HOLLOW SUPERCONDUCTORS IN A MAGNETIC FIELD

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Formulas are derived within the framework of the Ginzburg-Landau macroscopic theory of superconductivity to describe the behavior in a magnetic field of superconducting samples in the form of hollow spheres or hollow cylinders. The critical magnetic fields of such samples are found. The limits of possible "superheating" or "supercooling" are determined. Destruction of the superconductivity of a hollow cylinder by a current is also considered.

THE problem of the behavior of superconducting samples of various shapes in a magnetic field was solved within the framework of a local theory by a number of authors (see, for example, [1]). Recently, in connection with the problem of the quantization of the magnetic flux, the problem of a hollow superconducting cylinder in a magnetic field parallel to its surface was solved by Ginzburg.[2] Formulas are obtained in the present work which describe the behavior of a hollow superconducting sphere or cylinder in a field perpendicular to its surface; in particular, expressions are obtained for the magnetic moments of hollow superconductors. The problem of the destruction of superconductivity of such specimens by a field and by a current is also considered.

1. We begin with the equations of the macroscopic theory of Ginzburg-Landau (see [1,3]) which is suitable for London superconductors close to the critical temperature T_c. The equation describing the penetration of the field into the superconductor is written in this case in the form

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{A} = -\delta^{-2}\mathbf{A}, \qquad (1)^*$$

where $\mathbf{A}$ is the vector potential of the magnetic field, $\delta = \delta_0/\Psi$, δ_0 is the penetration depth of a weak field in the bulky superconductor; we shall consider the quantity Ψ to be a constant.

The solution of Eq. (1) has the form $A = i_\varphi A(r, \theta)$ ($r_1 \le r \le r_2$) for a hollow sphere in the spherical coordinates r, θ, φ; here

$$A(r, \theta) = \frac{3}{2} H_0 \delta^2 r_2 \frac{f_+(\xi - 1) e^{\xi-\xi_1} + f_-(\xi + 1) e^{\xi_1 - \xi}}{e^{\Delta} f_+ - e^{-\Delta} f_-} \frac{\sin\theta}{r^2},$$

$$f_\pm = \xi_1^2 \pm 3\xi_1 + 3, \quad \xi = r/\delta, \quad \xi_1 = r_1/\delta,$$

$$\xi_2 = r_2/\delta, \quad \Delta = \xi_2 - \xi_1,$$

$$H_1 = \frac{6 H_0 \xi_2}{e^{\Delta} f_+ - e^{-\Delta} f_-}, \quad M = -\frac{H_0}{2} r_2^3 \left[1 + \frac{3}{\xi_2^2} - \frac{3}{\xi_2} \frac{f_+ e^{\Delta} + f_- e^{-\Delta}}{f_+ e^{\Delta} - f_- e^{-\Delta}}\right]. \qquad (2)$$

In this case r_1 is the internal radius of the hollow sphere, r_2 is the external radius of the sphere, H_0 is the value of the external magnetic field far from the sphere, H_1 is the value of the field in the hollow interior of the sphere, and M is the magnetic moment of the hollow sphere.

In the case $r_1 = 0$, we get from (2) the well-known formulas for the field and the magnetic moment of a continuous superconducting sphere (see [1]). In the case $\xi_1 \gg 1$, $\Delta = \xi_2 - \xi_1 \gg 1$, we have

$$H_1 = 6 H_0 \xi_2 \xi_1^{-2} e^{-\Delta}, \quad M = {}^1/_2 H_0 r_2^3 (1 - 3\delta/r_2 + 3\delta^2/r_2^2), \qquad (3)$$

that is, the field inside a hollow sphere with a thick wall is exponentially small, and its moment is identical with the moment of a solid sphere of the same radius.

Finally, for a sphere with a thin wall, we find (for $\Delta \ll 1$, $\xi_1 \gg 1$)

$$H_1 = \frac{H_0}{1 + \xi_1\Delta/3}, \qquad M = -\frac{H_0 r_2^3}{2}\left(1 - \frac{1}{1 + \xi_1\Delta/3}\right). \qquad (4)$$

It is evident from Eqs. (4) that when $\xi_1\Delta \gg 1$ a large thin-walled sphere behaves in a weak magnetic field as a continuous bulky sample, even if its wall is a thin film, much less in thickness than the penetration depth δ (in connection with this remark, see also [2,4,5]). If $\xi_1\Delta \ll 1$, we get $M = -H_0 r_2^3 \xi_1 \Delta/6$, that is, the moment approaches zero linearly with decrease in $\xi_1\Delta$.

