THEORY OF SUPERFLUID FERMI LIQUID. APPLICATION TO THE NUCLEUS

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A method is given for treating systems of strongly interacting particles, in which the observable quantities are expressed in terms of several constants which are introduced into the theory, in the way that the masses and charges of particles are introduced in the theory of dispersion relations. The Landau theory of the Fermi liquid is extended to the case of superfluidity for systems of finite size. An equation is obtained which makes it possible to find the probabilities of electromagnetic transitions in nuclei.

I. INTRODUCTION

In all real many-particle systems the interaction is not small, and therefore in the derivation of quantitative relations one cannot proceed, as is often done, by combining some part of the diagrams of perturbation theory.

One must single out the set of those diagrams which have important variations in the region of energies and momenta to be considered and give expression to their functional dependences by means of several constants which are not calculable in the theory, in the same way as one introduces the masses and charges of the particles in the theory of dispersion relations.

For Fermi systems without the Cooper pair correlation, a program of this sort was carried through by Landau in his theory of the Fermi liquid.[1] It was shown that the spectrum of one-particle excitations near the Fermi surface is characterized by a single constant, the effective mass, and the spectrum of two-particle excitations (zero sound) and the reactions of the system to an external field are determined by a single function Γ^{ω}, the forward scattering amplitude, which depends only on the spin variables and on the angle between the momenta of the quasi-particles.

For systems with pairing it is well known that an important part is played by transitions of a particle to a hole and a condensate pair, which lead to the appearance of gaps in the spectrum of the one-particle excitations. In all real systems with pairing (superconductors, atomic nuclei) the gap energy is much smaller than the Fermi limiting energy. Owing to this one can sort the sets of diagrams into two types: a set of those that change appreciably over the width of the gap, and those for which the characteristic range of variation is the Fermi limiting energy (or, in the case of superconductors, the Debye temperature). Here one must introduce, in addition to Γ^{ω}, one other function of the angles between the momenta of the quasi-particles, Γ^{ξ}; the spherical harmonic of this function is connected with the width of the energy gap. It is natural to expect that the functions Γ^{ω} and Γ^{ξ} depending on the angles between the momenta of the quasi-particles will be well described by two or three terms of the expansion in Legendre polynomials. The coefficients of the Legendre polynomials are constants introduced into the theory; in real problems their calculation is practically impossible because of the strong interaction between the particles.

For the application of the theory to the nucleus it is necessary to use the smallness of the range of the forces in comparison with the dimensions of the system. The functions Γ^{ω} and Γ^{ξ} are given by sets of diagrams in which the integration over regions far from the Fermi surface is important. Therefore these quantities are insensitive to fluctuations in the distribution of levels near the Fermi surface and are the same for all nuclei except the light ones. It may be supposed that Γ^{ω} and Γ^{ξ} are constant to the same accuracy as the density of particles in nuclei is constant. This remark applies to all quantities which are determined by integrals over regions far from the Fermi surface, such as the effective mass, the renormalization of the Green's function, and the depth of the effective potential well.

As for quantities determined by the behavior of the quasi-particles near the Fermi surface, calculations for them reduce to the determination of the renormalized Green's functions.

Thus the problem of finding the spectrum of the one-particle excitations reduces to the determination of the renormalized one-particle Green's function, and, as was shown in [2], this is equiva-

lent to the solution of the Schrödinger equation for one particle with the effective mass in a potential well.

The two-particle spectrum and the reaction of the system to an external field are determined by the two-particle Green's function, and it is shown below that the finding of these quantities reduces to the solution of a simple equation analogous to the Schrödinger equation for two interacting particles in a potential well.

II. THE ONE-PARTICLE GREEN'S FUNCTION

1. The Green's Function in a System with Pairing. The Green's function is defined, as usual, by the relation

$$G(1,2) = -i\langle T\psi(1)\psi^{+}(2)\rangle$$

and represents the set of all diagrams which begin and end with one line.

To find the Green's function in the strong-interaction case it is necessary to sort out those diagrams which vary rapidly near the Fermi surface. For systems without pairing this is accomplished by the Dyson equation

$$G = G_0 + G_0\Sigma G. \tag{1}$$

A proper-energy part $\Sigma(p)$ [$p = (p, \epsilon)$] contains blocks connected by three or more lines, and the integration over the momenta of these lines makes Σ a slowly varying function near $\epsilon = \mu$ and $p = p_0$, where p_0 is the momentum at the Fermi surface and μ is the chemical potential.

By expanding Σ in series around $\epsilon = \mu$ and $p = p_0$, one can show that

$$G = \frac{a}{\varepsilon - \varepsilon_p + i\alpha\varepsilon|\varepsilon|} + G_{reg}. \tag{2}$$

Here and in what follows ϵ is measured from the value μ, $a^{-1} = (\partial G^{-1}/\partial\epsilon)_{\epsilon=0,p=p_0}$, $\epsilon_p = v(p - p_0)$, v is the velocity at the Fermi surface, given by $v = -a\partial G^{-1}/\partial p_\alpha \equiv p_\alpha/m^*$, and m^* is the effective mass: the quantity G_{reg} is slowly varying near the Fermi surface.

