RESTRICTIONS ON THE VALUES OF THE COUPLING CONSTANTS AND THE VERTEX PART FOR THE INTERACTION OF THREE PARTICLES IN QUANTUM FIELD THEORY. II.

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It is shown that the omission of the assumption made earlier[1] for the fermion case, that there is no zero of the fermion Green's function, does not alter the conclusion about the restriction on the coupling constant.

The boson case is treated with the same assumptions as in [1], and it is shown that the vertex part $\Gamma(\kappa^2)$ lies between limits $\Gamma_{\min}(\kappa^2) \le \Gamma(\kappa^2) \le \Gamma_{\max}(\kappa^2)$. Explicit expressions are obtained for the functions $\Gamma_{\min}(\kappa^2)$ and $\Gamma_{\max}(\kappa^2)$, which depend on the value of the renormalized coupling constant g^2.

1. THE RESTRICTIONS ON THE COUPLING CONSTANT

In our previous paper[1] (hereafter cited as I) restrictions were derived on the value of the coupling constant g^2 of three fields a, b, c. These restrictions were found on the assumption that the Green's function of the corresponding particle satisfies the Lehmann-Källen representation and that the vertex part $\Gamma(\kappa^2)$ for the transition a $\to$ b + c is an analytic function of κ^2 in the complex plane of κ^2 cut along the real axis from $(m_b + m_c)^2$ to infinity. In the fermion case, in which particles a and b are fermions with spin $1/2$ and c is a boson with spin 0, in order to obtain the restriction on g^2 we needed the additional assumption that the function $f_1(x)$ which appears in the expression for the Green's function of fermion a [1)]

$$G(\hat{p}) = \hat{p} f_1(x) + m_a f_2(x), \tag{1}$$

has no zeros. We shall show here that omitting this assumption does not change our conclusions about the restriction on g^2.

Let us write $f_1(x)$ and $f_2(x)$ in the Lehmann-Källen representation[2]:

$$f_1(x) = \frac{1}{x-\alpha} - \int_1^\infty \frac{\rho_1(x')}{x'-x-i\delta}\,dx',$$
$$f_2(x) = \frac{1}{x-\alpha} - \frac{1}{\sqrt{\alpha}}\int_1^\infty \frac{\sqrt{x'}\rho_1(x')-\rho_2(x')}{x'-x-i\delta}\,dx', \tag{2}$$

where $\rho_1 \ge 0$, $2x^{1/2}\rho_1(x) \ge \rho_2(x) \ge 0$. From the expression (2) for $f_1(x)$ it follows at once that $f_1(x)$ cannot have more than one zero, and the zero would have to be on the real axis in the range $\alpha < x_0 < 1$. Suppose $f_1(x)$ is equal to zero at the point x_0. Then

$$\int_1^\infty \frac{\rho_1(x')}{x'-x_0}\,dx' = \frac{1}{x_0-\alpha}. \tag{3}$$

If in $\rho_1(x)$ we confine ourselves to the contribution of the two-particle states of particles b + c, the equality in Eq. (3) becomes an inequality. Substituting for $\rho_{1\,\text{two-part}}(x)$ its expression from I, we get the following restriction on the value of the coupling constant

$$g^2/2\pi < 1/\Phi, \tag{4}$$

$$\Phi = (x_0-\alpha)\int_1^\infty \frac{dx}{x(x-x_0)}\frac{\sqrt{(x-1)(x-\lambda)}}{(x+\sqrt{\lambda})(x-\alpha)^2}$$
$$\times\left\{\frac{(x-1)(x-\lambda)}{x}|w_1(x)|^2 + (1+\sqrt{\lambda})^2|w_2(x)|^2\right\}, \tag{5}$$

where the functions $w_1(x)$ and $w_2(x)$ can be expressed in terms of $f_1(x)$ and $f_2(x)$ and the vertex parts $\Gamma_1(x)$ and $\Gamma_2(x)$ (see I):

$$w_1(x) = \tfrac{1}{2}\sqrt{\alpha}\,[f_1(x)\Gamma_2(x)x + \sqrt{\alpha}f_2(x)\Gamma_1(x)](x-\alpha),$$
$$w_2(x) = \tfrac{1}{2}\left\{\left[\frac{x+\sqrt{\lambda}}{1+\sqrt{\lambda}}f_1(x) \pm \sqrt{\alpha}f_2(x)\right]\Gamma_1(x) + \left[\frac{x+\sqrt{\lambda}}{1+\sqrt{\lambda}}\sqrt{\alpha}\,f_2(x) \pm x f_1(x)\right]\Gamma_2(x)\right\}(x-\alpha). \tag{6}$$

[1)]We use the notations adopted in I: $x = p^2/(m_b+m_c)^2$, $\alpha = m_a^2/(m_b+m_c)^2$, $\lambda = (m_b-m_c)^2/(m_b+m_c)^2$.

