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INVARIANT PARAMETRIZATION OF LOCAL OPERATORS

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A general method is developed by means of which the matrix elements of local operators of arbitrary tensor or spinor dimensionality, taken between states with either one or several particles of arbitrary masses and spins, can be expressed in terms of invariant form-factors. Such an invariant parametrization is necessary for the study of various dispersion relations, and also for the realization of a previously indicated program [1] for the study of the structure of elementary particles and of the limits of applicability of present ideas about the structure of space-time.

1. In present relativistic quantum theory a large part is played by the matrix elements of various local operators (fields, currents), taken in the Heisenberg representation between states of free (in the infinite past or future) particles. Each of these matrix elements can be parametrized, i.e., expressed in terms of a finite number of invariant functions, called form-factors. There has hitherto been no general method for this sort of parametrization. While in the simplest cases of spins 0 and $1/2$ the parametrization can be done with primitive methods, without a general method it is practically impossible to construct, for example, the matrix element of the energy-momentum tensor operator for the photodisintegration of the deuteron or the matrix element of the electromagnetic current for a nucleus with spin $9/2$. Yet this sort of parametrization is necessary for the theoretical study of processes of production and scattering of elementary particles by means of dispersion relations and the associated methods. A general method of parametrization is also necessary for carrying out a previously indicated program [1] for the study of the structure of elementary particles and the limits of applicability of present ideas about the structure of space-time.

The purpose of the present paper is the parametrization of matrix elements of local quantities of arbitrary tensor dimensionality, taken between states with either one or several particles of arbitrary masses and spins. The basis is the technique used by the writers for the parametrization of the scattering matrix. [2] It has been necessary, however, to develop this technique much further, since when we go over to the parametrization of local operators there are two additional difficulties which were not present in the analogous problem for the scattering matrix. The first is that in the parametrization of matrix elements of local operators one must deal not only with timelike four-momenta, but also with spacelike ones. The second difficulty is connected with the possible tensor or spinor indices of local operators.

In Section 2 we carry out the parametrization of the matrix elements of a scalar local quantity between one-particles states with arbitrary spin. In Section 3 the results are extended to the case of a local quantity of nontrivial tensor dimensionality (the vector and the second-rank symmetric tensor are considered). In Section 6 we give the parametrization of matrix elements of local operators for a system of particles.

2. The motion of a free particle is described by a state vector

$$|\,\mathbf{p}, \varkappa, j, m, \alpha\rangle, \tag{1}$$

where $\mathbf{p}$, κ, j, m are respectively the momentum, the mass, the spin, and the projection of the spin on the z axis, and α represents the other possible invariant characteristics of the particle (for example, the charge). We adopt for the state vectors (1) the normalization

$$\langle \mathbf{p}, \varkappa, j, m, \alpha|\,\mathbf{p}', \varkappa, j, m', \alpha'\rangle = \delta_{mm'}\delta^3(\mathbf{p}-\mathbf{p}')\,\delta_{\alpha\alpha'}. \tag{2}$$

The invariant index α is unimportant for the present work and will be omitted hereafter.

Because of translational invariance we have for a local operator A(x) of any tensor dimensionality the relation

$$\langle \mathbf{p}, \varkappa, j, m|A(x)|\,\mathbf{p}', \varkappa, j, m'\rangle = e^{-i(p-p')x}\langle \mathbf{p}, \varkappa, j, m|A(0)|\,\mathbf{p}', \varkappa, j, m'\rangle, \tag{3}$$

so that it suffices to carry out the parametrization of the operator $A(0)$. If the operator $A(x)$ is a scalar and the spin of the particle is zero, the one-particle matrix element can be expressed in terms of one invariant form-factor $f(t)$, which depends on the invariant t:

$$\langle \mathbf{p}, \varkappa \,|\, A(0)\,|\mathbf{p}', \varkappa\rangle = f(t)/(2\pi)^3 \sqrt{4p_0p_0'}, \quad (4)$$

$$t = -(p-p')^2 \equiv (p_0 - p_0')^2 - (\mathbf{p}-\mathbf{p}')^2, \qquad p_0 = \sqrt{\mathbf{p}^2 + \varkappa^2}. \quad (5)$$

The factor $(4p_0p_0')^{1/2}$ in the denominator of Eq. (4) appears owing to the choice of the normalization in the form (2).

The parametrization of a scalar operator for a particle with spin differs from the simplest parametrization (4) in two respects. First, the number of form-factors is larger, being $j+1$ for integer j and $j+\frac{1}{2}$ for half-integer j. (If we do not take parity conservation into account, the number of form-factors is $2j+1$.) Second, the matrix element of the operator $A(x)$ will contain the operator $D(\alpha, \beta, \gamma)$ of a certain three-dimensional rotation of the spin. The appearance of this three-dimensional rotation (first introduced in [3]) is due to the fact that the Lorentz transformation for the spin depends on the momentum of the particle (cf. e.g., [3,4]). Therefore in a matrix element of the operator $A(0)$ which is nondiagonal in the momentum the spin rotations for the right-hand and left-hand spin indices are different. Figuratively speaking, the right-hand and left-hand spin indices "sit" on different momenta. Therefore to get the invariant parametrization it is necessary to "set" both spins on the same momentum, and this brings in the three-dimensional spin rotation in question.