In the case of pairing in a system the following states combine: one particle with a background of N particles and one hole with a background of N + 2 particles. In other words, a "condensate" of paired particles is formed, and therefore there is a nonvanishing amplitude for transition of a particle to a hole and a "condensate" pair, analogous to transitions in a Bose system. Let us represent the irreducible amplitude of such a transition graphically:

$\hat{\Delta}^{(1)}$ = . (3)

The block here represents the set of all diagrams which take a particle into a hole and do not contain parts joined by one line of any direction. Analogously

$\hat{\Delta}^{(2)}$ = . (4)

The spin of the condensate pairs is zero, and therefore $\hat{\Delta}^{(1),(2)}$ must be antisymmetric matrices with respect to the spinor indices:

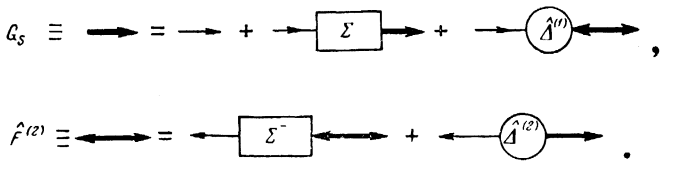
$$\hat{\Delta}^{(1)}_{\alpha\beta} = \Delta^{(1)} g_{\alpha\beta}, \qquad \hat{\Delta}^{(2)}_{\alpha\beta} = \Delta^{(2)} g_{\alpha\beta}, \tag{5}$$

where

$$\hat{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \hat{g}^2 = -1.$$

As will be shown below, $\Delta^{(1),(2)}$ are slowly varying functions of $p = (p, \epsilon)$.

The equation for the Green's function in a system with pairing is of the form



(6)

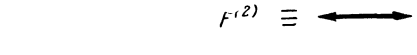
Here → ≡ G_0, the Green's function of a free particle,

$$G_0 = \frac{1}{\varepsilon - \varepsilon_p^0 - i\delta\,\mathrm{sign}\,\varepsilon}, \qquad \varepsilon_p^0 = \frac{p^2}{2m} - \frac{p_0^2}{2m}; \tag{7}$$

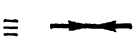
← ≡ G_0^-, the Green's function of a free hole, which is obtained from the Green's function of a particle by replacing p by −p (and interchange of the spinor indices, when G_0 is not diagonal in the spins),

$$G_0^- = G_0(-\varepsilon, -p) = \frac{-1}{\varepsilon + \varepsilon_p^0 - i\delta\,\mathrm{sign}\,\varepsilon}.$$

The quantity $\Sigma \equiv$ →  → does not contain blocks connected by one line of any direction. Since integration over distant regions is important in Σ, and in such regions the pairing does not change the Green's function much, Σ coincides (to accuracy $\sim\Delta^2/\mu^2$) with the analogous quantity in the problem without pairing. Also we have Σ^- = ← ← ≡ $\Sigma(-\epsilon, -p)$.

$\hat{F}^{(2)} \equiv$ 

is the complete set of all diagrams which take a hole into a particle. In analogy with this we introduce

$\hat{F}^{(1)} \equiv$ 

Unlike Δ, F includes the entering and emerging lines and contains parts connected by a single line. Equations of this type were first obtained by Belyaev [3] in the study of a system of Bose particles with strong interaction, and have been applied in the theory of superconductivity by Gor'kov.[4]

Let us write Eq. (6) in analytical form:

$$G_s = G_0 + G_0 \Sigma G_s + G_0 \hat{\Delta}^{(1)} \hat{F}^{(2)},$$
$$\hat{F}^{(2)} = G_0^- \Sigma^- \hat{F}^{(2)} + G_0^- \hat{\Delta}^{(2)} G_s. \qquad (6')$$

From Eq. (6') we get

$$G_s = G/(1 + GG^-\Delta^{(2)}\Delta^{(1)}),$$
$$\hat{F}^{(2)} = G^-\hat{\Delta}^{(2)}G_s = G^-\Delta^{(2)}G_s g \equiv F^{(2)}\hat{g}, \qquad (8)$$
$$\hat{F}^{(1)} = G\hat{\Delta}^{(1)}G_s(-p) \equiv F^{(1)}\hat{g},$$

where G is defined by Eq. (1) and is given by

$$G = \frac{1}{\varepsilon - \varepsilon_p^0 - \Sigma(\varepsilon, p)},$$
$$G^- = G(-p) = \frac{-1}{\varepsilon + \varepsilon_p^0 + \Sigma(-\varepsilon, -p)}. \qquad (9)$$

From time-reversibility it follows that

$$\Delta^{(2)} = (\Delta^{(1)})^*.$$

Therefore the system of equations (8) is invariant under the transformations

$$\Delta^{(1)} \to e^{i\varphi}\Delta^{(1)}, \qquad F^{(1)} = e^{i\varphi}F^{(1)}.$$

This allows us to regard $\Delta^{(1)}$ as real and equal to $\Delta^{(2)}$:

$$\Delta^{(1)} = \Delta^{(2)}, \qquad F^{(1)} = F^{(2)}.$$

Using the formula (2) for G, we find from Eq. (8) that near the Fermi surface

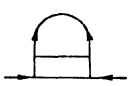
$$G_s = a\frac{\varepsilon + \varepsilon_p}{\varepsilon^2 - \varepsilon_p^2 - \Delta^2 + i\delta}, \quad F = -a\frac{\Delta}{\varepsilon^2 - \varepsilon_p^2 - \Delta^2 + i\delta}, \qquad (10)$$

where a and ϵ_p are the same as in Eq. (2).

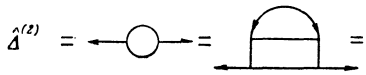
The expressions (10) were obtained in [2], where the equations (8) were found by separating out the pole diagrams in Σ (here F differs by a factor i from the F introduced in [2]).

2. The Equation for Δ. The equations (10) must be supplemented with the equation for Δ, which has been found by Gor'kov for a weak interaction between the particles [4] and can easily be extended to the case of an arbitrary interaction.[2]

The graphical equations for $\hat{\Delta}^{(1)}$ and $\hat{\Delta}^{(2)}$ are



$$= \hat{v}\hat{F}^{(1)} = \hat{v}G\hat{\Delta}^{(1)}G_s^-,$$



$$= \hat{v}\hat{F}^{(2)} = \hat{v}G^-\hat{\Delta}^{(2)}G_s, \qquad (11)$$

where, as before, the index "minus" means that p is replaced by $-p$. Integration over momenta $d\tau = d\epsilon d^3p/(2\pi)^4 i$ and summation over the spin variables are understood here.