The restriction on g^2 can be written in a somewhat different form if, as in I, we make use of the fact that the function $f_1^{-1}(x)$ is an R-function in the complex plane of x. From the most general form for $f_1^{-1}(x)$ which has a pole at the point x_0,

$$f_1^{-1}(x) = \int_1^\infty \frac{\rho_1(x')}{|f_1(x')|^2}\left(\frac{1}{x'-x} - \frac{1}{x'-\alpha}\right)dx' + R\left(\frac{1}{x_0 - x} - \frac{1}{x_0-\alpha}\right) + A(x-\alpha), \quad (7)$$

with $R > 0$, $A > 0$, and the condition $[f_1^{-1}(x)]'_{x=\alpha} = 1$ we have the inequality

$$\int_1^\infty \frac{\rho_1(x')}{|f_1(x')|^2}\frac{dx'}{(x'-\alpha)^2} + \frac{R}{(x_0-\alpha)^2} \leqslant 1. \quad (8)$$

On the other hand, the same condition has the consequence that $f_1^{-1}(x)$ can also be written in the form

$$\frac{1}{x-\alpha} f_1^{-1}(x) = 1 + \frac{R}{x_0-\alpha}\left(\frac{1}{x_0-x} - \frac{1}{x_0-\alpha}\right) + \int_1^\infty \frac{\rho_1(x')}{|f_1(x')|^2}\frac{dx'}{x'-\alpha}\left(\frac{1}{x'-x} - \frac{1}{x'-\alpha}\right). \quad (9)$$

The function $f_1^{-1}(x)/(x-\alpha)$ has no zeroes. At $x = x_0 + 0$ we have $f_1^{-1}(x) = -\infty$, and consequently in the entire interval $x_0 < x < 1$ the ratio $f_1^{-1}(x)/(x-\alpha) < 0$.

Since the functions in the right member of Eq. (9) are monotonic, the condition that the right member be negative will be strongest at $x = 1$. Writing this inequality for $x = 1$, and obtaining from it an expression for R and substituting in Eq. (8), we get the following restriction:

$$\int_1^\infty \frac{\rho_1(x')}{|f_1(x')|^2}\frac{x'-x_0}{x'-1}\frac{dx'}{(x'-\alpha)^2} \leqslant \frac{x_0-\alpha}{1-\alpha}, \quad (10)$$

or, after substituting ρ_1 two-part(x):

$$\frac{g^2}{2\pi}\int_1^\infty \frac{dx}{x(x-\alpha)^4}\frac{x-x_0}{x+\sqrt{\lambda}}\sqrt{\frac{x-\lambda}{x-1}}\left\{\frac{(x-1)(x-\lambda)}{x}|w_1(x)|^2 + (1+\sqrt{\lambda})|w_2(x)|^2\right\}\frac{1}{|f_1(x)|^2} < \frac{x_0-\alpha}{1-\alpha}. \quad (11)$$

In virtue of the assumptions regarding $\Gamma_1(x)$ and $\Gamma_2(x)$ which were made in I, the functions $w_1(x)$ and $w_2(x)$ are holomorphic functions in the complex plane of x with a cut from 1 to ∞, and obey definite conditions at the points $x = \alpha$ and $x = -\lambda^{1/2}$.

Let us examine the relations (4), (5). At first glance it seems from these relations that for $x_0 \to \alpha$, i.e., when the zero of $f_1(x)$ comes close enough to the pole, the restriction on g^2 disappears. It might also seem that this same conclusion can be drawn from Eq. (11). If in Eq. (11) we introduce instead of $w_1(x)$ and $w_2(x)$ new functions $F_1(x)$ and $F_2(x)$ defined by the relations

$$\frac{1}{x-\alpha}\frac{w_1(x)}{f_1(x)} = \frac{\alpha-x_0}{x-x_0}F_1(x), \qquad \frac{1}{x-\alpha}\frac{w_2(x)}{f_1(x)} = \frac{\alpha-x_0}{x-x_0}F_2(x), \quad (12)$$

the functions $F_1(x)$ and $F_2(x)$ are holomorphic in the plane of x cut from 1 to ∞ and obey the same conditions as $w_1(x)$ and $w_2(x)$ at the points $x = \alpha$ and $x = -\lambda^{1/2}$. When we substitute Eq. (12) in Eq. (11) we see that in the right member of the inequality there is a factor $1/(x_0-\alpha)$ analogous to that in Eq. (4).