We now obtain formulas applying to a round hollow superconducting cylinder, parallel to the z axis and placed in a magnetic field H_0 parallel to

*rot = curl.

the x axis. We have $\mathbf{A} = \mathbf{i}_z A(r, \varphi)$ (r, z, φ are the cylindrical coordinates, $r_1 \le r \le r_2$),

$$A = 2H_0\delta \frac{I_1(\xi) K_2(\xi_1) + I_2(\xi_1) K_1(\xi)}{I_0(\xi_2) K_2(\xi_1) - I_2(\xi_1) K_0(\xi_2)} \sin\varphi,$$

$$H_1 = \frac{2H_0}{\xi_1^2 [I_0(\xi_2) K_2(\xi_1) - I_2(\xi_1) K_0(\xi_2)]},$$

$$M = -\frac{1}{2} H_0 r_2^2 \left[1 - \frac{2}{\xi_2} \frac{I_1(\xi_2) K_2(\xi_1) + I_2(\xi_1) K_1(\xi_2)}{I_0(\xi_2) K_2(\xi_1) - I_2(\xi_1) K_0(\xi_2)}\right]. \quad (5)$$

Here H_1 is the field in the hollow part of the cylinder, M is the moment of a unit length of the cylinder, $r_1 = \xi_1\delta$ is the inner and $r_2 = \xi_2\delta$ is the outer radius; I_n, K_n are the well-known Bessel functions, and $\xi = r/\delta$.

In the case $\xi_1 = 0$, the results of Silin [6] follow from (5). For $\xi_1 \gg 1$, $\Delta \gg 1$ (a hollow cylinder with thick walls), we have

$$H_1 = \frac{4H_0}{\xi_1} \sqrt{\frac{\xi_2}{\xi_1}} e^{-\Delta}, \qquad M = -\frac{1}{2} r_2^2 \left(1 - \frac{2}{\xi_2}\right) H_0,$$

$$\Delta = \xi_2 - \xi_1.$$

In the case $\xi_1 \gg 1$, $\Delta \ll 1$, we get

$$H_1 = \frac{H_0}{1 + \xi_1\Delta/2}, \qquad M = -\frac{1}{2} H_0 r_2^2 \left(1 - \frac{1}{1 + \xi_1\Delta/2}\right) \quad (6)$$

and, finally, for $\xi_1 \gg 1$, $\Delta\xi_1 \ll 1$, we have $H_1 = H_0$, $M = -\frac{1}{4} H_0 r_2^2 \xi_1 \Delta$.

Finally, we write out the formulas for the field H_1 and the magnetic moment M of a hollow cylinder in a field parallel to its surface. Making use of the solution of Ginzburg [2] (see also [7]), we find

$$H_1 = \left[\frac{n\Phi_0}{\pi r_1^2} + \frac{2H_0\xi_1^{-2}}{K_0(\xi_1) I_0(\xi_2) - I_c(\xi_1) K_0(\xi_2)}\right]$$

$$\times \left[1 + \frac{2}{\xi_1} \frac{K_0(\xi_2) I_1(\xi_1) + I_0(\xi_2) K_1(\xi_1)}{K_0(\xi_1) I_0(\xi_2) - I_0(\xi_1) K_0(\xi_2)}\right]^{-1},$$

$$M = -\frac{1}{4} \frac{r_2^2 [aI_2(\xi_2) - bK_2(\xi_2)] - r_1^2 [aI_2(\xi_1) - bK_2(\xi_1)]}{K_0(\xi_1) I_0(\xi_2) - I_0(\xi_1) K_0(\xi_2)},$$

$$a = H_0 K_0(\xi_1) - H_1 K_0(\xi_2), \qquad b = H_0 I_0(\xi_1) - H_1 I_0(\xi_2). \quad (7)$$

Here n is a quantum number which takes into consideration the presence inside the cylinder of a magnetic flux, $\Phi_0 = hc/2e \approx 2 \times 10^{-7}$ G-cm^2 is the flux "quantum." For $\xi_1 \gg 1$ and $\Delta \ll 1$ we get from (7)

$$H_1 = \frac{H_0 + (n\Phi_0/\pi r_1^2)\xi_1\Delta/2}{1 + \xi_1\Delta/2}, \qquad M = -\frac{r_2^2}{4} \frac{\xi_1\Delta}{2} \frac{H_0 - n\Phi_0/\pi r_1^2}{1 + \xi_1\Delta/2}. \quad (8)$$

Let us consider the case in which there is no magnetic flux inside the cylinder ($n = 0$) and the field H_0 is produced by a solenoid wound on the cylinder so that the field H_1 inside the cavity arises only because of the penetration of the external field H_0 through the lateral surface of the cylinder. Here, as is seen from (8), the field inside the thin cylinder and its moment will be the same as for a bulky specimen if $\xi_1\Delta \gg 1$. As $\xi_1\Delta \to 0$, of course, $M \to 0$.