It is obvious that the block v does not contain parts joined by two vertical lines, since F is the complete set of diagrams with two entering (or emerging) lines. The number of functions $\hat{F}^{(1)}$ that are constituents of $\hat{\Delta}^{(1)}$ is larger by unity than the number of functions $\hat{F}^{(2)}$. Each pair $\hat{F}^{(1)}$ and $\hat{F}^{(2)}$ in the block v is integrated over a wide range of energies and makes a contribution $\sim \Delta^2/\mu^2$. Therefore to accuracy Δ^2/μ^2 the quantity v is equal to the irreducible four-pole diagram of a system without pairing. By the Pauli principle $\hat{v}$ satisfies the relations

$$\hat{v}_{\alpha\beta\gamma\delta}(p, p', q) \equiv \hat{v}_{\alpha\beta\gamma\delta}(p_1p_2p_3p_4) = -\hat{v}_{\beta\alpha\gamma\delta}(p_2p_1p_3p_4)$$
$$= -\hat{v}_{\alpha\beta\delta\gamma}(p_1p_2p_4p_3),$$
$$p_{1,2} = q/2 \pm p, \qquad p_{3,4} = q/2 \pm p', \qquad q = (k, \omega).$$

The interchange $p_1 \leftrightarrow p_2$ means $p \to -p$; thus $\hat{v}_{\alpha\beta\gamma\delta}$ is the sum of a term even in p and antisymmetric in α and β and a term odd in p and symmetric in α and β. Instead of $\hat{v}_{\alpha\beta\gamma\delta}$ it is convenient to introduce the four-pole quantity v defined by the relation

$$\hat{v}_{\alpha\beta\gamma\delta} = -v_{\alpha\nu\gamma\mu}g_{\nu\beta}g_{\delta\mu}, \qquad (12)$$

where $\hat{g}$ is defined in Eq. (5).

If there is no spin-orbit interaction between the particles in the system, v can be written in the form

$$v_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta}v_1 + \sigma_{\alpha\beta}\sigma_{\delta\gamma}v_2; \qquad (13)$$

for $k \ll p_0$ the quantities v_1 and v_2 depend on p^2, p'^2, and the angle between p and p', and v_1 is an even and v_2 an odd function of this angle. The first term in Eq. (13) describes the interaction of two particles with zero total spin, and the second, that of two particles with total spin unity. The v in Eq. (11) is that for q = 0.

From Eqs. (5) and (8) we have $\hat{\Delta}_{\alpha\beta} = \Delta g_{\alpha\beta}$, $\hat{F}_{\alpha\beta} = Fg_{\alpha\beta}$, and therefore, using Eq. (12), we can rewrite Eq. (11) in the form

$$\Delta = 2v_1F = 2v_1G\Delta G_s^-. \qquad (11')$$

The formulas (11') involve integrations far from the Fermi surface. We can renormalize v in such a way that the integration in Eq. (11) will

be over regions near to the Fermi surface, which will allow us to use the expressions (10) for G_s and F. To do this we introduce a quantity Γ^ξ defined by the equation

$$\Gamma^\xi_{\alpha\beta\gamma\delta}(p, p') = v(1 + \theta_\xi GG^- \Gamma^\xi) = v_{\alpha\beta\gamma\delta}(p, p') + \int v_{\alpha\beta\mu\nu}(p, p_1) G(p_1) G^-(p_1) \theta(\xi) \Gamma^\xi_{\mu\nu\gamma\delta}(p_1, p') \frac{d^4p}{(2\pi)^4 i}, \quad (14)$$

where $\theta(\xi) = 1$ for $|\epsilon_p| > \xi$, and $\theta(\xi) = 0$ for $|\epsilon_p| < \xi$.

The quantity ξ does not appear in the final expressions, and the only condition on it is

$$\Delta \ll \xi \ll \varepsilon_0.$$

Γ^ξ, like v, is of the form

$$\Gamma^\xi_{\alpha\beta\gamma\delta} = \Gamma^\xi_1 \delta_{\alpha\beta}\delta_{\gamma\delta} + \Gamma^\xi_2 \sigma_{\alpha\beta}\sigma_{\delta\gamma}. \quad (13')$$

Thus Eq. (12) contains only a region of integration far from the Fermi surface.

Let us find the dependence of Γ^ξ on ξ. To do this we write Eq. (12) in symbolic form in two different ways:

$$\Gamma^\xi = (1 + \Gamma^\xi \theta_\xi GG) v, \qquad \Gamma^{\xi'} = v(1 + GG\theta_{\xi'}\Gamma^{\xi'}).$$

To eliminate the distant regions we multiply the first of these equations by the operator $1 + GG\theta_{\xi'}\Gamma^{\xi'}$ on the right and find

$$\Gamma^\xi = \Gamma^{\xi'} + \Gamma^\xi GG(\theta_\xi - \theta_{\xi'})\Gamma^{\xi'}.$$

Since $\xi, \xi' \ll \mu$, the integration in this equation goes over regions close to the Fermi surface, where Γ^ξ depends only on the angles and G can be replaced by its pole part.

Expanding Γ^ξ and $\Gamma^{\xi'}$ in series of spherical functions and using Eq. (13), we find without difficulty that

$$\Gamma^\xi_l - \Gamma^{\xi'}_l = a^2 p \Gamma^\xi_l \ln(\xi'/\xi)\Gamma^{\xi'}_l,$$

where ρ is the density of levels at the Fermi surface:

$$p = 2\int \frac{d^3p\delta(\varepsilon_p - \mu)}{(2\pi)^3} = \frac{p_0 m^*}{\pi^2}. \quad (15)$$

We must take Γ^ξ_l to mean the harmonic component of Γ^ξ_1 for even l and that of Γ^ξ_2 for odd l. This functional equation has the unique solution

$$a^2 p \Gamma^\xi_l = 1/\ln(\xi/c_l), \quad (16)$$

where c_l is a constant introduced into the theory.