The transfer of the left-hand spin index from the momentum $\mathbf{p}$ to the momentum $\mathbf{p}'$ is accomplished by the transformation

$$\langle \mathbf{p}, \varkappa, j, m\,|\,A(0)\,|\,\mathbf{p}', \varkappa, j, m'\rangle = \sum_{m''} D^j_{mm''}(\mathbf{p}, \mathbf{p}')\langle \mathbf{p}, \varkappa, j, m''\,|\,A(0)\,|\,\mathbf{p}', \varkappa, j, m'\rangle, \quad (6)$$

where $D^j_{mm''}(\mathbf{p}, \mathbf{p}')$ is the matrix for the three-dimensional rotation (cf. [5]) with the Euler angles α, β, γ given by

$$\alpha = \operatorname{arc\,tg}\left\{\frac{[(p_0+\varkappa)(p_0'+\varkappa) - \mathbf{p}\mathbf{p}']\, r_x + r_y r_z}{[(p_0+\varkappa)(p_0'+\varkappa) - \mathbf{p}\mathbf{p}']\, r_y - r_x - r_z}\right\},$$

$$\beta = \arccos\left\{1 - \frac{r_x^2 + r_y^2}{(p_0+\varkappa)(p_0'+\varkappa)(p_0p_0' - \mathbf{p}\mathbf{p}' + \varkappa^2)}\right\},$$

$$\gamma = \operatorname{arc\,tg}\left\{\frac{r_y r_z - r_x[(p_0+\varkappa)(p_0'+\varkappa) - \mathbf{p}\mathbf{p}']}{r_x r_z + r_y[(p_0+\varkappa)(p_0'+\varkappa) - \mathbf{p}\mathbf{p}']}\right\} \quad (7)^*$$

$(\mathbf{r} = [\mathbf{p} \times \mathbf{p}'])$. For spin $\frac{1}{2}$ the matrix $D^{1/2}$ is calculated in [3] and is given by

$$D^{*1/2} = \frac{(p_0+\varkappa)(p_0'+\varkappa) - (\mathbf{p}\sigma)(\mathbf{p}'\sigma)}{\{2(p_0+\varkappa)(p_0'+\varkappa)[p_0p_0' - \mathbf{p}\mathbf{p}' + \varkappa^2]\}^{1/2}}, \quad (8)$$

where σ is the Pauli matrices.

In the matrix element in Eq. (6) the left-hand and right-hand indices are already "sitting" on the same momentum, so that, for example, the quantity $\delta_{m''m}$ is a scalar under Lorentz transformations. For the same reason the spin operator $\Gamma(\mathbf{p}')$ written to the right of the rotation $D^j(\alpha\beta\gamma)$ (cf. [6])

$$\Gamma(\mathbf{p}') = \varkappa\mathbf{j} + \mathbf{p}'(\mathbf{p}'\mathbf{j})/(p_0'+\varkappa), \qquad \Gamma_0(\mathbf{p}') = \mathbf{p}'\mathbf{j},$$

$$[j_i j_k]_- = i\varepsilon_{ikl} j_l, \qquad \mathbf{j}^2 = j(j+1), \quad (9)$$

behaves like a four-vector. Furthermore, to complete the invariant parametrization of the one-particle scalar matrix element, it is necessary to list all of the linearly independent scalars which can be formed from the unit operator and the vectors p_μ, p'_μ, $\Gamma_\mu(\mathbf{p}')$. Since $p'_\mu\Gamma_\mu(\mathbf{p}') = 0$, the linearly independent scalars under rotations (i.e., the scalars and pseudoscalars) will be the $2j+1$ quantities $\{p_\mu\Gamma_\mu(\mathbf{p}')\}^n$, where $n = 0, 1, 2, \ldots, 2j$. Furthermore the even values of n correspond to scalars, and the odd to pseudoscalars. To each linearly independent scalar operator there will correspond an invariant form-factor $f_n(t)$, so that the formula for the parametrization of the operator $A(0)$ can be written in the form

$$\langle \mathbf{p}, \varkappa, j, m\,|\,A(0)\,|\,\mathbf{p}', \varkappa, j, m'\rangle = (2\pi)^{-3}(4p_0p_0')^{-1/2}\sum_{m''} D^j_{mm''}(p, p') \times \sum_{n=0}^{2j}\langle m''\,|\,\{ip_\mu\Gamma_\mu(\mathbf{p}')\}^n\,|\,m'\rangle f_n(t), \quad (10)$$

where n runs through the even values from 0 to $2j$ for a scalar operator, and through the odd values for a pseudoscalar operator.

