THE THEORY OF SPIRAL AMPLITUDES

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The method of spiral amplitudes developed by Wick and Jacob becomes very lucid when formulated in a relativistic velocity space having a Lobachevsky metric. In this space the relativistic properties of the spiral amplitudes become obvious. As an example we treat the problem of transforming spin amplitudes from one channel of a binary reaction to another (the crossing transformation of field theory).

1. INTRODUCTION

THE formulas of relativistic kinematics simplify considerably if reactions are described in a three-dimensional velocity space whose geometry is isomorphic to the Lobachevsky geometry.[1,2] Such a description is especially convenient for studying reactions involving particles with spin. The simplicity of the method is related to the fact that both velocities and spins are kept as three-dimensional vectors (and not 4-vectors, as in the usual treatments) and that relativistic effects only cause a change in the metric of the velocity space, i.e., the law of composition of vectors.

A convenient representation, which has become widely used lately, is the picture of Wick and Jacob,[3,4] in which the spin of the particle is characterized by its projection along the particle momentum in its rest frame. At first glance such a definition does not seem to be covariant, since the scalar product of the 4-spin and 4-momentum is zero by definition, while the projection of the vector part of the spin on the 3-momentum is not an invariant. In order to make the covariance manifest it is more convenient to define the spirality as the projection of the spin 3-vector on the direction of the relative velocity of the particle and the system of the center of inertia (we call this the s-system). Obviously such a projection is a relativistic scalar. We note that such a quantization procedure can be described in terms of a bundle of rays emerging from some selected point in velocity space (the point 1 in our case). The usual quantization method can be described as quantization in terms of a parallel beam of rays (a beam passing through the point at infinity in non-relativistic velocity space). It is obvious that giving a point in velocity space and a beam of rays corresponding to it completely describes the quantization, and is actually covariant (since the definition of the beam does not use the theorem of parallels and, so to speak, is part of absolute geometry). A more detailed discussion of this question will be given in another paper.

Here we shall discuss the transition from the system of the center of inertia of the colliding particles - the s-system, to the system of the center of inertia of the cross reaction- the t-system (cf. below). This transformation is easily found by using kinematic diagrams.

2. KINEMATIC DIAGRAMS AND NOTATION

We consider a reaction of the type (s-channel)

$$1 + 2 \rightarrow 3 + 4 \qquad (2.1)$$

(the numbers label the particles). The conservation law in the s-channel has the form

$$p_1 + p_2 = p_3 + p_4 \qquad (2.2)$$

(p_1 is the 4-momentum with components ϵ_1 and $\mathbf{p}_1$, etc).

The velocities of the particles are described by points on the upper sheet of a three-dimensional hyperboloidal surface.[1)] If we take the plane of the

[1)]The points on the lower sheet of the hyperboloid correspond to antiparticles, which are gotten from the particles by the transformation $\bar{p} = -p$ (change in sign of all components of the 4-momentum). The scalar product of the momenta of particle and antiparticle is obviously $\bar{p}p = -m^2$. Since the hyperbolic cosine of the distance between the points p_1/m and p_2/m in Lobachevsky space is equal to the scalar product of the two vectors, the arc between the points corresponding to particle and antiparticle is $\cosh^{-1}(-1) = i\pi$. Thus the transition from particle to antiparticle is described by a hyperbolic rotation through angle $i\pi$.

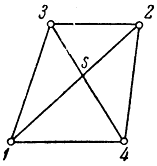

FIG. 1.

paper as a map of this surface, the particle velocities can be marked by points 1 . . . 4 on the map (cf. Fig. 1). These points can be pictured as the ends of vectors drawn from some origin whose exact location is of no interest. The fact that our arguments are independent of the coordinate origin enables us to treat all possible coordinate systems on one drawing, just as the geographic hemisphere shows all coordinate origins which one can choose on the earth. Thus the transition from one coordinate system to another is described on our map of velocity space as the change from projection along one direction to projection along another.