Let us now obtain an equation for Δ which contains integrals only over regions close to the Fermi surface. Multiplying Eq. (11) by the operator $1 + \Gamma^\xi GG^- \theta_\xi$ on the left, we get, on using the relation $\Delta^{(1)} = \Delta^{(2)} = \Delta/a$,

$$\Delta = \Gamma^\xi G(G_s^- - G^-\theta_\xi)\Delta. \quad (17)$$

For $\epsilon_p^2 \gg \Delta^2$ the function G_s coincides with G (except in a narrow region $\epsilon^2 - \epsilon_p^2 \sim \Delta^2$, and also the region $\epsilon_p^2 < \xi^2$, $\epsilon^2 > \xi^2$, which give contributions of relative magnitude $\sim\Delta/\mu$), and therefore in Eq. (17) the integration is only over a region close to the Fermi surface: $\epsilon^2, \epsilon_p^2 < \xi^2$. Here F can be replaced by the simple expression (10). For $p = p_0$, $\epsilon = 0$ we get for $\Delta = a\Delta^{(1)}$

$$\Delta = a\Gamma^\xi F \theta_\xi. \quad (18)$$

Γ^ξ depends only on the angle between p and p'.

Substitution of Eq. (10) in Eq. (18) gives

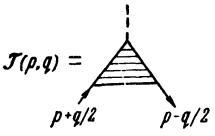
$$1 = a^2\Gamma^\xi_0 p \int_{-\xi}^{\xi} \frac{d\varepsilon_p}{2\pi i} \int_{-\infty}^{\infty} \frac{d\varepsilon}{\varepsilon^2 - \varepsilon_p^2 - \Delta^2} = a^2\Gamma^\xi_0 p \ln\frac{2\xi}{\Delta}, \quad (19)$$

where Γ^ξ_0 is a spherical harmonic of Γ^ξ. Comparing Eqs. (19) and (16), we find $\Delta = c_0/2$.

The formula (19) connects Δ with the zeroth harmonic of the amplitude Γ^ξ, which is also involved in the two-particle spectrum of the system and in the polarization operator. Moreover, the introduction of Γ^ξ into the formula for Δ is particularly important for the application to the nucleus, since, as can be seen from Eq. (12), the region near the Fermi surface is unimportant in Γ^ξ, and Γ^ξ must be the same for all nuclei except the light ones, whereas for finite systems Δ, as determined from a formula of the type of Eq. (18), will change appreciably from element to element.

III. REACTION OF THE SYSTEM TO AN EXTERNAL FIELD. GAUGE INVARIANCE

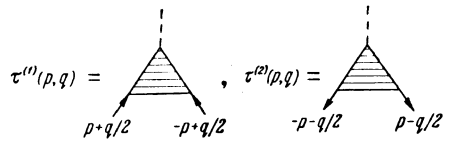
1. The Vertex in an External Field. The absorption of the field in the system is determined by the vertex



(20)

For frequencies of the external field which coincide with frequencies of collective oscillations $\mathcal{T}(p, q)$ has poles, and the residues at these poles give the excitation amplitudes of the oscillations.

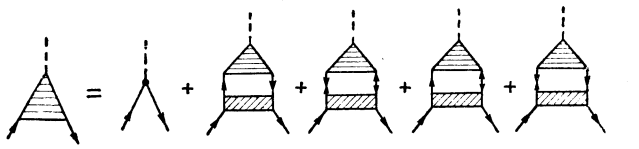
For the description of a system with pairing in an external field we must introduce, in addition to the normal vertex $\mathcal{T}$ for the production of a particle and a hole, the vertices for production of two particles and of two holes:



(21)

Our problem is to find the functional dependence of these vertices on the wave vector $q = (k, \omega)$ of the external field, when ω, kv are small in comparison with μ but can be comparable with Δ.

The only diagrams for which the dependence on q is important are those that contain parts joined by two lines, since they are expressed in terms of integrals which in the region ω, kv, $\Delta < \epsilon$, $\epsilon_p \ll \mu$ take the form $\int d\epsilon d\epsilon_p/(\epsilon^2 - \epsilon_p^2)$ and are sensitive to the value of the lower limit. Diagrams which contain blocks joined by four lines are expressed by integrals of the type $\int d^6\epsilon/\epsilon^4$; they are not sensitive to the lower limit and in them we can set $\omega = kv = \Delta = 0$. When, as before, we separate out the slowly varying blocks, we get the graphical equation



(22)

or in analytical form

$$\mathcal{T} = \mathcal{T}_0 + UG_s\mathcal{T}G_s + U\hat{F}\mathcal{T}^-\hat{F} + UG_s\tau^{(1)}\hat{F}^{(2)} + U\hat{F}^{(1)}\tau^{(2)}G_s. \quad (22')$$

The form of $\mathcal{T}_0$ depends on the type of field applied: for a scalar field $\mathcal{T}_0 = 1$, for a vector field $\mathcal{T}_0 = p_\alpha$, and for a field which acts on the spin of a particle $\mathcal{T}_0 = \sigma_\alpha$.

To explain the symbolic way Eq. (22′) is written, we write the second term in the right member out in full:

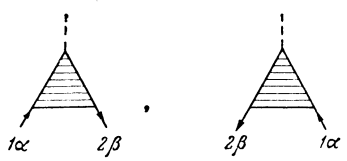
$$\int \sum_{\gamma\delta} U_{\alpha\beta\cdot\gamma\delta}(p, p_1)\, G_s(p_1 + q/2)\, \mathcal{T}_{\gamma\delta}(p_1, q)\, G_s(p_1 - q/2)\, \frac{d^4p}{(2\pi)^4 i}.$$

The irreducible four-pole quantity U does not contain parts joined by two lines, and therefore is slowly varying, so that we can set $q = 0$.

