamental changes we could also take the exponential $\exp\{ib(\omega^2 - m^2)^{1/2}\}$, fixing the physical sheet of the root by the condition $i(\omega^2 - m^2)^{1/2} \to -\infty$, $|\operatorname{Im}\omega| \to \infty$.

[4)]Such a nonlocal model has been discussed by Lehmann.[6]

from the physical point of view is that it is analytic, real on the real axis, and has singularities in the complex plane (asymptotic commutator of the type (11) or (12). A model in which $R(\omega)$ has only singularities of the indicated type has been investigated by Kirzhnits.[7]

Finally, in theories with nonanalytic functions $R(\omega)$ causality is apparently violated in a physically inadmissible way.

It is curious that the asymptotic forms (10) and (11) are both exponentially decreasing. The results differ only in that in the second case the commutator decreases less rapidly in a certain sector of directions (but oscillates here). When $\mathrm{Re}\, z = 0$, the function (11) decreases more slowly than the function (10) only in one direction of the (r, t) plane, but this is sufficient for the appearance of a pole. We note also that if the amplitude has several poles, several terms of identical type will appear on the right-hand side of (11), each of which represents its pole. Not a single one of them can be discarded on account of smallness in comparison with the others, since this would mean throwing away part of the singularities of the amplitude.

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