Consider Fig. 1. The point of intersection of the diagonals (12) and (34) is the velocity of the system of the center of inertia. It corresponds to a 4-vector with components

$$(\varepsilon_1 + \varepsilon_2)/\sqrt{s}, \qquad (\mathbf{p}_1 + \mathbf{p}_2)/\sqrt{s}, \tag{2.3}$$

where in the usual notation

$$s = (p_1 + p_2)^2 = (\varepsilon_1 + \varepsilon_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2. \tag{2.4}$$

In field theory one usually also introduces the two quantities

$$u = (p_1 - p_3)^2 = (p_2 - p_4)^2, \tag{2.5}$$

$$t = (p_1 - p_4)^2 = (p_2 - p_3)^2, \tag{2.6}$$

where

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{2.7}$$

We can also associate velocity 4-vectors with these quantities:

$$(\varepsilon_1 - \varepsilon_3)/\sqrt{t}, \quad (\mathbf{p}_1 - \mathbf{p}_3)/\sqrt{t}, \tag{2.8}$$

$$(\varepsilon_1 - \varepsilon_4)/\sqrt{u}, \quad (\mathbf{p}_1 - \mathbf{p}_4)/\sqrt{u}. \tag{2.9}$$

The vectors (2.8) and (2.9) are frequently spacelike. Then the points t and u corresponding to them in velocity space may be imaginary, i.e., they may lie on the hypersurface of the other hyperboloid $u_0^2 - u^2 = -1$. Such points are also treated in the Lobachevsky geometry, and we shall return to them later.

Let us determine the sides and diagonals of the kinematic quadrilateral (1423). Using the fact that

$$p_1 p_2 = m_1 m_2 \operatorname{ch}(12) \text{ etc.}, \tag{2.10*}$$

*ch = cosh.

we get, by expanding the parentheses in (2.4), (2.5) and (2.6);

$$\begin{aligned}
\operatorname{ch}(14) &= (2m_1m_4)^{-1}(t - m_1^2 - m_4^2),\\
\operatorname{ch}(23) &= (2m_2m_3)^{-1}(t - m_2^2 - m_3^2),\\
\operatorname{ch}(42) &= (2m_4m_2)^{-1}(u - m_4^2 - m_2^2),\\
\operatorname{ch}(31) &= (2m_3m_1)^{-1}(u - m_3^2 - m_1^2),\\
\operatorname{ch}(12) &= (2m_1m_2)^{-1}(s - m_1^2 - m_2^2),\\
\operatorname{ch}(34) &= (2m_3m_4)^{-1}(s - m_3^2 - m_4^2).
\end{aligned} \tag{2.11}$$

Applying the law of cosines, we can determine all the angles in the diagram. For example, for the angle (213) we find

$$\cos(213) = \frac{\operatorname{ch}(31) + \operatorname{ch}(12) - \operatorname{ch}(23)}{\operatorname{sh}(31)\operatorname{sh}(12)}. \tag{2.12*}$$

After this has been done, Fig. 1 determines all the directions which are usually chosen as the coordinate axes. Consider, for example, nucleon-nucleon scattering. Then in the center-of-mass system one usually chooses the axes $\mathbf{v}_1 + \mathbf{v}_1'$ and $\mathbf{v}_1 - \mathbf{v}_1'$, where $\mathbf{v}_1$ and $\mathbf{v}_1'$ are the 3-velocities of one of the nucleons before and after the collision. In the kinematic diagram there correspond to these directions two perpendicular lines bisecting the angles formed by the diagonals at the point s.