As can be seen from the diagram, $\mathcal{T}^-$ has the sign of p changed and the spinor indices interchanged:

$$\mathcal{T}^-_{\alpha\beta}(p, q) = \mathcal{T}_{\beta\alpha}(-p, q).$$

In fact, in the coordinate representation the diagrams



(21′)

are exactly the same (each means the vanishing of a particle at point 1 with spin projection α and the creation of a particle at point 2 with spin projection β). It can be seen from this that the vertices in the second and third terms of Eq. (22) differ by the interchanges $1 \leftrightarrow 2$, $\alpha \leftrightarrow \beta$. When we go over to the momentum representation the corresponding interchanges are $q \to q$, $p \to -p$, $\alpha \leftrightarrow \beta$.

Let us now carry out a renormalization of the block U so that only an integration near the Fermi surface will remain in Eq. (22′). We note that the only distant integrations are those in the second term of the right member of Eq. (22′) (far from the Fermi surface G_sF falls off like $1/\epsilon^3$, and therefore the two-dimensional integral over $d\epsilon d\epsilon_p$ converges).

Let us introduce the scattering amplitude of the problem without pairing:

$$\Gamma = U + UGG\Gamma; \quad (23)$$

Up to terms of order Δ^2/μ^2 this U is equal to the quantity introduced in Eq. (22′). For the renormalization of U we need the quantity introduced by Landau,[1] $\Gamma^\omega = \Gamma|_{kv \ll \omega, \omega \ll \mu}$, which satisfies the equation

$$\Gamma^\omega = U + U(GG)^\omega\Gamma^\omega = U + \Gamma^\omega(GG)^\omega U. \quad (24)$$

Multiplying Eq. (23) by the operator $1 + \Gamma^\omega(GG)^\omega$ on the left, we get

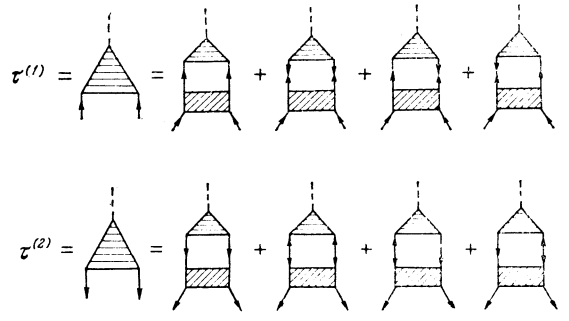
$$\mathcal{T} = \mathcal{T}^\omega + \Gamma^\omega[(G_sG_s - (GG)^\omega)\mathcal{T} + \hat{F}^{(1)}\mathcal{T}^-\hat{F}^{(2)} + G_s\tau^{(1)}\hat{F}^{(2)} + \hat{F}^{(1)}\tau^{(2)}G_s], \quad (25)$$

where the symbol $\mathcal{T}^\omega$ means

$$\mathcal{T}^\omega = \mathcal{T}_0 + \Gamma^\omega(GG)^\omega\mathcal{T}_0. \quad (26)$$

In Eq. (25) the only important region is that near the Fermi surface, where Γ^ω can be regarded as depending only on the angle between the momenta p and p′.

Let us write the graphical equations for $\tau^{(1)}$ and $\tau^{(2)}$:



(27)

In analytical form we have

$$\hat{\tau}^{(1)} = \hat{v}\,(G_s\hat{\tau}^{(1)}G_s + \hat{F}\hat{\tau}^{(2)}\hat{F} + G_s\hat{\mathcal{J}}\hat{F} - \hat{F}\hat{\mathcal{J}}^{-}G_s^{-}),$$

$$\hat{\tau}^{(2)} = \hat{v}\,(G_s^{-}\hat{\tau}^{(2)}G_s + \hat{F}\tau^{(1)}\hat{F} + \hat{F}\hat{\mathcal{J}}G_s - G_s^{-}\hat{\mathcal{J}}^{-}\hat{F}). \qquad (28)$$

The integrals over d^4p of the quantities $G_S(p+q/2)G_S^{-}(p-q/2)$ $= G_S(p+q/2)G_S(q/2-p)$ and $G_S^{-}(p+q/2)G_S(p-q/2)$ are equal, and the integrals of G_SF and FG_S differ only in sign [cf. Eq. (33) below], and therefore we get from Eq. (28)

$$\hat{\tau}^{(1)} = -\hat{\mathcal{J}}\hat{g}, \qquad \hat{\tau}^{(2)} = \hat{g}\hat{\mathcal{J}}. \qquad (29)$$

From Eqs. (28) and (29) and the formula (12) for $\hat{v}$ we get

$$\hat{\mathcal{J}} = v\,(G_sG_s^{-} + FF)\hat{\mathcal{J}} - (G_sF + FG_s^{-}\hat{P})\mathcal{J}. \qquad (30)$$

Here v is the same irreducible four-pole quantity as in the equation (14) for Γ^{ξ}; the operator $\hat{P}$ is defined by the relation

$$\hat{P}\mathcal{J} = -\hat{g}\mathcal{J}^{-}\hat{g} = \pm\mathcal{J}(-p), \qquad (31)$$

where the minus sign corresponds to vertices $-\sigma$ and the plus sign to vertices which do not change the spinor indices.