2. For an analysis of the problem of the destruction of the superconductivity of specimens by a magnetic field, it is convenient to employ the superconductivity theory of Ginzburg-Landau.[1,3] In this theory, a certain quasi-wave function Ψ is introduced; under definite conditions, this function can be considered a constant. We shall consider these conditions to be satisfied. The value of Ψ, which plays the role of the relative concentration of superconducting electrons, is found from the condition of a minimum difference Φ of the free energies in the superconducting and normal states. In our case, the quantity Φ can be rewritten in the form (see [8])

$$\Phi = \Psi^4 - 2\Psi^2 - \frac{4\pi}{H_{cm}^2 V} \mathbf{M}\mathbf{H}_0. \quad (9)$$

Here, V is the superconducting volume, $\mathbf{M}$ is the moment of the sample in the external field $\mathbf{H}_0$, and H_{cm} is the critical magnetic field of the bulky continuous specimen.

From an analysis of the dependence of Φ on Ψ and H_0, one can draw a number of conclusions on the behavior of the superconductor in the magnetic field. For example, we consider the case in which the destruction of the superconductivity by the field takes place through the action of a second-order phase transition. In this case, the value of $\Psi \to 0$, and from the condition $\partial\Phi/\partial\Psi = 0$ (which corresponds to a minimum free energy) we find the value H_{c1}—the critical fields for second-order transitions. By using concrete expressions for the momenta M from Sec. 1, we get:

in the case of a hollow sphere with an internal radius r_1 and external radius $r_2 = r_1 + d$:

$$\frac{H_{c1}}{H_{cm}} = 2\sqrt{5}\,\delta_0 \sqrt{\frac{r_2^3 - r_1^3}{r_2^5 - r_1^5}};$$

$$\text{for } \frac{d}{\delta_0} \ll 1 \text{ we have } \frac{H_{c1}}{H_{cm}} = 2\sqrt{3}\frac{\delta_0}{r_1}; \quad (10)$$

in the case of a hollow cylinder with radii r_1 and $r_2 = r_1 + d$, in a transverse field:

$$\frac{H_{c1}}{H_{cm}} = \frac{2\sqrt{2}\,\delta_0}{\sqrt{r_2^2 + r_1^2}}; \text{for } \frac{d}{\delta_0} \ll 1 \text{ we have } \frac{H_{c1}}{H_{cm}} = \frac{2\delta_0}{r_1}; \quad 11)$$

in the case of a hollow cylinder where the external field is parallel to its surface and the "frozen" flux is absent ($n = 0$), so that the field penetrates in the internal cavity of the cylinder only through the lateral walls,

$$\frac{H_{c1}}{H_{cm}}=\frac{4\delta_0}{\sqrt{r_2^2+r_1^2}};\text{for } \frac{d}{\delta_0}\ll 1 \text{ we have } \frac{H_{c1}}{H_{cm}}=2\sqrt{2}\frac{\delta_0}{r_1};\quad (12)$$

finally, in the case in which the external field is absent outside the cylinder[1)] but there is a "frozen" magnetic flux in the internal cavity in the direction of H_n,

$$\frac{H_{nc1}}{H_{cm}}=\frac{2\delta_0}{r_1^2}\sqrt{\frac{r_2^2-r_1^2}{\ln(r_2/r_1)}};$$

$$\text{for } \frac{d}{\delta_0}\ll 1 \text{ we have } \frac{H_{nc1}}{H_{cm}}=2\sqrt{2}\frac{\delta_0}{r_1}. \quad (13)$$

We note that the critical fields H_{c1} (10)—(13) for $d/\delta_0 \ll 1$ are significantly less in value than for the case of a plane film in which $H_{c1}/H_{cm} = 2\sqrt{6}\delta_0/d$.