We emphasize that the use of the operator $\Gamma_\mu(\mathbf{p})$ instead of $\Gamma_\mu(\mathbf{p}')$ gives nothing new, since, according to [3], these operators are connected by the relation

$$\Gamma_\mu(\mathbf{p})D(p, p') = D(p, p')\left\{\Gamma_\mu(\mathbf{p}') + \frac{p_\mu + p_\mu'}{\varkappa^2 - p_\lambda p_\lambda'}\, p_\nu\Gamma_\nu(\mathbf{p}')\right\} \quad (11)$$

and since the operator $\Gamma_\mu(\mathbf{p})\,[\Gamma_\mu(\mathbf{p}')]$ is covariant only if it stands on the left (right) of $D(\mathbf{p}, \mathbf{p}')$. We note that according to Eq. (11)

$$p_\mu'\Gamma_\mu(\mathbf{p})D(p, p') = -D(p, p')\,p_\mu\Gamma_\mu(\mathbf{p}'), \quad (12)$$

so that for real form-factors all of the terms in Eq. (10) are real.

3. The extension of the parametrization (10) to the case of an operator which has a tensor dimen-

*tg = tan.

sionality is not difficult if we use the fact that from the variables p, p′, m, m′ we can form only one axial vector $\Gamma_\mu(\mathbf{p}')$ and three independent vectors

$$K_\mu, \; K'_\mu, \; R_\mu = \varepsilon_{\mu\nu\lambda\sigma} p_\nu p'_\lambda \Gamma_\sigma(p'),$$

where

$$K_\mu = p_\mu - p'_\mu, \qquad K^2 = -t,$$
$$K'_\mu = p_\mu + p'_\mu, \qquad K^2_\mu = -u, \quad u + t = 4\varkappa^2. \qquad (13)$$

From these vectors we must form all the independent expressions that have the tensor dimensionality of the operator in question, and substitute them in Eq. (10) on the right of the matrix D(p, p′). Thus the parametrization of a vector operator $J_\mu(x)$ can be written in the form

$$\langle \mathbf{p}, \varkappa, j, m | J_\mu(x) | \mathbf{p}', \varkappa, j, m' \rangle$$
$$= \frac{\exp(-iK_\lambda x_\lambda)}{(2\pi)^3 (4p_0 p'_0)^{1/2}} \sum_{m''} D^j_{mm''}(p, p') \langle m'' | F_1 K'_\mu$$
$$+ F_2 \{\Gamma_\mu + (K_\mu/t + K'_\mu/u) p_\nu \Gamma_\nu\} + iF_3 K_\mu + iF_4 K_\mu | m' \rangle, \qquad (14)$$

where

$$F_i = \sum_n f_{in}(t) Q^n, \qquad Q \equiv ip_\mu \Gamma_\mu, \qquad (15)$$

and the summation over n is restricted by parity conservation (if it holds) and by the fact that each term can contain not more than 2j factors Γ_μ. Therefore in F_1 and F_4 the summation goes over even n in the range $2j \geq n \geq 0$, in F_3 over even n in the range $2j - 1 \geq n \geq 0$, and in F_2 over odd n in the range $2j - 1 \geq n \geq 0$. By means of Eq. (11) one can verify that the coefficient of each form-factor in Eq. (14) is a Hermitian operator. If $J_\mu(x)$ is the operator of a conserved current, it satisfies the conservation law

$$J_\mu(x) K_\mu = 0. \qquad (16)$$

In this case $F_4 = 0$ in Eq. (14) and the quantities $f_{1,0}(0)$, $f_{3,0}(0)$, $\sigma F'_{1,0}(0)$ are respectively the charge, the magnetic moment, and the mean square radius of the charge distribution.

In analogy with this, we can parametrize the matrix element of a second-rank symmetric tensor $T_{\mu\nu}(x)$, which for definiteness we shall take to be the energy-momentum tensor, in the following way:

$$\langle \mathbf{p}, \varkappa, j, m | T_{\mu\nu}(0) | \mathbf{p}', \varkappa, j, m' \rangle$$
$$= \frac{1}{(2\pi)^3 (4p_0 p'_0)^{1/2}} \sum_{m''} D^j_{mm''}(p, p') \langle m'' | \tau_{\mu\nu}(0) | m' \rangle; \qquad (17)$$

$$\tau_{\mu\nu}(0) = \tfrac{1}{2} G_1 K'_\mu K'_\nu + G_2 \Gamma'_\mu \Gamma'_\nu + G_3 (K'_\mu \Gamma'_\nu + \Gamma'_\mu K'_\nu)$$
$$+ iG_4 (K'_\mu R_\nu + R_\mu K'_\nu) + iG_5 (\Gamma'_\mu R_\nu + R_\mu \Gamma'_\nu)$$
$$+ G_6 (t\delta_{\mu\nu} + K_\mu K_\nu) + G_7 (t\delta_{\mu\nu} - K_\mu K_\nu)$$
$$+ iG_8 (K_\mu \Gamma'_\nu + \Gamma'_\mu K_\nu) + iG_9 (K_\mu K'_\nu + K'_\mu K_\nu)$$
$$+ G_{10} (K_\mu R_\nu + R_\mu K_\nu),$$
$$\Gamma'_\mu \equiv \Gamma_\mu(\mathbf{p}') + (K_\mu/t + K'_\mu/u) p_\lambda \Gamma_\lambda(\mathbf{p}'). \qquad (18)$$

Here the scalars G_i ($i = 1, 2, \ldots, 10$) are connected with the invariant real form-factors by the relations

$$G_i = \sum_n g_{in}(t) Q^n, \qquad (19)$$

where the summation is over even n in G_1, G_2, G_4, G_6, G_7, G_9, G_{10} and over odd n in the other G_i, and the ranges are $2j \geq n \geq 0$ for G_1, G_6, G_7, G_9, $2j - 1 \geq n \geq 0$ for G_3, G_4, G_8, G_{10}, and $2j - 2 \geq n \geq 0$ for G_2, G_5. If we do not require parity conservation, the summation is to be over both even and odd n in all cases.