Repeating the operation used in the replacement of U by Γ^{ω}, we find from Eqs. (12), (25), and (30)

$$\mathcal{J} = \mathcal{J}^{\omega} + \Gamma^{\omega}\{[G_sG_s - (GG)^{\omega} - FF\hat{P}]\mathcal{J} + 2G_sF\hat{\mathcal{J}}\},$$

$$\hat{\mathcal{J}} = \Gamma^{\xi}\{[G_sG_s^{-} - GG^{-}\theta_{\xi} + FF]\hat{\mathcal{J}} - [G_sF - FG_s^{-}\hat{P}]\mathcal{J}\}. \qquad (32)$$

In the equations (32) the integration is only near the Fermi surface, and the function Γ^{ξ}, and also Γ^{ω}, can be regarded as depending only on the angle between **p** and **p′**. Therefore in Eq. (32) we can perform the integrations over $d\omega$ and $d\epsilon_p$, leaving only the integration over the angle. The integrals used in this have been calculated by the Feynman method in the appendix to a paper by Vaks and others.[5] We have:

$$\mathcal{J} = \mathcal{J}^{\omega} + \Gamma^{\omega}(\mathcal{L}\mathcal{J} + \mathcal{M}\hat{\mathcal{J}}),$$

$$\hat{\mathcal{J}} = \Gamma^{\xi}(a^2\rho\ln(2\xi/\Delta)\hat{\mathcal{J}} + \mathcal{N}\hat{\mathcal{J}} + O\hat{\mathcal{J}}). \qquad (32')$$

Here

$$\mathcal{L} = \rho\int\frac{d\varepsilon\,d\varepsilon_p}{2\pi i}[G_sG_s - (GG)^{\omega} - FF\hat{P}]$$

$$= \rho a^2\left[\frac{kv}{\omega - kv}(1 - g(x)) - g(x)\frac{1+\hat{P}}{2}\right],$$

$$\mathcal{M} = 2\rho\int\frac{d\varepsilon\,d\varepsilon_p}{2\pi i}G_sF = -2\rho\int\frac{d\varepsilon\,d\varepsilon_p}{2\pi i}FG_s$$

$$= \rho a^2\frac{\omega + kv}{2\Delta}g(x),$$

$$\mathcal{N} = \rho\int\frac{2\varepsilon\,d\varepsilon_p}{2\pi i}[G_sG_s^{-} - GG^{-}\theta_{\xi} + FF^{-}] - a^2\rho\ln\frac{2\xi}{\Delta}$$

$$= -\rho a^2x^2g(x),$$

$$O = -\rho\int\frac{d\varepsilon\,d\varepsilon_p}{2\pi i}[G_sF - FG_s^{-}\hat{P}]$$

$$= -\rho a^2\left(\frac{\omega + kv}{4\Delta} + \frac{\omega - kv}{4\Delta}\hat{P}\right);$$

$$g(x) = \operatorname{arcsh} x/x\sqrt{1+x^2},\ x^2 = [(kv)^2 - \omega^2]/4\Delta^2 \qquad (33)^*$$

with a determined from Eq. (2) and ρ given by Eq. (15).

The dependence of the amplitudes Γ^{ω} and Γ^{ξ} on the spin variables is given (if the spin-orbit interaction is small) by a spin-exchange term

$$a^2\rho\Gamma^{\omega} = f^{\omega}(\mathbf{nn}') + g^{\omega}(\mathbf{nn}')(\boldsymbol{\sigma\sigma}'),$$
$$a^2\rho\Gamma^{\xi} = f^{\xi}(\mathbf{nn}') + g^{\xi}(\mathbf{nn}')(\boldsymbol{\sigma\sigma}'). \qquad (34)$$

Replacing the inhomogeneous term in Eq. (22′) by σ_{α}, we obtain equations analogous to Eq. (32) for the spinor vertex $\mathcal{J}_{S\alpha}$ (which corresponds to an interaction with the external field of the form $\sigma_{\alpha}H_{\alpha}$). If the total spin of the system is conserved, it can be shown [6] that

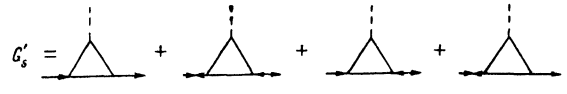
$$\mathcal{J}^{\omega}_{s\alpha} = \sigma_{\alpha} + \mathrm{Sp}_{\sigma'}\sigma'_{\alpha}\Gamma^{\omega}(GG)^{\omega} = \sigma_{\alpha}\,\partial G^{-1}/\partial\varepsilon.$$

In the equations (32) for the spherical harmonic the term containing $\ln(2\xi/\Delta)$ cancels with the left member by the relation (19). It follows from this same relation that the quantity $a^2\rho\Gamma^{\xi}_0$ is logarithmically small. It is natural to expect that the further harmonics of Γ^{ξ} will be smaller than the zeroth harmonic because of the effect of the centrifugal potential. It is easy to see that the criterion for negligibility of further harmonics of $\hat{\mathcal{J}}$ is that the quantity $a^2\rho\Gamma^{\xi}_l$ be small for all $l \neq 0$. The zeroth harmonic of $\hat{\mathcal{J}}$ cannot be neglected, on the other hand, even when $a^2\rho\Gamma^{\xi}_0 \ll 1$, because of the cancellation we have mentioned. Therefore the second of the equations (32) becomes, to logarithmic accuracy,

$$\hat{\mathcal{J}}_i = -v\overline{\hat{\mathcal{J}}_i}/\overline{N}, \qquad (35)$$

where the bar means averaging over the angle.

2. The Change of the Green's Function in an External Field. The change of the Green's function in an external field can be found from the graphical equation



(36)

or the same equation in analytical form

*arcsh = $\sinh^{-1}$

$$G_s' = \{G_s \mathcal{T} G_s + \hat{F}^{(1)} \mathcal{T} \hat{F}^{(2)} + G_s \tau^{(1)} \hat{F}^{(2)} + \hat{F}^{(1)} \tau^{(2)} G_s\} A. \quad (37)$$

A may be a field of any type. Proceeding just as in the derivation of Eq. (30), we get

$$G_s' = \{(G_s G_s - F\hat{F}\hat{P})\, \mathcal{T} + (G_s F - F G_s)\, \tilde{\mathcal{T}}\} A. \quad (38)$$

The expression (38) will be used below in the calculation of the polarization operator.