3. Although we found Eqs. (10)—(13) formally for the critical fields of a second-order transition, it must be kept in mind that actually a second-order transition can take place only under certain limitations on the dimensions of the specimen. In fact, from the condition $\partial\Phi/\partial\Psi = 0$, one can find the value of the magnetic field $H(\Psi)$ corresponding to an equilibrium transition. It is easy to establish the fact that the condition which distinguishes between first and second order transitions has the form (compare[1,9])

$$\partial^2 H/\partial\Psi^2|_{\Psi=0}=0. \quad (14)$$

Equation (14) also gives a limitation on the dimensions of the specimen that is of interest to us. From (14) we find the critical dimension L_c: for a hollow sphere,

$$L_c\equiv\left[\frac{r_2^7-{}^7/_2 r_2^2 r_1^5+{}^5/_2 r_1^7}{r_2^5-r_1^5}\right]^{1/2}=\frac{\sqrt{21}}{2}\delta_0;\text{for } d\ll r_1 \text{ we have}$$

$$L_c\equiv\sqrt{r_1 d}=\frac{\sqrt{6}}{2}\delta_0; \quad (15)$$

for a hollow cylinder in a transverse field,

$$L_c\equiv\left[\frac{r_2^6-3r_2^2r_1^4+2r_1^6}{r_2^4-r_1^4}\right]^{1/2}$$

$$=\sqrt{3}\,\delta_0;\ \text{for } d\ll r_1 \text{ we have } L_c\equiv\sqrt{r_1 d}=\delta_0; \quad (16)$$

in the case of a hollow cylinder in a parallel field (in the absence of a frozen field inside) the expression for L_c is also given by Eq. (16); for a hollow cylinder in the absence of an external field, but in the presence of a frozen flux inside the cylinder,

$$L_c\equiv\left[\frac{r_2^2-r_1^2}{2\ln(r_2/r_1)}-r_1^2\right]^{1/2}=\delta_0;$$

$$\text{for } d\ll r_1 \text{ we have } L_c\equiv\sqrt{r_1 d}=\delta_0. \quad (17)$$

If the characteristic dimensions r_1 and r_2 of the samples are such that the value of L_c is less than the value appearing on the right hand side of Eqs. (15)—(17), respectively, then we have a second-order transition; in the opposite case, we have a first-order transition. It follows from Eqs. (15)—(17) that the conditions for observation of a second-order transition are difficult to achieve for hollow specimens, inasmuch as it is actually required that the product $r_1 d \lesssim \delta_0^2$ be small, which is difficult to realize. We recall that in the case, for example, of a film in a field parallel to its surface, a second-order transition takes place when its thickness is sufficiently small, that is, for $d < \sqrt{5}\,\delta_0$.[9]

Thus the destruction of superconductivity of a thin spherical or cylindrical film under the given conditions should take place via a first-order transition in contrast to a plane film of such a thickness where a second-order transition takes place.

4. We shall now find the value of the critical magnetic field H_c in the case of a first-order phase transition and, correspondingly, the value of Ψ for which an equilibrium transition exists. These quantities are found from the conditions [1,9]

$$\Phi=0,\quad \partial\Phi/\partial\Psi=0 \quad (18)$$

with account of the exact expressions given above for the magnetic moments of the specimens. In a number of limiting cases, these expressions are simplified and we shall use the following formulas.

In the case of a hollow sphere for $r_1/\delta_0 \gg 1$, $d/\delta_0 \ll 1$[2)]

$$\frac{H_c^2}{H_{cm}^2}=\frac{6\delta_0^2}{r_1^2}\left(1+\frac{1}{3}\frac{r_1 d}{\delta_0^2}+\frac{3}{4}\frac{\delta_0^2}{r_1 d}\right),\quad \Psi_c^2=1-\frac{3}{2}\frac{\delta_0^2}{r_1 d};\quad (19)$$

for $r_1 d \gg \delta_0^2$ we have $H_c^2/H_{cm}^2 = 2d/r_1$. If now $r_1/\delta_0 \gg 1$, $d/\delta_0 \gtrsim 1$, then Eqs. (18) take the form

$$\frac{H_c^2}{H_{cm}^2}=\frac{2\Psi^2(2-\Psi^2)(r_2^3-r_1^3)}{3r_2^3}\left[1-\frac{3\delta_0}{r_2\Psi}\operatorname{cth}\frac{\Psi d}{\delta_0}\right]^{-1}, \quad (20a)^*$$

[1)]In this latter case, it is convenient to write immediately the expression for $\partial\Phi/\partial\Psi$ in place of (9), by analogy to what was done in Sec. 6 to obtain Eq. (36).