3. TRANSFORMATION OF SPIRAL AMPLITUDES

In the s system the transition matrix element for a particle with spin is given by (formula 1 of [3])

$$\langle E'J'M'\lambda_3\lambda_4 | S | EJM\lambda_1\lambda_2\rangle = \delta(E - E')\,\delta_{JJ'}\delta_{MM'}\langle\lambda_3\lambda_4 | S^J(E) | \lambda_1\lambda_2\rangle, \tag{3.1}$$

where the spiral amplitude S^J (where J is the orbital angular momentum) is related to the amplitude for spins quantized along the direction of the incident beam (quantum numbers μ_1, μ_2, μ_3, μ_4):

$$\langle\lambda_3\lambda_4 | S^J | \lambda_1\lambda_2\rangle = \sum_{\mu_3\mu_4}\langle\mu_3\mu_4 | S^J | -\mu_1\mu_2\rangle \times D^{s_3}_{\mu_3\lambda_3}(-\varphi, \vartheta, \varphi)\, D^{s_4}_{\mu_4\lambda_4}(-\varphi, \pi - \vartheta, \varphi). \tag{3.2}$$

Formula (3.2) has a simple interpretation. The quantum numbers $\lambda_1, \ldots$ determine the projections of the spins along the four segments joining the point s of Fig. 1 to the vertices of the quadrilateral. In the same Figure, the direction of the incident beam is from s to particle 2. When we transform to spiral amplitudes, the sign of the projection for particle 1 changes, since the velocity of particle 1 is directed from s to 1. For particles 3 and 4 in the final state we can turn the

*sh = sinh.

axes of quantization from (s2) to (13) for particle 3, and from (s2) to (s4) for particle 4. The angles of rotation are obviously equal to the scattering angles ϑ and $\pi-\vartheta$ respectively. The two factors D in (3.2) correspond to the rotations of the axis of quantization through these angles.

The angle φ can be taken equal to zero since all the directions lie in a plane. The spiral quantization is usually described in the s-system. One can also carry over the quantization axes to any other system; as shown earlier,[1] to do this requires one to carry out a parallel displacement between the corresponding points in velocity space. Thus any transformation of axes will consist of a parallel displacement to some other point and an ordinary rotation about the new point.

For quantization along the relative velocity the projection obviously remains unchanged if we move the spin along the direction of the velocity. Thus the spirality λ_1 of particle 1 will be the same in the s system or in the rest systems of particle 1 or 2, or in any other system located along the non-Euclidean line passing through points 1 and 2.

Nowhere in our considerations did we use the fact that the particles move with a velocity less than that of light. Since on the projective Lobachevsky plane, the point at infinity is not distinguished, all our arguments apply to the photon or neutrino also. For these the angle at the corresponding vertex will simply go to zero. The spin of the photon (its polarization) is also quantized along the mutual velocity of the photon and the coordinate system. Because the angle at the vertex is zero, the axis of quantization does not change when we go over to another system.

Thus we see that the spins can be treated independently of the coordinate parts of the wave functions; they can be abstracted from the particles, just as in nonrelativistic problems. The only difference in our case is that the rotation angles are computed using formulas of hyperbolic geometry. In particular we see from this that the spirality representation is not distinguished in any way in its relativistic properties as compared to other quantizations which can be gotten by using the kinematic diagram of Fig. 1.

4. s-AND t-SYSTEMS. CROSSING TRANSFORMATION

Let us consider a somewhat more complicated transformation—the transformation to the cross reaction. The cross reaction is the reaction obtained from (2.1) by interchanging particles 2 and 3 together with the corresponding replacement of particle by antiparticle. Thus, for example, if the reaction (2.1) is

$$p + \pi^- \rightarrow n + \pi^0,$$

the cross reaction will be

$$p + \bar{n} \rightarrow \pi^+ + \pi^0.$$

The problem arises: how are the spiral amplitudes for the cross reaction related to those for the original reaction?

To answer this question we construct the center-of-mass system of the cross reaction, the t-system. Since the conservations laws for the cross reaction have the form

$$p_1 - p_4 = - p_2 + p_3, \tag{4.1}$$

this system will have the four-momentum $p_1 - p_4$.