The energy-momentum tensor satisfies the conservation law

$$T_{\mu\nu} K_\nu = 0. \qquad (20)$$

Substituting Eq. (20) in Eqs. (17) and (8), we get

$$G_7 = G_8 = G_9 = G_{10} = 0. \qquad (21)$$

In particular, it follows from Eqs. (18)—(21) that the matrix element of the energy-momentum tensor for a particle with spin 0 reduces to two form-factors (cf. [1]), that for a particle with spin ½ to three, and that for a particle with spin 1 to seven.

For example, for spin ½ the parametrization of $T_{\mu\nu}(x)$ is of the form

$$\langle \mathbf{p}, \varkappa, {}^1/_2, m, | T_{\mu\nu}(x) | \mathbf{p}', \varkappa, {}^1/_2, m' \rangle$$
$$= (2\pi)^{-3} (4\, p_0 p'_0)^{-1/2} e^{-i(p-p')x} \sum_{m''} D^{1/2}_{mm''}(p, p')$$
$$\times \langle m'' | \{{}^1/_2 g_{1,0}(t) K'_\mu K'_\nu + ig_{4,0}(t) (K'_\mu R_\nu + R_\mu K'_\nu)$$
$$+ g_{6,0}(t) (t\delta_{\mu\nu} + K_\mu K_\nu)\} | m' \rangle. \qquad (22)$$

As in the case of the current, the values of the form-factors and their derivatives at zero argument have direct physical meanings: $g_{1,0}(0) = 1$ describes the mass. The form-factor $g_{6,0}(t)$ describes the internal forces (for example, pressure) in the particle. This form-factor does not contribute to the mass, but does contribute, for example, to the mean square radius $r_\varkappa$ of the particle:

$$r^2_\varkappa = 3\varkappa^{-2} \{\varkappa^2 g'_{1,0}(0) + g_{6,0}(0)\}. \qquad (23)$$

The description (1) of the motion of a free particle in terms of the momentum and the spin projection on the z axis is not suitable when the rest mass is zero. In this case the state vector is taken in the form

$$|\,\mathbf{p},\lambda\rangle, \qquad (24)$$

where λ is the helicity, i.e., the spin projection along the momentum, which for any nonvanishing spin j runs through the two values $\lambda = \pm j$. The helicity is invariant under proper Lorentz transformations, but changes sign when the space axes are inverted.[1] Here the axial vector $\Gamma_\mu(p')$ is not independent, but simply equal to $\lambda p'_\mu$. This simplifies the finding of the independent form-factors, and in general reduces their number.

For the case of zero rest mass the three-dimensional rotation $D(p, p')$ of Eq. (10) is replaced by a phase factor $W_{\lambda\lambda''}(p, p')$:

$$W_{\lambda\lambda''}(p, p') = e^{-i\lambda\chi(p,p')}\delta_{\lambda\lambda''}, \qquad (25)$$

where

$$\operatorname{tg}\chi = \frac{\sin(\theta'-\theta)\sin(2\varphi-\varphi')}{\sin\theta\cos(\theta'-\theta)+\cos\theta\sin(\theta'-\theta)\cos(2\varphi-\varphi')}, \qquad (26)$$

θ, φ and θ', φ' are the spherical angles of the vectors $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$ and $\mathbf{n}' = \mathbf{p}'/|\mathbf{p}'|$.

It follows from what has been said that for zero rest mass the invariant parametrizations for the matrix elements of a scalar, a conserved current, and a conserved energy-momentum tensor are of the forms

$$\langle \mathbf{p},\lambda|A(0)|\mathbf{p}',\lambda\rangle = \frac{e^{-i\lambda\chi}}{(2\pi)^3(4p_0p_0')^{1/2}}f_{\lambda\lambda'}(t), \qquad (27)$$

$$\langle \mathbf{p},\lambda|J_\mu(0)|\mathbf{p}',\lambda'\rangle = \frac{e^{-i\lambda\chi}}{(2\pi)^3(4p_0p_0')^{1/2}}F_{\lambda\lambda'}(t)K'_\mu, \qquad (28)$$

$$\langle \mathbf{p},\lambda|T_{\mu\nu}|(0)|\mathbf{p}',\lambda'\rangle = \frac{e^{-i\lambda\chi}}{(2\pi)^3(4p_0p_0')^{1/2}} \times\left\{\frac{1}{2}g^{(1)}_{\lambda\lambda'}(t)K'_\mu K'_\nu + g^{(2)}_{\lambda\lambda'}(t)(t\delta_{\mu\nu}+K_\mu K_\nu)\right\}. \qquad (29)$$

Here, in order for the energy-momentum tensor to have the right connection with the total energy and momentum, we must impose the condition

$$g^{(1)}_{\lambda\lambda'}(0) = \delta_{\lambda\lambda'}. \qquad (30)$$

The requirement of parity conservation leads to the relations

$$f_{\lambda,\lambda}(t) = f_{-\lambda,-\lambda}(t), \qquad f_{\lambda,-\lambda}(t) = f_{-\lambda,\lambda}(t) \qquad (31)$$

and analogous relations for $F_{\lambda\lambda'}(t)$, $g^{(1)}_{\lambda\lambda'}(t)$, and $g^{(2)}_{\lambda\lambda'}(t)$.