[2)]Actually, here and below it is further required that the condition $r_1/\delta \gg 1$ be satisfied or $d/\delta \ll 1$, where $\delta = \delta_0/\Psi$; that is, it is assumed that Ψ is not a small quantity. In the region of a first-order transition, as is seen from the formulas, $\Psi \sim 1$, so that these conditions are equivalent.

*cth = coth, csch = cosech.

$$\frac{H_c^2}{H_{cm}^2} = \frac{8\Psi(1-\Psi^2)(r_2^3-r_1^3)}{9r_2^3} \times \left[\frac{\delta_0}{r_2\Psi^2}\operatorname{cth}\frac{\Psi d}{\delta_0} + \frac{d}{r_2\Psi}\operatorname{csch}^2\frac{\Psi d}{\delta_0}\right]^{-1}, \tag{20b}$$

while the values of Ψ_c and $H_c(\Psi_c)$ are found graphically at the point of intersection of the curves $H_c(\Psi)$, which are given by (20a) and (20b).

Finally, in the case $r_1/\delta_0 \gg 1$, $d/\delta_0 \gg 1$ (thick-walled sphere) we have

$$\frac{H_c^2}{H_{cm}^2} = \frac{2}{3}\frac{r_2^3-r_1^3}{r_2^3}\left(1+\frac{3}{2}\frac{\delta_0}{r_2}\right)^2, \quad \Psi_c = 1-\frac{3}{8}\frac{\delta_0}{r_2}. \tag{21}$$

In the case of a hollow cylinder in a transverse field for $r_1/\delta_0 \gg 1$, $d/\delta_0 \ll 1$,

$$\frac{H_c^2}{H_{cm}^2} = \frac{2\delta_0^2}{r_2^2}\left(1+\frac{1}{2}\frac{r_1 d}{\delta_0^2}+\frac{1}{2}\frac{\delta_0^2}{r_1 d}\right), \quad \Psi_c^2 = 1-\frac{\delta_0^2}{r_1 d}; \tag{22}$$

for $r_1 d/\delta_0^2 \gg 1$, we have $H_c^2/H_{cm}^2 = d/r_1$. If $r_1/\delta_0 \gg 1$, $d/\delta_0 \gtrsim 1$, then Eqs. (18) take the form

$$\frac{H_c^2}{H_{cm}^2} = \frac{\Psi^2(2-\Psi^2)(r_2^2-r_1^2)}{2r_2^2}\left[1-\frac{2\delta_0}{r_2\Psi}\operatorname{cth}\frac{\Psi d}{\delta_0}\right]^{-1}, \tag{23a}$$

$$\frac{H_c^2}{H_{cm}^2} = \frac{\Psi(1-\Psi^2)(r_2^2-r_1^2)}{r_2^2}\left[\frac{\delta_0}{r_2\Psi^2}\operatorname{cth}\frac{\Psi d}{\delta_0}+\frac{d}{r_2\Psi}\operatorname{csch}^2\frac{\Psi d}{\delta_0}\right]^{-1}, \tag{23b}$$

while the values of Ψ_c and $H_c(\Psi_c)$ are found graphically at the point of intersection of the curves of $H_c(\Psi)$ which are given by (23a, b).

Finally, in the case $r_1/\delta_0 \gg 1$, $d/\delta_0 \gg 1$ (thick-walled cylinder), we have

$$\frac{H_c^2}{H_{cm}^2} = \frac{r_2^2-r_1^2}{2r_2^2}\left(1+\frac{\delta_0}{r_2}\right)^2, \quad \Psi_c = 1-\frac{1}{4}\frac{\delta_0}{r_2}. \tag{24}$$

For a hollow cylinder in a parallel field (in the absence of a frozen flux) the equations (22)—(24) hold for all the cases considered; here only the right hand sides of the equations for H_c^2 must be multiplied by 2.

In the case of a hollow cylinder with a frozen field inside and in the absence of an external field, for the critical values of H_{nc} and Ψ_c we have Eqs. (22)—(24), where the right hand side of H_c^2 must also be multiplied by 2; furthermore, the substitution $r_2 \rightarrow r_1$ must be made in all formulas (in the factors $r_2^2 - r_1^2$ the substitution need not be made).