3. Gauge Invariance. To obtain the conditions imposed by the requirement of gauge invariance, let us find the change of G under the action of a fictitious vector field $A_i = \partial f/\partial x_i = (\partial f/\partial x_\alpha, \partial f/\partial t)$. The quantized operators $\psi(x)$ involved in the definition of G then acquire factors $e^{if(x)}$, and when we go to the momentum representation this gives (cf. the analogous derivation of Ward's identity in quantum electrodynamics):

$$G' = [G(p - q/2) - G(p + q/2)]\, f. \quad (39)$$

Comparing this with Eq. (38), with $\mathcal{T} A = \mathcal{T}_i A_i = \mathcal{T}_i q_i f$, we get the condition of gauge invariance:

$$G_s(p - q/2) - G_s(p + q/2) = (G_s G_s - F\hat{F}\hat{P})\, \mathcal{T}_i q_i + (G_s F - F G_s)\, \tilde{\mathcal{T}}_i q_i. \quad (40)$$

For $\omega, kv \gg \Delta$ the condition (40) goes over into the condition of gauge invariance for a system without pairing:

$$G(p - q/2) - G(p + q/2) = G(p + q/2)\, G(p - q/2)\, \mathcal{T}_i q_i$$

or in another form

$$G^{-1}(p + q/2) - G^{-1}(p - q/2) = \mathcal{T}_i q_i. \quad (41)$$

From Eq. (41) we find for $\mu \gg \omega, kv \gg \Delta$

$$\mathcal{T}^\omega = \partial G^{-1}/\partial \varepsilon, \qquad \mathcal{T}^k_\alpha = -\partial G^{-1}/\partial p_\alpha. \quad (42)$$

These conditions were first obtained by Pitaevskiĭ [7] (cf. also [6]).

Substituting Eq. (10) in Eq. (40) and equating the functions of ϵ and ϵ_p in the left and right members, we can easily get the gauge condition in a system with pairing:

$$\mathcal{T}_i q_i = (\omega - kv)\, \partial G^{-1}/\partial \varepsilon, \qquad \tilde{\mathcal{T}}_i q_i = 2\Delta/a. \quad (43)$$

The conditions (43) are very useful for checking the correctness of expressions one has derived. By some not very cumbersome algebra one can verify that the equations (32) satisfy the conditions (43).

IV. THE SPECTRUM OF TWO-PARTICLE (COLLECTIVE) EXCITATIONS

As has already been mentioned, the spectrum of two-particle excitations is determined by the poles of the vertex $\mathcal{T}$. The residues at the poles give the amplitudes for excitation of these oscillations by an external field. Thus to find the spectrum and the excitation amplitudes it is necessary to solve the system of equations (32).

We shall confine ourselves to the determination of the acoustical branch of the excitations for $\omega, kv \ll \Delta$. The investigation of more complicated cases can be carried out just as in the paper by Vaks and others,[5] where the calculations were made for a weak interaction between the particles.

Let us find the solution of the equations (32) for $\omega, kv \ll \Delta$. Assuming $x^2 \ll 1$, $g = 1$ in Eq. (32), we get

$$\mathcal{T} = \mathcal{T}^\omega - a^2 p \Gamma^\omega \left[\frac{1 + \hat{P}}{2} \mathcal{T} - \frac{\omega + kv}{2\Delta} \tilde{\mathcal{T}}\right],$$

$$\tilde{\mathcal{T}} = a^2 p \Gamma^\xi \left[\left(\ln \frac{2\xi}{\Delta} - x^2\right) \tilde{\mathcal{T}} - \frac{1}{4\Delta} [(\omega + kv) + (\omega - kv)\hat{P}]\, \mathcal{T}\right]. \quad (44)$$

Let us first consider the case of a scalar external field. We shall look for the solution for $\mathcal{T}$ in the form

$$\mathcal{T} = A + B(kv).$$

From the second of the equations (44) we get for the spherical harmonic, using Eq. (19),

$$(\omega^2 - k^2 v^2/3)\, \tilde{\mathcal{T}} = \Delta \{[\omega + (kv) + (\omega - kv)\hat{P}]\, \mathcal{T}\}_0 = 2\Delta(\omega A + k^2 v^2 B/3). \quad (45)$$

The remaining harmonics of $\tilde{\mathcal{T}}$ are smaller by factors $(\omega/\Delta)^2$, $(kv/\Delta)^2$. Therefore the expression (45) can be substituted for $\tilde{\mathcal{T}}$ in the first of the equations (44). After simple but lengthy calculations we get

$$\mathcal{T} = \frac{\partial G^{-1}}{\partial \varepsilon} \frac{\omega^2 - c_1^2 k^2 + f_1 \omega (kv)/3}{\omega^2 - c^2 k^2}, \quad (46)$$

where

$$c_1^2 = v^2 (1 + f_1/3)/3, \qquad c^2 = v^2 (1 + f_0)(1 + f_1/3)/3, \quad (47)$$

and f_0 and f_1 are determined by the relation

$$\frac{1}{2} \mathrm{Sp}\, a^2 p \Gamma^\omega = \sum f_l P_l \left(\frac{\mathbf{pp'}}{p^2}\right). \quad (48)$$

The pole of the expression (46) determines the spectrum of the long-wave acoustical oscillations.