In these arguments, however, it has been tacitly assumed that for x_0 arbitrarily close to α the functions $w_1(x)$ and $w_2(x)$ are not subject to any additional conditions. Actually such conditions do arise. To convince ourselves of this, we return to the formulas (6), which define $w_1(x)$ and $w_2(x)$ in terms of the vertex parts $\Gamma_1(x)$ and $\Gamma_2(x)$. We assume that $\Gamma_1(x)$ and $\Gamma_2(x)$ have no poles in the cut plane of x. Since $w_1(x)$ and $w_2(x)$ do not have any poles in this region, this means that if we use the linear equations (6) to express $\Gamma_1(x)$ and $\Gamma_2(x)$ in terms of $w_1(x)$ and $w_2(x)$, the determinant must not have any zeroes except on the cut and at the point $x = \alpha$, where the equations (6) are not independent. If the determinant is zero at some point x_1 in the complex plane of x which does not lie on the cut, then in order for $\Gamma_1(x)$ and $\Gamma_2(x)$ to be regular at this point we must impose on the functions $w_1(x)$ and $w_1(x)$ the conditions $w_1(x_1) = w_2(x_1) = 0$.

The determinant of the equations (6) is proportional to the function

$$Q(x) = (x-\alpha)^2[f_2^2(x) - x\alpha^{-1}f_1^2(x)]. \quad (13)$$

It is easy to see that if $f_1(x_0) = 0$, then $Q(x)$ has at least one other zero besides $x = \alpha$ which lies in the interval $0 < x < x_0$. In fact, $Q(0) \geq 0$, $Q(\alpha) = 0$, and $Q(x_0) \geq 0$. This is possible only if there is a point x_1 such that $Q(x_1) = 0$, $0 \leq x_1 \leq x_0$, $x_1 \neq \alpha$, or else if the function $Q(x)$ has a zero of second (or higher) order at the point $x = \alpha$. In the latter case it is also necessary to impose on $w_1(x)$ and $w_2(x)$ the conditions $w_1(x_1) = w_2(x_1) = 0$ for $x_1 = \alpha$. It is convenient to take these conditions into account in looking for the minimum of the expression (5) or of the integral in Eq. (11) by setting

$$w_1(x) = \frac{x-x_1}{\alpha-x_1}\bar{w}_1(x), \qquad w_2(x) = \frac{x-x_1}{\alpha-x_1}\bar{w}_2(x). \quad (14)$$

Then $\bar{w}_1(x)$, $\bar{w}_2(x)$ will obey the same conditions as $w_1(x)$, $w_2(x)$ at the points $x = \alpha$ and $x = -\lambda^{1/2}$.

Substituting Eq. (14) in Eq. (5) and comparing the result with the functional of I, Eq. (49), whose minimum was looked for in I, we see that the minima of both functionals are defined on the same class of functions, and that the functional (5) differs from I, Eq. (49) only by the additional factor

$$\left(\frac{x-x_1}{x_1-\alpha}\right)^2\frac{x_0-\alpha}{x-x_0}$$

in the weight function. As is seen from I, Eqs. (69) — (71), the minimum of this functional is given (for real x_1) by

$$\Phi_{min}=\Phi_{0,\,min}\frac{x_0-\alpha}{(x_1-\alpha)^2}\times D^{-2}\,[(x-x_0);\,0]\,D^2\,[(x-x_1)^2;\,0], \tag{15}$$

where $\Phi_{0,min}$ is the value of the minimum of the functional Φ of I, Eq. (49) [which is what we would have for the case in which $f_1(x)$ has no zeroes]. In Eq. (15) $D[f(\theta);\,0]$ denotes the function $D(z)$ of I, Eq. (69), taken at the point $z = 0$ $(x = \alpha)$ and corresponding to the weight function $f(\theta)$ [for $f(x) = x - x_0$ we have $f(\theta) = 1 - x_0 + (1-\alpha)\tan^2\theta/2$]. Calculating $D[x - x_0;\,0]$ and $D[(x - x_1)^2;\,0]$ in accordance with I, Eq. (69), we get $D[x - x_0;\,0] = (1-\alpha)^{1/2} + (1-x_0)^{1/2}$ and

$$\Phi_{min}=\Phi_{0,\,min}\frac{x_0-\alpha}{(x_1-\alpha)^2}\frac{(\sqrt{1-\alpha}+\sqrt{1-x_1})^4}{(\sqrt{1-\alpha}+\sqrt{1-x_0})^2}. \tag{16}$$

For the case in which $Q(x)$ has two real zeroes x_1 and x_2 ($x_1 < 1$, $x_2 < 1$), we get in an obvious way instead of Eq. (16)

$$\Phi_{min}=\Phi_{0,\,min}\frac{x_0-\alpha}{(x_1-\alpha)^2}\frac{(\sqrt{1-\alpha}+\sqrt{1-x_1})^4}{(\sqrt{1-\alpha}+\sqrt{1-x_0})^2}\times\frac{(\sqrt{1-\alpha}+\sqrt{1-x_2})^4}{(x_2-\alpha)^2}. \tag{17}$$

It is not hard to verify that the inequality (11) leads to the same expressions (16), (17).