We note that the expressions obtained for the critical field H_c in the case $d/\delta_0 \ll 1$ lead to much smaller values than for a plane film where $H_c/H_{cm} = 2\sqrt{6}\,\delta_0/d$.

5. It is known [1,3,9] that in the case of the first-order phase transition the existence of a metastable normal state is possible in a field $H_0 < H_c$ ("supercooling") and a metastable superconducting state in a field $H_0 > H_c$ ("superheating"). It is not difficult to determine the limits of the regions of "supercooling" and "superheating." The field H_{c1} below which the normal state can generally not exist is determined by the condition

$$\partial\Phi/\partial\Psi|_{\Psi=0} = 0, \tag{25}$$

which leads to Eqs. (10)—(13).

The field H_{c2} above which the superconducting state with the given field configuration cannot exist[3)] is determined by the conditions

$$\partial\Phi/\partial\Psi = 0, \qquad \partial^2\Phi/\partial\Psi^2 = 0, \tag{26}$$

from which the corresponding value of Ψ_{c2} is simultaneously determined.

In the case of a sphere for $r_1/\delta_0 \gg 1$, $d/\delta_0 \ll 1$, we find from (26)

$$\frac{H_{c2}^2}{H_{cm}^2} = \frac{16}{9}\frac{\delta_0^2}{r_1^2}\left(1+\frac{1}{3}\frac{r_1 d}{\delta_0^2}\right)^2\left(1+\frac{3\delta_0^2}{r_1 d}\right), \quad \Psi_{c2}^2 = \frac{2}{3}\left(1-\frac{3}{2}\frac{\delta_0^2}{r_1 d}\right); \tag{27}$$

for $r_1 d/\delta_0^2 \gg 1$, we have $H_{c2}^2/H_{cm}^2 = 16d^2/81\delta_0^2$. If $r_1/\delta_0 \gg 1$, $d/\delta_0 \gtrsim 1$, then the first of Eqs. (26) reduces to Eq. (20b), while the second of Eqs. (26) takes the form

$$\frac{H_{c2}^2}{H_{cm}^2} = \frac{4(3\Psi^2-1)(r_2^3-r_1^3)}{9r_2^3}\left[\frac{\delta_0}{r_2\Psi^3}\operatorname{cth}\frac{\Psi d}{\delta_0}+\frac{d}{r_2\Psi^2}\operatorname{csch}^2\frac{\Psi d}{\delta_0} + \frac{d^2}{r_2\delta_0\Psi}\operatorname{csch}^2\frac{\Psi d}{\delta_0}\operatorname{cth}\frac{\Psi d}{\delta_0}\right]^{-1}. \tag{28}$$

Finally, in the case $r_1/\delta_0 \gg 1$, $d/\delta_0 \gg 1$, we have

$$\frac{H_{c2}^2}{H_{cm}^2} = \frac{16}{75}\sqrt{\frac{3}{5}}\,\frac{r_2^3-r_1^3}{r_2^2\delta_0}, \quad \Psi_{c2} = \sqrt{\frac{3}{5}}. \tag{29}$$

In the case of a hollow cylinder in a transverse field for $r_1/\delta_0 \gg 1$, $d/\delta_0 \ll 1$,

$$\frac{H_{c2}^2}{H_{cm}^2} = \frac{4}{27}\frac{\delta_0^2}{r_2^2}\left(2+\frac{r_1 d}{\delta_0^2}\right)^2\left(1+\frac{2\delta_0^2}{r_1 d}\right), \quad \Psi_{c2}^2 = \frac{2}{3}\left(1-\frac{\delta_0^2}{r_1 d}\right); \tag{30}$$

for $r_1 d/\delta_0^2 \gg 1$, we have $H_{c2}^2/H_{cm}^2 = 4d^2/27\,\delta_0^2$. If $r_1/\delta_0 \gg 1$, $d/\delta_0 \gtrsim 1$, the first of Eqs. (26) leads

3)In fields exceeding H_{c2}, a purely superconducting state, screening the external field and not permitting it inside the hollow, is thermodynamically inconvenient. Actually, the same specimen can undergo a transition to an intermediate state in which the field penetrates inside the cavity through the normal parts; here the superconductivity is frequently maintained up to much higher values of the field. Thus, in particular, a hollow superconducting sphere will screen the field inside the cavity only for a sufficiently small external field.