The parametrization in the helicity representation with nonzero mass has been obtained in [8].

[1]As is well known, the eigenvectors corresponding to states with a given helicity can contain an arbitrary phase factor.[2] In this paper this factor is chosen in accordance with our previous paper.[2] To change to the choice of factor made by Jacob and Wick[7] one must multiply the right member of Eq. (25) by $\exp\{-i\lambda(\varphi-\varphi')\}$.

To conclude this section we point out that for the practically important case of particles with spin $1/2$ the notation for the matrix elements used in this paper, namely

$$\langle \mathbf{p},\varkappa,1/2,m,\varepsilon|\,Q(x_\sigma)\,|\,\mathbf{p}',\varkappa,1/2,m',\varepsilon'\rangle,$$

where ϵ takes the values $+1$ (electron state) and and -1 (positron state) is related in a simple way to the usual notation which uses Dirac spinors $u^m(\mathbf{p})$ and $v^m(-\mathbf{p})$. For example, for the matrix elements of the electron, positron, and annihilation currents the connection is given by

$$\langle \mathbf{p},\varkappa,1/2,m,+1|J_\mu(x)|\mathbf{p}',\varkappa,1/2,m',+1\rangle = \frac{\exp\{-i(p-p')x}{(4p_0p_0')^{1/2}}\bar{u}^m(\mathbf{p}) \times\left\{\tilde{\varphi}_1^+(t)\gamma_\mu + \tilde{\varphi}_2^+(t)\frac{\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu}{2}(p_\nu-p'_\nu)\right\}u^{m'}(\mathbf{p}'), \qquad \text{(I)}$$

$$\langle \mathbf{p},\varkappa,1/2,m,-1|J_\mu(x)|\mathbf{p}',\varkappa,1/2\,m',-1\rangle = \frac{\exp\{-i(p-p')x\}}{(4p_0p_0')^{1/2}}\bar{v}^{m'}(-\mathbf{p}') \times\left\{\tilde{\varphi}_1^-(t)\gamma_\mu + \tilde{\varphi}_2^-(t)\frac{\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu}{2}(p_\nu-p'_\nu)\right\}v^m(-\mathbf{p}), \qquad \text{(II)}$$

$$\langle \mathbf{p},\varkappa,1/2,m,+1,\mathbf{p}',\varkappa,1/2,m',-1|J_\mu(x)|0\rangle = \frac{\exp\{-i(p-p')x\}}{(4p_0p_0')^{1/2}}\bar{u}^m(\mathbf{p}) \times\left\{\tilde{\varphi}_1(t')\gamma_\mu + \tilde{\varphi}_2(t')\frac{\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu}{2}(p_\nu+p'_\nu)\right\}v^{m'}(-\mathbf{p}'), \qquad \text{(III)}$$

$$u^{1/2}(\mathbf{p}) = A\begin{pmatrix}1\\0\\p_z/(|p_0|+\varkappa)\\(p_x+ip_y)/(|p_0|+\varkappa)\end{pmatrix},$$

$$u^{-1/2}(\mathbf{p}) = A\begin{pmatrix}0\\1\\(p_x-ip_y)/(|p_0|+\varkappa)\\-p_z(|p_0|+\varkappa)\end{pmatrix}$$

$$v^{1/2}(\mathbf{p}) = A\begin{pmatrix}-p_z/(|p_0|+\varkappa)\\(-p_x-ip_y)/(|p_0|+\varkappa)\\1\\0\end{pmatrix},$$

$$v^{-1/2}(\mathbf{p}) = A\begin{pmatrix}(-p_x+ip_y)/(|p_0|+\varkappa)\\p_z/(|p_0|+\varkappa)\\0\\1\end{pmatrix}, \qquad \text{(IV)}$$

where

$$t = -(p-p')^2, \qquad t' = -(p+p')^2,$$

$$A = [2|p_0|/(|p_0|+\varkappa)]^{1/2}.$$

4. The method for parametrization of one-particle matrix elements of local operators can be applied to matrix elements taken between states containing one particle of spin j and mass M and

an arbitrary number of scalar particles:

$$\langle k_1, \mu_1, \ldots, k_n, \mu_n, p, M, j, m | Q(x) | k_1', \mu_1', \ldots, k_l', \mu_l', p', M, j, m'' \rangle.$$