We have already mentioned that this vector may turn out to be spacelike, so that $t = (p_1 - p_4)^2 < 1$, as for example for the elastic scattering of identical particles. Therefore the velocity of the t-system may be greater than the velocity of light. But this causes no difficulties, since we are interested not in the point t itself but only in the direction toward it; the direction toward the point which is "at infinity" in the Lobachevsky plane corresponds to diverging lines.

After this remark it is easy to understand that just as the s-system is at the intersection of the lines joining the pairs of points (1, 2) and (3,4), similarly the t-system is at the intersection (real or imaginary) of the lines drawn through the pairs of points (1, 4) and (2, 3).

The intersection of the lines (1, 4) and (2, 3) should be regarded as their "outer" intersection. This means that these lines are regarded as arcs on a circle of infinite radius and, for example, one of the interesting segments goes from point 4 to infinity and then comes back "through infinity" to point 1. In the same way the other segment goes through the points (2, ∞, 3) (cf. Fig. 2).

The transition to the cross reaction is usually described by the transformation

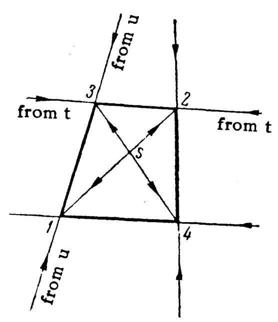

FIG. 2

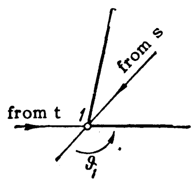

FIG. 3

$$s \to -t, \qquad t \to -s. \tag{4.2}$$

Under such a transformation, in Fig. 1 the point t moves inside the quadrilateral and the point s moves out of it, so that their roles are interchanged.

After this it is clear how one should change to new spiral amplitudes. To do this we bring the spin of particle 1 to point 1 and turn the axis of quantization into the direction (41), i.e., we turn it through the angle (412) (cf. Fig. 3). Using formula (2.8), the angle of rotation can be expressed in terms of t and s. We then carry out the transformation (4.2). It is clear that then the new amplitude is transformed into the normal spiral amplitude for the cross reaction. Particle 2 was quantized along the direction from s to 2. In order for it to be along the direction from t to 2, we must turn it through the angle π-(321). Similar rotations must also be carried out for the other two particles, displacing them into their systems and turning them. Then all the particles will be quantized along the direction from t to the rest frames of the particles.

Thus the transition from the s-channel to the t-channel can be formulated as follows: if the spinless amplitude for the cross reaction is gotten from the initial reaction by the transformation

$$S^{cr}(s, t) = S(-t, -s), \tag{4.3}$$

the amplitude including spins transforms as follows:

$$\langle \lambda_3 \lambda_2 | S^{cr} | \lambda_1 \lambda_4 \rangle = \sum_{\lambda_1' \lambda_2' \lambda_3' \lambda_4'} D^{s_1}_{\lambda_1 \lambda_1'}(0, \vartheta_1, 0)\, D^{s_2}_{\lambda_2 \lambda_2'}(0, \vartheta_2, 0)$$
$$\times D^{s_3}_{\lambda_3 \lambda_3'}(0, \vartheta_3, 0)\, D^{s_4}_{\lambda_4 \lambda_4'}(0, \vartheta_4, 0)\, \langle \lambda_3' \lambda_4' | S | \lambda_1' \lambda_2' \rangle. \tag{4.4}$$

The angles ϑ are determined from Fig. 1:

$$\cos \vartheta_1 = -\cos(412), \qquad \cos \vartheta_2 = -\cos(123),$$
$$\cos \vartheta_3 = -\cos(432), \qquad \cos \vartheta_4 = -\cos(143),$$
$$\cos(abc) = \frac{\operatorname{ch}(bc) + \operatorname{ch}(ab) - \operatorname{ch}(ac)}{\operatorname{sh}(bc)\operatorname{sh}(ab)}. \tag{4.5}$$

Formulas (4.4) and (4.5) solve the problem.

In conclusion I should like to thank A. Popov for helpful discussions of the manuscript.

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