to Eq. (23b), while the second of Eqs. (26) takes the form

$$\frac{H_{c2}^2}{H_{cm}^2}=\frac{(3\Psi^2-1)(r_2^2-r_1^2)}{2r_2^2}\left[\frac{\delta_0}{r_2\Psi^3}\operatorname{cth}\frac{\Psi d}{\delta_0}+\frac{d}{r_2\Psi^2}\operatorname{csch}^2\frac{\Psi d}{\delta_0}+\frac{d^2}{r_2\delta_0\Psi}\operatorname{csch}^2\frac{\Psi d}{\delta_0}\operatorname{cth}\frac{\Psi d}{\delta_0}\right]^{-1}. \tag{31}$$

Finally, in the case $r_1/\delta_0 \gg 1$, $d/\delta_0 \gg 1$, we have

$$\frac{H_{c2}^2}{H_{cm}^2}=\frac{6}{25}\sqrt{\frac{3}{5}}\frac{r_2^2-r_1^2}{r_2\delta_0},\quad \Psi_{c2}=\sqrt{\frac{3}{5}}. \tag{32}$$

The Eqs. (30)–(32) also hold for H_{c2} and Ψ_{c2} of a cylinder in a parallel field for the two cases considered—the presence and absence of a frozen flux—with limitations similar to those given at the end of Sec. 4 at Eq. (24).

We note that the fields H_{c2} obtained above for $d/\delta_0 \ll 1$ are appreciably smaller than in the case of a plane film.

6. Finally, let us consider the destruction of the superconductivity of a hollow cylinder by the current. If the total current I flows along the cylinder, then, as is not difficult to verify, the z component of the vector potential of the field inside the walls of the cylinder has the form

$$A_z(\xi)=\frac{\delta_0}{\Psi}H_I\frac{I_0(\xi)K_1(\xi_1)+K_0(\xi)I_1(\xi_1)}{I_1(\xi_1)K_1(\xi_2)-I_1(\xi_2)K_1(\xi_1)}, \tag{33}$$

where $I_l(\xi)$ and $K_l(\xi)$ are Bessel functions, $\xi=\Psi r/\delta_0$, $H_I = 2I/cr_2$, and the distribution of the field is found from the formula $H(\xi) = -\partial A_z/\partial r$. In the case $\xi_1 = 0$, we have $A_z(\xi) = -(\delta_0 H_I/\Psi)\times I_0(\xi)/I_1(\xi_2)$—the result of Silin.[6] The current density inside the superconductor is $j_z = -(c/4\delta^2)A_z$ and, of course, the integral over the cross section $\int j_z dS = I$.

If we write down the expression for the free energy of the superconductor with a current in the usual form (compare [1,3]), then the following equation for the function Ψ ensues from it:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Psi}{dr}\right)=\frac{\kappa^2}{\delta_0^2}\left\{-1+\Psi^2+\frac{1}{2\delta_0^2H_{cm}^2}A_z^2\right\}\Psi, \tag{34}$$

κ is some constant. By requiring the vanishing of the derivative $d\Psi/dr$ at $r = r_1$ and $r = r_2$ (see [1,3]), we get the following condition from (34):

$$\int_{r_1}^{r_2}\{-1+\Psi^2+A_z^2/2\delta_0^2H_{cm}^2\}\,r\,dr=0, \tag{35}$$

which it is convenient to transform to[4])

$$\Psi(1-\Psi^2)(r_2^2-r_1^2)=\tfrac{1}{2}H_{cm}^{-2}r_2H_I\,dA_z(\xi_2)/d\Psi. \tag{36}$$

Here we shall assume the quantity Ψ to be a constant. The expression (33) must be used for A_z.

It is easy to verify the fact that the relation (36) is nothing else than the vanishing of the derivative of the free energy $\partial\Phi/\partial\Psi = 0$.