Here k_i, μ_i are the energy-momentum and mass of the i-th scalar particle, and p, p′ are momenta of the particle with mass M and spin j. In this case the vectors we have available for the construction of the matrix elements of a local operator of given tensor dimensionality are

$$k_r, \quad k_q', \quad p, \quad p', \quad \Gamma(p'), \quad R_\mu^{a,b,c} \equiv \varepsilon_{\mu\nu\lambda\sigma} a_\nu b_\lambda c_\sigma,$$

(a, b, c are any of the vectors k_r, k_q', p, p′, Γ). We can take any four linearly independent vectors from among these as "basis" vectors, and form from them all possible tensors of the given rank. Since $\Gamma_\mu(p')p_\mu' = 0$, we can get all linearly independent scalar matrices $N_{n,m,l}$ from the three scalars

$$Q_1 = q_\mu^{(1)}\Gamma_\mu, \qquad Q_2 = q_\mu^{(2)}\Gamma_\mu, \qquad Q_3 = q_\mu^{(3)}\Gamma_\mu$$

($q_\mu^{(i)}$ are any three of the available vectors except p′, Γ) by means of the formula

$$N_{n,m,l} = \sum_{2j \geqslant n+m+l \geqslant 2j-1} Q_1^n Q_2^m Q_3^l. \tag{32}$$

As an example we give the parametrization of the matrix element of the operator of the conserved current for the photoproduction of a π meson on a nucleon:

$$\begin{aligned}
&\langle \mathbf{k}, \mu, \mathbf{p}, M, {}^1/_2, m | J_\nu(x) | \mathbf{p}', M, {}^1/_2, m' \rangle \\
&= (2\pi)^{-3/2} (8k_0 p_0 p_0')^{-1/2} e^{-iKx} \sum_{m''} D^{1/2}_{mm''}(p, p') \\
&\times \langle m'' | F^{(1)}(s,t,u) K_\nu^{(1)} F^{(2)}(s,t,u) \; K_\nu^{(2)} \\
&+ F^{(3)}(s,t,u) K_\nu^{(3)} | m' \rangle, \\
&F^{(i)}(s,t,u) = \sum_{1 \geqslant n+m+l \geqslant 0} f^{(i),n,m,l}(s,t,u) N_{n,m,l}; \\
&K_\mu = k_\mu + p_\mu - p_\mu', \qquad K_\mu^{(2)} = p_\mu + K_\mu (Kp)/t, \\
&K_\mu^{(1)} = \varepsilon_{\mu\nu\lambda\rho} k_\nu p_\lambda p_\rho', \qquad K_\mu^{(3)} = p_\mu' + K_\mu (Kp')/t, \\
&t = -K^2, \qquad s = -[K^{(2)}]^2, \qquad u = -[K^{(3)}]^2, \\
&Q_1 = k\Gamma, \qquad Q_2 = p\Gamma, \qquad Q_3 = K^{(1)}\Gamma.
\end{aligned} \tag{33}$$

Here parity conservation means that only the form-factors $f^{(1),0,0,0}$, $f^{(1),0,0,1}$, $f^{(2,3),1,0,0}$, $f^{(2,3),0,1,0}$ are different from zero.

5. Let us now consider the matrix element of most general form for the scalar operator A(x), between states with arbitrary numbers of particles of any sort. It follows from considerations of relativistic invariance that any physical system can be described by a state vector

$$|P_\mu, S, m, \alpha\rangle, \tag{34}$$

where P_μ is the four-momentum, S is the total intrinsic angular momentum, m is the projection of this angular momentum, and α represents the other invariants under proper four-dimensional rotations and displacements of the coordinates; here these can be not only the charges, but also, for example, orbital angular momenta of the particles in the center-of-mass system (c.m.s.), and so on.

A state vector for two or more particles, given in the form of a direct product of vectors of the type (1), can be reduced to the form (34) by means of one or more expansions of the Clebsch-Gordan type for the representations of the inhomogeneous Lorentz group. This expansion has been obtained in [9] (see also [10]).

For the special case of the scattering of two spinless particles the quantity (34) is simply a component of the partial-wave expansion in the center-of-mass system. Here it is convenient to take the normalization in the form

$$\langle P_\mu, S, m, \alpha | P_\mu', S', m', \alpha' \rangle = \delta^4(P_\mu - P_\mu') \delta_{SS'} \delta_{mm'} \delta_{\alpha\alpha'}. \tag{35}$$

The index α will be omitted hereafter.

A relation of the type (3) is of course still valid here, so that it is sufficient to consider the parametrization of the operator A(0). Just as before, it is helpful to go over from the vectors P, P′ to vectors K, K′ as defined in Eq. (13). Together with the invariant $t = -k^2$ we now use the invariants

$$s = -P^2, \qquad s' = -P'^2. \tag{36}$$

We call attention to the fact that in the matrix element

$$\langle P_\mu, S, m | A(0) | P_\mu', S', m' \rangle \tag{37}$$

the intrinsic angular momenta S and S′ are different, which excludes the possibility of constructing the operator Γ_μ (diagonal in the spin), and thus prevents the use of the methods developed in Secs. 2, 3.