As can be seen from Eq. (16), if we show that for $x_0 \to \alpha$ the point x_1 goes to α faster than $(x_0-\alpha)^{1/2}$ [or that there are two roots x_1 and x_2 which go to α so that $(x_1-\alpha)(x_2-\alpha) < (x_0-\alpha)^{1/2}$], this will prove that there is a restriction on g^2 even when there is a zero of the Green's function $f_1(x)$.

For the proof we establish some inequalities which the zeroes of $Q(x)$ must satisfy. We shall be interested in the zeroes of one of the factors contained in $Q(x)$:

$$Q_1(x)=(x-\alpha)\,[f_2(x)+\sqrt{x/\alpha}\,f_1(x)] \tag{18}$$

for $0 \le x \le 1$. If $Q_1(x)$ has a zero x_1 located to the left of α, then, as we see from Eqs. (16) and (17), the smallest value of Φ_{min} is obtained when for given x_0 the quantity $\alpha - x_1$ is as large as possible. If $Q_1(x)$ has a zero x_2 located to the right of α, the smallest value of Φ_{min} is obtained for the largest $x_2 - \alpha$ [in particular, the presence of zeroes of $Q(x)$ lying to the right of $x = 1$ does not lead to an increase of Φ_{min}].[2)]

By means of Eqs. (2) and (3) we can write Eq. (18) in the following way:

$$Q_1(x)=1+\sqrt{\frac{x}{\alpha}}-\frac{(x-\alpha)}{x_0-\alpha}b-\sqrt{\frac{x}{\alpha}}\frac{x-\alpha}{x_0-\alpha}-(x-\alpha)(x-x_0)\int_1^\infty dx'\frac{1}{(x'-x)(x'-x_0)}\times\left[\rho_1(x')\left(\sqrt{\frac{x'}{\alpha}}+\sqrt{\frac{x}{\alpha}}\right)-\frac{\rho_2(x')}{\sqrt{\alpha}}\right], \tag{19}$$

$$b=\frac{x_0-\alpha}{\sqrt{\alpha}}\int_1^\infty\frac{\sqrt{x'}\rho_1(x')-\rho_2(x')}{x'-x_0}dx'. \tag{20}$$

From Eq. (19), by using the condition $0 \le \rho_2(x) \le 2x^{1/2}\rho_1(x)$ and Eq. (3), we can easily get the inequality

$$Q_1(x)\leqslant 1+\sqrt{\frac{x}{\alpha}}-\frac{x-\alpha}{x_0-\alpha}\left(b+\sqrt{\frac{x}{\alpha}}\right)+\frac{(x-\alpha)(x-x_0)}{\sqrt{\alpha}}\int_1^\infty\frac{\rho_1(x')\,dx'}{(\sqrt{x'}+\sqrt{x})(x'-x_0)}$$
$$\leqslant 1+\sqrt{\frac{x}{\alpha}}-\frac{x-\alpha}{x_0-\alpha}\left(b+\sqrt{\frac{x}{\alpha}}\right)+\frac{(x-\alpha)(x-x_0)}{\sqrt{\alpha}}\frac{1}{(1+\sqrt{x})(x_0-\alpha)}\equiv\frac{\sqrt{x}+\sqrt{\alpha}}{(x_0-\alpha)\sqrt{\alpha}(1+\sqrt{x})}u(x). \tag{21}$$

For $x = \alpha$ we have $Q_1(\alpha) = 2$. Therefore the zero x_1 of the function $Q_1(x)$ that is closest to α satisfies the condition $|x_1 - \alpha < |\tilde{x}_1 - \alpha|$, where $\tilde{x}_1$ is the zero of $u(x)$.

Introducing the notations $x^{1/2} - \alpha^{1/2} = y$, $x_0 - \alpha = \epsilon$, we can rewrite the function $u(y)$ in the form

$$u(y)=-y^2(1+b\sqrt{\alpha})-y\sqrt{\alpha}(1+\sqrt{\alpha})(1+b)+(1+\sqrt{\alpha})\,\varepsilon. \tag{22}$$

The equation $u(y) = 0$ has two roots:

$$y_{1,2}=-\frac{\sqrt{1+\sqrt{\alpha}}}{2(1+b\sqrt{\alpha})}\left\{\sqrt{\alpha}(1+b)\sqrt{1+\sqrt{\alpha}}\pm[\alpha(1+\sqrt{\alpha})(1+b)^2+4(1+b\sqrt{\alpha})\,\varepsilon]^{1/2}\right\}. \tag{23}$$

Our problem now is to find the minimum of the functional (16) or (17), in which x_1 and x_2 are the values given by Eq. (23), for various values of the parameter b. One cannot solve this simple problem by direct substitution of Eq. (23) in Eq. (16)