For $\xi_1 \gg 1$, $\xi_2 - \xi_1 \ll 1$, in particular, we get at once from (36)

$$\frac{H_I^2}{H_{cm}^2}=\frac{2d^2}{\delta_0^2}\Psi^4(1-\Psi^2), \tag{37}$$

where $d = r_2 - r_1$ is the thickness of the wall of the cylinder. The critical field H_{Ic} is determined from the condition $dH_I/d\Psi = 0$, whence

$$\Psi_c=\sqrt{\frac{2}{3}},\quad \frac{H_{Ic}}{H_{cm}}=\frac{2}{3}\sqrt{\frac{2}{3}}\frac{d}{\delta_0}. \tag{38}$$

This value of the critical field is twice as large as for destruction of the superconductivity of a plane film by a current, in correspondence with the result of Ginzburg.[10]

7. A characteristic peculiarity of the expressions obtained above for the critical magnetic fields H_{ci} is their smallness in comparison with the case of a plane film. In order to understand better the reason for such a difference, we shall consider the distribution of the field and current over the radius of the equator of a thin hollow sphere (in the case of a cylindrical film, similar formulas exist). Assuming $r = r_1 + x$, where $0 \le x \le d$, and assuming $\Psi \sim 1$, $d/\delta_0 \ll 1$, $r_1/\delta_0 \gg 1$, we get from (2)

$$H_{eq}=H_0\frac{1+r_1x/2\delta_0^2}{1+r_1d/3\delta_0^2},\quad \mathbf{j}_{eq}=-\mathbf{i}_\varphi H_0\frac{cr_1}{8\pi\delta_0^2}\frac{1+x^2/2\delta_0^2}{1+r_1d/3\delta_0^2}. \tag{39}$$

It is then evident that the screening current flowing along the film is practically constant over the thickness while the current in the film changes rapidly (for $r_1d \gg \delta_0^2$) from a value $\sim H_0$ on the outer surface to the value $\sim\delta_0^2H_0/r_1d \ll H_0$ on the inner surface, in spite of the fact that the film itself can be very thin, $d \ll \delta_0$. Here the behavior of the hollow superconducting film is completely analogous to the ordinary solenoid in which the current flowing along it creates a magnetic field inside the solenoid; this magnetic field changes discontinuously in passing through the surface current.

Now the reason for the difference in the critical fields mentioned above becomes obvious. In the case of a hollow film, there is a large jump in the value of the magnetic field inside and outside the cavity, just as would be the case for a continuous sample of the same dimensions. But, if in the case of a continuous sample, the non-uniformity of the

4)In this case the equation $(\text{curl curl } A)_z = -\delta^{-2}A_z$ is used; this must be differentiated with respect to Ψ and later integrated by parts.

configuration of the magnetic field (which leads to a positive contribution to the free energy) is compensated by the presence of a negative difference of the potential energies $(F_{s0} - F_{n0})V$ of the superconducting and normal states of the specimen, then in the case of a thin film this negative contribution is much smaller inasmuch as there is only the volume of the external thin layer in place of the hollow volume V. Therefore, in a weak field $H \ll H_{cm}$, the purely superconducting state of the hollow film is shown to be thermodynamically inconvenient. A plane film, as is well known, can exist in the superconducting state in fields much larger than H_{cm}.

What has been said above also applies to hollow samples for which $d \gg \delta_0$, $r_1 \gg d$. In this case, as is seen from the formulas given above, $H_c/H_{cm} \ll 1$ also. We note that in the work of Kontarev [11] penetration of the external magnetic field inside the cavity of thick cylindrical films was initially observed for $H \sim \frac{1}{2} H_{cm}$, i.e., in a field exceeding the equilibrium critical field H_c of a thin-walled cylinder as obtained above. In the light of what has been said, one must interpret this fact as the appearance of a metastable "superheated" state which is observed with difficulty under ordinary conditions.

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[1] V. L. Ginzburg, UFN 42, 169 (1950).

[2] V. L. Ginzburg, JETP 42, 299 (1962), Soviet Phys. JETP 15, 207 (1962).

[3] V. L. Ginzburg and L. D. Landau, JETP 20, 1064 (1950).

[4] G. F. Zharkov, JETP 42, 1397 (1962), Soviet Phys. JETP 15, 968 (1962).

[5] G. F. Zharkov and Hsü Lung-tao, FMM, 15, 12 (1963).

[6] V. P. Silin, JETP 21, 1330 (1951).

[7] Lipkin, Peshkin, and Tassie, Phys. Rev. 126, 116 (1962).

[8] V. L. Ginzburg, JETP 34, 113 (1958), Soviet Phys. JETP 7, 78 (1958).

[9] V. L. Ginzburg, UFN 68, 25 (1952).

[10] V. L. Ginzburg, DAN SSSR 118, 464 (1958), Soviet Phys. Doklady 3, 102 (1959).

[11] V. Ya. Kontarev, ZhTF 31, 854 (1961), Soviet Phys. Tech. Phys. 6, 617 (1962).

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332