For the invariant parametrization of the matrix element (37) we make a Lorentz transformation from the original (laboratory) coordinate system to the Breit system (B.s.), i.e., the system in which

$$K_\mu' = (0, 0, 0, \sqrt{-K'^2}).$$

When the vectors P and P′ are equal, the B.s. goes over into the c.m.s. Then

$$\langle P_\mu, S, m | A(0) | P'_\mu, S', m' \rangle = [4P_0P'_0]^{-1/2} \sum_{\widetilde{m}\widetilde{m}'} D^S_{m\widetilde{m}}(P, w) \times \langle \widetilde{P}_\mu, S, \widetilde{m} | A(0) | \widetilde{P}'_\mu, S', \widetilde{m}' \rangle D^{*S'}_{m'\widetilde{m}'}(P', w). \quad (38)$$

Here $w_\mu = K'_\mu(-K'^2)^{-1/2}$ is the four-velocity corresponding to the Lorentz transformation in question; $D^S(P, w)$ are matrices for relativistic spin rotations, taken in the form given in [2] [Eqs. (33), (34)]; and

$$\widetilde{P}_\mu = (\mathbf{q}, \widetilde{P}_0), \qquad \widetilde{P}'_\mu = (-\mathbf{q}, \widetilde{P}'_0),$$

with

$$\mathbf{q} = \tfrac{1}{2}\{\mathbf{K} + \mathbf{w}(\mathbf{wK})/(w_0 + 1) - \mathbf{w}K_0\},$$

$$|\mathbf{q}|^2 = [s^2 + s'^2 + t^2 - 2(ss' + st + s't)]/4[2(s + s') - t],$$

$$\widetilde{P}_0^2 = s + |\mathbf{q}|^2, \qquad \widetilde{P}'_0 = s' + |\mathbf{q}|^2. \quad (39)$$

It is obvious that

$$\langle \widetilde{P}_\mu, S, \widetilde{m} | A(0) | \widetilde{P}'_\mu, S', \widetilde{m}' \rangle = \langle S, \widetilde{m} | B(s, s', t, \mathbf{n}) | S' \widetilde{m}' \rangle, \quad (40)$$

where $\mathbf{n} = \mathbf{q}/|\mathbf{q}|$. Expanding the operator B in spherical functions of $\mathbf{n}$, we get

$$\langle S, \widetilde{m} | B(s, s', t, n) | S', \widetilde{m}' \rangle = \sum_{LM} Y_{LM}(\mathbf{n}) \langle S, \widetilde{m} | B_{LM}(s, s', t) | S', \widetilde{m}' \rangle. \quad (41)$$

We note that the vector $\mathbf{n}$ is the unit vector in the direction of the vector K in the B.s., where it is distinguished from K_0 in an invariant way. Therefore the expansion (41) in spherical harmonics is of an invariant nature. In the special case $\mathbf{q} = 0$ the vector n is undefined, but in this case $L = 0$, since a state with the momentum equal to zero is an eigenstate of the angular-momentum operator with the eigenvalue zero (to verify this, it suffices to consider the commutation relation for the angular-momentum and momentum operators).

Since all of the angular momenta in Eq. (41) are now given in the B.s., we can use the Wigner-Eckardt theory, according to which the matrix element of the operator B_{LM} must be a scalar multiplied by a Clebsch-Gordan coefficient (cf. e.g.,[5])

$$\langle S, \widetilde{m} | B_{LM}(s, s', t) | S', \widetilde{m}' \rangle = \langle S'\widetilde{m}'LM | S\widetilde{m} \rangle F^L_{SS'}(s, s', t)|. \quad (42)$$

The formulas (38)—(42) give the desired parametrization for a scalar matrix element in terms of the invariant form-factors $F^L_{SS}(s, s', t)$. We note that if the operator $A(0)$ is Hermitian, all of the form-factors are real. Depending on the parities of the operator $A(0)$ and of the states between which the matrix element is taken, the summation in Eq. (42) goes over only even or only odd L in the range $S + S' \geq L \geq |S - S'|$. The physical meaning of the transformation (38) is that, as in Eq. (6), all of the spins of the particles are "set" onto the same momentum [in Eq. (38) onto the momentum K'], by means of the matrices $D^S(P, w)$ and $D^{S'}(P, w)$. An important point is that not only the spins of the particles, but also the angular momentum L are "sitting" on the momentum K'; i.e., under Lorentz transformations the quantities $Y_{LM}(\mathbf{n})$ transform like wave functions of a particle with the intrinsic angular momentum L and the momentum K'.

6. The parametrization of a matrix element of an operator with a tensor or spinor dimensionality, taken between two arbitary systems, calls for the conquest of one further specific difficulty, connected with the fact that the tensor (or spinor) indices of the operator do not sit on any momentum, but are covariant in themselves. These tensors (or spinors), however, can also be set onto the momentum K' (or onto any other timelike four-vector H which is a linear combination of the given momentum P, P'), when we subject them to the Lorentz transformation from the laboratory system to the B.s. (or to any other coordinate system which corresponds to setting them on a vector H).