[2)]If $\alpha < x_2 < x_0$, then it is not hard to see that $\Phi_{min} > \Phi_{0min}$.

or (17). Therefore we shall first give a number of auxiliary arguments. The two roots of Eq. (22) are real; for $b > -\alpha^{-1/2}$ one root is positive and the other negative, and for $b < -\alpha^{-1/2}$ both roots are negative. It can be seen from Eqs. (22) and (23) that when $\epsilon = 0$, depending on the value of b either one of the roots goes to zero, proportional to ϵ, or else both roots go to zero with the product proportional to ϵ. Thus for $x_0 \rightarrow \alpha$ either the function has one zero x_1, which goes to zero proportional to $x_0 - \alpha$, or else it has two zeroes x_1 and x_2, which approach α so that $(x_1 - \alpha)(x_2 - \alpha) \sim (x_0 - \alpha)$. It both cases it is clear from Eqs. (16), (17) that for sufficiently small values of $x_0 - \alpha$ the restriction on g^2 not only exists, but even leads to arbitrarily small values $g^2 \sim (x_0 - \alpha)$. Thus in principle our assertion is proved.

To get a quantitative estimate of Φ_{min}, we note that

$$\partial u(y, b)/\partial b = -y\sqrt{\alpha}\,(1 + \sqrt{\alpha} + y) \tag{24}$$

is positive for $y < 0$ and negative for $y > 0$ (the only values of the root that have meaning are those for which $y > -\alpha^{1/2}$. It follows from Eq. (24) that when Eq. (22) has one positive and one negative root, the roots both move to the left when b increases. If, on the other hand, both roots are negative, then as b increases the root nearer to $y = 0$ moves to the left. Suppose that for a certain value of b both roots are negative. Then it helps (decreases Φ_{min}) to increase b up to the point at which a positive root appears. Indeed, as b increases the root nearer to $y = 0$ moves away from zero, and consequently Φ_{min} decreases. After a positive root appears it is still advantageous to increase b, up to the point at which the positive root, which is decreasing, reaches the value $y_2 = 1 - \alpha^{1/2}$ $(x_2 = 1)$, since for $y_2 > 1 - \alpha^{1/2}$ $(x_2 > 1)$ the zero of $u(x)$ lies on the cut, and the inequality (21) is meaningless. The value $b = b_0$ at which the positive root y_2 is equal to $1 - \alpha^{1/2}$ is

$$b_0 = -\frac{1}{2}\frac{1+\alpha}{\sqrt{\alpha}} + \frac{1}{2\sqrt{\alpha}}\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}\,\varepsilon. \tag{25}$$

For $b = b_0$ the negative root has the value

$$y_1 = -2\varepsilon/[(1-\sqrt{\alpha})^2 + \varepsilon]. \tag{26}$$

It can be seen from Eq. (26) that for $\epsilon > \epsilon_0 = (1 - \alpha^{1/2})\alpha^{1/2} \cdot (2 - \alpha^{1/2})^{-1}$ we have $y_1 = -\alpha^{1/2}$, i.e., $x_1^{1/2} < 0$. Since we have proved in the general case that the function $Q(x)$ has a zero in the interval $0 < x < x_0$, this means that for $\epsilon > \epsilon_0$ we must take the worst value of this zero, $x_1 = 0$. For $\epsilon < \epsilon_0$ we can no longer take a value of b such that one root of Eq. (22) is to the right of $1 - \alpha^{1/2}$ and the other to the left of $-\alpha^{1/2}$. It can be shown (we shall not take space for this) that for not very small α $(\alpha > 1/2)$ the smallest value of Φ_{min} is obtained when as ϵ is decreased the positive root remains at $y_2 = 1 - \alpha^{1/2}$ and the negative one approaches zero in accordance with Eq. (26). Thus the minimum value of Φ_{min} for a given $x_0 - \alpha < \epsilon_0$ is given by Eq. (16), in which we must put for x_1 the value

$$x_1 = \left[\sqrt{\alpha} - \frac{2\varepsilon}{(1-\sqrt{\alpha})^2 + \varepsilon}\right]^2, \qquad \varepsilon = x_0 - \alpha. \tag{27}$$

Substituting Eq. (27) in Eq. (16) and Eq. (16) in Eq. (4), we have

$$\frac{g^2}{g_0^2} < 16(1-\sqrt{\alpha})^2\varepsilon \times \frac{[(1-\sqrt{\alpha})^2\sqrt{\alpha} - \varepsilon]^2\,[\sqrt{1-\alpha} + \sqrt{1-\alpha-\varepsilon}]^2}{[(1-\sqrt{\alpha})^2 + \varepsilon]^4} \times \left\{\sqrt{1-\alpha} + \left[1 - \left(\sqrt{\alpha} - \frac{2\varepsilon}{(1-\sqrt{\alpha})^2+\varepsilon}\right)^2\right]^{1/2}\right\}^{-4}, \tag{28}$$

where g_0 is the maximum value of the coupling constant which is obtained on the assumption that $f_1(x)$ has no zeroes.

To find the maximum value of g^2 we still have to determine the maximum of the right member of Eq. (28) as a function of ϵ in the interval $0 < \epsilon < 1 - \alpha$. Since we have not been able to get a general expression for the maximum of g^2 in closed form, we shall not carry the calculations further, but shall only indicate some properties of this maximum. It is obvious that it exists and is finite. Furthermore it is clear that it lies at small ϵ, less than ϵ_0. An explicit expression for the right member of Eq. (28) can be written for the case of small $1 - \alpha$. For this case we have

$$\frac{g^2}{g_0^2} < \frac{16}{81}\frac{1}{(1-\sqrt{\alpha})^2}.$$

For the case of the pion-nucleon interaction constant, when we substitute the actual mass values and find the maximum of the function of ϵ, we find numerically $g^2/g_0^2 < 16$.

We have shown that in the fermion case there is a restriction on the size of the coupling constant even when the Green's function $f_1(x)$ has a zero. Still, our result cannot be regarded as entirely satisfactory: first, in practically interesting cases the quantity g^2_{max} is extremely large, and second, in the nonrelativistic case $(\Delta = m_b + m_c - m_a \ll m_a$, i.e., $1 - \alpha \ll 1)$ we find g^2_{max} proportional to $\Delta^{-3/2}$ (Δ is the binding energy) instead of the usual law in nonrelativistic theory $g^2_{max} \sim \Delta^{-1/2}$. These last two facts show that the restriction we have used is inadequate. The reason is that in determining the positions of the zeros of the function $Q_1(x)$ we have used an extremely crude procedure. Namely, in going from Eq. (19) to Eq. (21) we actually assumed that the integral (3) is concentrated near $x = 1$ and furthermore that $\rho_2(x) = 2x^{1/2}\rho_1(x)$ (which is the worst case), and then looked for the minimum over all values of b. In actual fact the integrand in the integral (20), which determines b near $x = 1$, is expressed in terms of the same functions $w_1(x)$ and $w_2(x)$ that occur in the definition of the functional Φ, and consequently b cannot be regarded as completely independent. We are inclined to think that when all of this is taken into account the presence of a zero of $f_1(x)$ will lead to a stronger restriction on g^2 than one had without the zero.

2. RESTRICTIONS ON THE VERTEX PART FOR THE INTERACTION OF THREE PARTICLES

In I we showed that the coupling constant of three fields a, b, and c must be bounded above: $g^2 \le g^2_{max}$, where g^2_{max} is a known function of the masses of the particles, m_a, m_b, and m_c. This restriction was derived on the following assumptions (we shall here consider the case in which the particles a, b, and c are bosons with spin zero):

I. A Lehmann-Källen representation[2] holds for the Green's function of particle a.

II. Particles b and c are the nearest particles (in terms of the sum of their masses) into which particles a can be converted;

III. The vertex part $\Gamma(\kappa^2, m_b^2, m_c^2) \equiv \Gamma(\kappa^2)$ is an analytic function of κ^2 in the complex plane of κ^2 with a cut along the real axis from $\kappa^2 = (m_b + m_c)^2$ to infinity. On the real axis to the left of $\kappa^2 = (m_b + m_c)^2$ the function $\Gamma(\kappa^2)$ is real, and at infinity $\Gamma(\kappa^2)$ does not increase faster than a power of its argument. At the point m_a^2 the value is $\Gamma(m_a^2) = 1$.

On the basis of these assumptions it was proved in I that g^2 is restricted by the inequality

$$\frac{g^2}{4\pi} \leqslant (m_b + m_c)^2 \frac{1}{\Phi},$$

$$\Phi = \int_1^\infty \frac{\sqrt{(x-1)(x-\lambda)}}{x(x-\alpha)^2} |\Gamma(x)|^2 dx; \qquad x = \frac{\kappa^2}{(m_b+m_c)^2},$$

$$\alpha = \frac{m_a^2}{(m_b+m_c)^2}, \qquad \lambda = \frac{(m_b-m_c)^2}{(m_b+m_c)^2}. \tag{29}$$

The minimum of the functional Φ over the class of functions $\Gamma(x)$ satisfying condition III exists and is given by

$$\Phi_{min} = \frac{\pi}{4} \frac{\sqrt{1-\lambda} + \sqrt{1-\alpha}}{\sqrt{1-\alpha}(1+\sqrt{1-\alpha})^2}, \tag{30}$$

so that the maximum value of g^2 is given by

$$g^2_{max}/4\pi = (m_b + m_c)^2/\Phi_{min}. \tag{31}$$

In the present paper we shall treat the problem inverse to that treated in I. We shall regard the coupling constant as given (from experiment), and on the basis of the assumptions I—III we shall find the limits between which the function $\Gamma(x)$ can vary at a point x_0 (we shall consider real values $x_0 < 1$; the extension to the case of complex x_0 can be made without difficulty). It is clear that such limits exist, since for $g^2 \to g^2_{max}$ the quantity $\Gamma(x)$ goes to the function $\Gamma_{min}(x)$ which gives the minimum of the functional Φ.