The respective forms of the transformation to the B.s. for a vector J_μ and a spinor τ_λ ($\lambda = 1/2, -1/2$) are

$$\mathbf{J} = \widetilde{\mathbf{J}} + \mathbf{w}(\widetilde{\mathbf{J}}\mathbf{w})/(w_0 + 1) + \mathbf{w}\widetilde{J}_0, \qquad J_0 = \widetilde{J}_0 w_0 + \widetilde{\mathbf{J}}\mathbf{w}; \quad (43)$$

$$\tau_\lambda = \Lambda_{\lambda\lambda'}\tau_{\lambda'}, \quad \tau_{\dot\lambda} = \Lambda^{-1}_{\dot\lambda\dot\lambda'}\tau_{\dot\lambda'},$$

$$\Lambda = (\omega_0 + 1 + \sigma\mathbf{w})/\sqrt{2(\omega_0 + 1)}. \quad (44)$$

The transformation for a Dirac spinor ψ is

$$\psi = W\widetilde{\psi}, \qquad W = (w_0 + 1 + \boldsymbol{\alpha}\mathbf{w})/\sqrt{2(w_0 + 1)}, \quad (45)$$

where $\boldsymbol{\alpha}$ is the Dirac matrix.

The component $\widetilde{J}_0$ of the transformed vector $\widetilde{J}_\mu$ is a four-dimensional scalar, and the three-dimensional components form a vector sitting on the momentum K'. After the standard change to the canonical basis:

$$\widetilde{J}_i = a_{i\mu}\widetilde{J}^1_\mu, \qquad a_{i\mu} = \sqrt{\frac{2\pi}{3}} \left\| \begin{matrix} -1 & 0 & 1 \\ i & 0 & i \\ 0 & \sqrt{2} & 0 \end{matrix} \right\|, \quad (46)$$

the unit angular momentum described by this vector can be combined with the angular momenta S, S′, and L, by means of the usual coefficients for vector composition. The vanishing of the sum of all four angular momenta gives (together with the terms corresponding to the scalar $\tilde{J}_0$) the complete parametrization of the matrix element of the vector operator $J_\nu(0)$, which thus can be written in the form

$$\langle P_\nu, S, m|\, \tilde{J}^1_\mu(0)|\, P'_\nu, S', m'\rangle$$
$$= [4P_0P'_0]^{-1/2} \sum_{j, \lambda, \tilde{m}, \tilde{m}', L, M} D^S_{m\tilde{m}}(P, w)$$
$$\times D^{*S'}_{m'\tilde{m}'}(P, w)\, \langle S', \tilde{m}' j\lambda\,|\, S\tilde{m}\rangle$$
$$\times \langle 1\mu\, LM\,|\, j\lambda\rangle\, Y_{LM}(\mathbf{n})\, G^{1, L, j}_{S, S'}(s, s', t),$$
$$\langle P_\nu, S, m|\, \tilde{J}_0(0)|\, P'_\nu, S', m'\rangle$$
$$= [4P_0P'_0]^{-1/2} \sum_{\tilde{m}, \tilde{m}', L', M'} D^S_{m\tilde{m}}(P, w)\, D^{*S'}_{m'\tilde{m}'}(P', w')$$
$$\times \langle S', \tilde{m}' L' M'\,|\, S\tilde{m}\rangle\, Y_{L'M'}(\mathbf{n})\, G^{0, L'}_{S, S'}(s, s', t). \qquad (47)$$

The conservation law (16) leads to the relations

$$4\pi\, [s^2 + s'^2 + t^2 - 2\,(ss' + st + s't)]^{1/2}$$
$$\times \{[I/(2I-1)]^{1/2}\, G^{1, I-1, I}_{S, S'}$$
$$- [(I+1)/(2I+3)]^{1/2}\, G^{1, I+1, I}_{S, S'}\} = 9\,(s - s')\, G^{0, I}_{S, S'}. \qquad (48)$$

The parametrization of a spinor operator is accomplished in an analogous way. It is based on the fact that the spinor $\tilde{\tau}_\lambda$ sits on the momentum K′, so that the angular momentum ½ which corresponds to it can be combined with the angular momenta S, S′, and L. Therefore the parametrization of a spinor operator $\tau_\lambda(x)$ can be written in the form

$$\langle P_\mu, S, m|\, \tau_\lambda(x)|\, P'_\mu, S', m'\rangle$$
$$= \frac{e^{-ikx}}{(4P_0P'_0)^{1/2}} \sum_{\lambda'} \Lambda_{\lambda\lambda'}(w) \cdot \sum_{j, \mu, \tilde{m}, \tilde{m}', L, M} D^S_{m\tilde{m}}(P, w)$$
$$D^{*S'}_{m'\tilde{m}'}(P', w)\langle S'\tilde{m}' j\mu\,|\, S\tilde{m}\rangle\langle 1/2\, \lambda' LM\,|\, j\mu\rangle Y_{LM}(\mathbf{n})\Phi^{1/2, L, j}_{S, S'}(s, s', t). \qquad (49)$$

It is also not hard to write the formula for the general case of an operator which transforms according to an arbitrary finite-dimensional representation of the Lorentz group. We shall not do this, since for practical purposes the formulas (47) and (49) are sufficient. The method we have given obviously applies to one-particle matrix elements, but for these the less general method developed in Secs. 2 and 3 is more convenient.

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