In order to find these limits, we first solve an auxiliary problem: we find the minimum of the functional Φ (1) over the class of functions $\Gamma(x)$ satisfying condition III and the condition $\Gamma(x_0) = a$. This problem can be solved by the same method as used in I (cf. also [3-5]).

We make a conformal transformation which takes the plane of x cut from 1 to ∞ into the interior of the unit circle:

$$z = -(\sqrt{x-1} - i\sqrt{1-\alpha})/(\sqrt{x-1} + i\sqrt{1-\alpha}). \tag{32}$$

This takes the point $x = \alpha$ into $z = 0$ and $x = x_0$ into $z = z_0$. The integral (29) becomes

$$\Phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,|\Gamma(z)|^2 d\theta, \qquad z = e^{i\theta},$$

$$f(\theta) = \frac{\pi}{\sqrt{1-\alpha}} \frac{u\sqrt{1-\lambda+(1-\alpha)u}}{[1+(1-\alpha)u](1+u)}, \qquad u = \mathrm{tg}^2\frac{\theta}{2}. \tag{33*}$$

$\Gamma(z)$ is an analytic function inside the unit circle and can be expanded in a series of polynomials $p_n(z)$ orthogonal with respect to the weight function $f(\theta)$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, p_n^*(e^{i\theta})\, p_m(e^{i\theta})\, d\theta = \delta_{mn}. \tag{34}$$

When the expansion

$$\Gamma(z) = \sum c_n p_n(z) \tag{35}$$

is substituted and Eq. (34) is used, Φ takes the form

$$\Phi = \sum c_n^2. \tag{36}$$

[In our case, because $f(\theta)$ is an even function, the coefficients $p_n(z)$ are real, and by condition III the c_n are also real.] $\Gamma(z)$ satisfies the following two conditions:

*tg = tan.

$$\Gamma(0) = \sum c_n p_n(0) = 1, \tag{37}$$

$$\Gamma(z_0) = \sum c_n p_n(z_0) = a. \tag{38}$$

When we determine the minimum of Φ, Eq. (36), under the supplementary conditions (37) and (38), we arrive at the equation

$$c_n = \tfrac{1}{2}\left[\nu_1 p_n(0) + \nu_2 p_n(z_0)\right] \tag{39}$$

(ν_1, ν_2 are Lagrange multipliers). Substituting Eq. (39) in Eqs. (37) and (38), we get equations for ν_1, ν_2:

$$\begin{aligned} &\tfrac{1}{2}\nu_1 \sum p_n^2(0) + \tfrac{1}{2}\nu_2 \sum p_n(0)\, p_n(z_0) = 1, \\ &\tfrac{1}{2}\nu_1 \sum p_n(0) p_n(z_0) + \tfrac{1}{2}\nu_2 \sum p_n^2(z_0) = a. \end{aligned} \tag{40}$$

To calculate the sums of orthogonal polynomials we use the formulas[4]

$$\sum_n p_n^*(z_1)\, p_n(z_2) = \frac{1}{1 - z_1^* z_2}\, \frac{1}{D^*(z_1)}\, \frac{1}{D(z_2)}, \tag{41}$$

$$D(z) = \exp\left\{\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln f(\theta)\, \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\, d\theta\right\}. \tag{42}$$

The equations (40) can then be written in the form

$$\begin{aligned} &\nu_1 D^{-2}(0) + \nu_2 D^{-1}(0)\, D^{-1}(z_0) = 2, \\ &\nu_1 D^{-1}(0)\, D^{-1}(z_0) + \nu_2 D^{-2}(z_0)/(1 - z_0^2) = 2a, \end{aligned} \tag{43}$$

and the minimum of the functional (36) takes the form

$$\Phi_{min}^{(x_0)} = \frac{1}{4}\left[\nu_1^2 D^{-2}(0) + 2\nu_1\nu_2 D^{-1}(0)\, D^{-1}(z_0) + \nu_2^2 D^{-2}(z_0)/(1 - z_0^2)\right]. \tag{44}$$

Solving the equations (43) for ν_1, ν_2 and substituting the solutions in Eq. (44), we find

$$\Phi_{min}^{(x_0)} = \Phi_{min}\left\{1 + \frac{1 - z_0^2}{z_0^2}\left[1 - a\frac{D(z_0)}{D(0)}\right]^2\right\}. \tag{45}$$

By means of the expression (45) for the minimum of the functional Φ under the condition $\Gamma(x_0) = a$ we can now easily find the limits within which the function $\Gamma(x)$ can vary at the point x_0. By Eqs. (28) and (31) we have for fixed g^2 the inequality

$$\Phi^{(x_0)}/\Phi_{min} \leqslant g_{max}^2/g^2, \tag{46}$$

from which we get

$$\left[1 - a\frac{D(z_0)}{D(0)}\right]^2 \leqslant \frac{z_0^2}{1 - z_0^2}(\gamma - 1), \qquad \gamma = \frac{g_{max}^2}{g^2}. \tag{47}$$

For the $f(\theta)$ of Eq. (33) the function $D(z)$ can be calculated easily. Substituting it in Eq. (47) and expressing z in terms of x, we get the following formula for the maximum and minimum values of the function $\Gamma(x)$ at the point x_0:

$$\begin{aligned} \Gamma_{\substack{max\\min}}(x_0) = {} & \frac{1}{2}\,\frac{(\sqrt{1-\alpha} + \sqrt{1-\lambda})^{1/2}}{1 + \sqrt{1-\alpha}} \\ & \times \frac{(1 + \sqrt{1-x_0})(\sqrt{1-\alpha} + \sqrt{1-x_0})}{\sqrt{1-x_0}\,(\sqrt{1-\lambda} + \sqrt{1-x_0})^{1/2}} \\ & \times \left[1 \pm \sqrt{\gamma - 1}\, \frac{\sqrt{1-x_0} - \sqrt{1-\alpha}}{2(1-x_0)^{1/4}(1-\alpha)^{1/4}}\right]. \end{aligned} \tag{48}$$

From Eq. (48) we see that for $g^2 = g_{max}^2$ there remains for $\Gamma(x_0)$ simply the coefficient of the square brackets, and, as was to be expected, this agrees with the value[5] of the function $\Gamma(x)$ which minimizes the functional $\Phi(1)$. We note that if at x_0 $\Gamma(x)$ takes its maximum or minimum value (48), then at other points $\Gamma(x)$ will be uniquely determined and given by the equation

$$\begin{aligned} \Gamma(x) = {} & \frac{1}{2}\,\frac{(\sqrt{1-\alpha} + \sqrt{1-\lambda})^{1/2}}{1 + \sqrt{1-\alpha}} \\ & \times \frac{(1 + \sqrt{1-x})(\sqrt{1-\alpha} + \sqrt{1-x})}{\sqrt{1-x}\,(\sqrt{1-\lambda} + \sqrt{1-x})^{1/2}} \\ & \times \left[1 \pm \sqrt{\gamma - 1}\, \frac{(\sqrt{1-x} - \sqrt{1-\alpha})(1-x_0)^{1/4}}{(1-\alpha)^{1/4}(\sqrt{1-x} + \sqrt{1-x_0})}\right]. \end{aligned} \tag{49}$$

In the nonrelativistic case, in which the binding energy $m_b + m_c - m_a$ is small, $\Delta \ll m_a$, and the quantity $1 - x_0$ can also be treated as small, Eq. (48) goes over into

$$\begin{aligned} \Gamma_{\substack{max\\min}}(E_0) = {} & \frac{1}{2}\left(1 + \sqrt{\frac{\Delta}{-E_0}}\right) \\ & \times \left\{1 \pm \frac{1}{2}\sqrt{\gamma - 1}\left[\left(-\frac{E_0}{\Delta}\right)^{1/4} - \left(\frac{\Delta}{-E_0}\right)^{1/4}\right]\right\}, \end{aligned} \tag{50}$$

where E is the kinetic energy ($E < 0$).

The ratio g_{max}^2/g^2 occurs in Eqs. (48), (50) in the form $(\gamma - 1)^{1/2}$, and therefore even for rather large values of γ the limits on the variation of $\Gamma(x_0)$ are comparatively narrow. For the deuteron (a is the deuteron, and b and c are the proton and neutron) we have from experiment (cf. [1]) $g_{max}^2/g^2 = 1.3$ and, as follows, for example, from Eq. (50), for $E_0 = -4\Delta$ the quantity $\Gamma(E_0)$ cannot differ from $\frac{1}{2}[1 + (\Delta/(-E_0))^{1/2}]$ by more than 20 percent.

We note that the problem we have treated is a special case of a more general one: given the values of the function $g\Gamma(x)$ at one or more points $x_1, x_2, \ldots x_n$, it is required to determine the limits within which $g\Gamma(x)$ can vary at the point x_0. It is obvious that this more general problem can be solved by a trivial extension of the method used here.

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