As Landau has shown, $1 + f_1/3 = m^*/m$, and the expression (47) for c^2 coincides with the usual thermodynamic formula for the speed of sound:

$$c^2 = m^{-1} n\, d\mu/dn.$$

The expression for the vector vertex can be obtained in a similar way. Assuming for $\mathcal{T}_\alpha$ the

expression

$$\mathcal{T}_\alpha = A_1 p_\alpha + A_2 k_\alpha + (B_1 k_\alpha + B_2 p_\alpha)\,\mathrm{kv},$$

we can find

$$\mathcal{T}_\alpha = \frac{\partial G^{-1}}{\partial \varepsilon}\left\{p_\alpha + \frac{m^* v^2 k_\alpha\,[f_0\omega + {}^1/_3\,\mathrm{kv}\,(1+f_0)]}{3\,(\omega^2 - c^2k^2)}\right\}. \quad (49)$$

Here we have used a relation derived by Pitaevskiĭ [7]:

$$\mathcal{T}^\omega_\alpha = \frac{m^*}{m}\,\mathcal{T}^k_\alpha = \frac{\partial G^{-1}}{\partial \varepsilon}\,p_\alpha. \quad (50)$$

V. THE POLARIZATION OPERATOR

The flux of particles which arises in the system under the action of a weak external field A_i is given by the relation $\langle j_i \rangle = \mathcal{P}_{ik}A_k$, where $\langle j_i \rangle$ is the mean value of the current density $j_i = (\delta n, j_\alpha)$; δn is the change of density of the particles in the field, and $\mathcal{P}_{ik}$ is the polarization operator.

To find $\mathcal{P}_{ik}$, we express the current density in terms of the Green's function. It follows from the definition of the Green's function that

$$\langle \delta n \rangle = \int \frac{d^4p}{(2\pi)^4\,i}\,(\tilde{G} - G),$$

$$\langle j_\alpha \rangle = \int \frac{d^4p}{(2\pi)^4\,i}\left(p_\alpha - \frac{1}{c}A_\alpha\right)\tilde{G} = \int \frac{d^4p}{(2\pi)^4\,i}\,p_\alpha\,\tilde{G} - \frac{1}{c}\,\tilde{n}A_\alpha, \quad (51)$$

where $\tilde{G}$ is the Green's function in the external field. We shall confine ourselves to the approximation linear in the field; we then get

$$j_i' \equiv (\delta n,\ j_\alpha + c^{-1}A_\alpha n) = (p_i G') = \mathcal{P}'_{ik}A_k,$$

where G′ is determined from Eq. (38). The parentheses denote the integral over $d\tau \equiv d^4p/(2\pi)^4 i$.

Using Eqs. (32) and (38), we find

$$\mathcal{P}'_{ik} = \overline{p_i\,(\mathcal{L}\mathcal{T}^k + \mathcal{M}\tilde{\mathcal{T}}_k)} + (p_i\,(GG)^\omega \mathcal{T}^k). \quad (52)$$

As before the bar indicates averaging over the angular and spin variables. In the last term this formula contains an integration over regions far from the Fermi surface.

We now transform this expression to a form where the only integration is over the Fermi surface. Using the first of the equations (32), we get

$$(p_i\,(GG)^\omega \mathcal{T}_k) = (p_i\,(GG)^\omega \boldsymbol{\mathcal{T}}^\omega_k) + p_i\,(GG)^\omega \Gamma^\omega\,[\mathcal{L}\mathcal{T}_k + \mathcal{M}\tilde{\mathcal{T}}_k].$$

By using the formulas (42) and (50) for $\mathcal{T}^\omega$ and $\mathcal{T}^\omega_\alpha$ we can write the first term in the form $(p_i p_k \partial G/\partial \epsilon)$, so that it gives zero when integrated over ϵ. In the second term we use Eq. (26) to make the substitution $p_i (GG)^\omega \Gamma^\omega = p_i \partial G^{-1}/\partial \epsilon - p_i$. The result is

$$\mathcal{P}'_{ik} = (p_i \partial G^{-1}/\partial \varepsilon)\,(\mathcal{L}\mathcal{T}_k + \mathcal{M}\tilde{\mathcal{T}}_k). \quad (53)$$

Let us verify the gauge invariance of $\mathcal{P}_{ik}$. Multiplying Eq. (53) by q_k and using the relation (43), we get

$$\mathcal{P}'_{ik}q_k = a^{-2}\,(p_i\,(\mathcal{L}v_k q_k + 2\Delta\mathcal{M})),$$

from which and Eq. (33) we have

$$\mathcal{P}'_{0k}q_k = 0, \qquad \mathcal{P}'_{\alpha k}q_k = nk_\alpha$$

or $\mathcal{P}_{ik}q_k = 0$.

For the case $\omega, \mathrm{kv} \ll \Delta$, by using Eqs. (46) and (49) one easily gets

$$\mathcal{P}_{00} = \frac{nk^2}{\omega^2 - k^2c^2}, \quad \mathcal{P}_{\alpha 0} = \mathcal{P}_{0\alpha} = \frac{n\omega k_\alpha}{\omega^2 - k^2c^2},$$

$$\mathcal{P}_{\alpha\beta} = \frac{nc^2 k_\alpha k_\beta}{\omega^2 - k^2c^2} + n\delta_{\alpha\beta}, \quad (54)$$

where c is the speed of sound given by Eq. (47).

The expressions (54) show that the London diamagnetic anomaly can be expressed in terms of the density for an arbitrary interaction between the particles.

In addition to the polarization operator $\mathcal{P}_{ik}$ we can introduce the analogous quantity $\mathcal{P}^s_{\alpha\beta}$, which enables us to find the mean spin density in a magnetic field:

$$\langle \sigma_\alpha \rangle = \mathrm{Sp}\,\sigma_\alpha G' = \mathcal{P}^s_{\alpha\beta}H_\beta.$$

If the total spin of the system is conserved, it can be shown [6] that

$$\mathcal{T}^\omega_{s\alpha} = \sigma_\alpha \partial G^{-1}/\partial \varepsilon. \quad (55)$$

From Eqs. (38) and (55) we get an expression analogous to Eq. (53):

$$\mathcal{P}^s_{\alpha\beta} = \sigma_\alpha \partial G^{-1}/\partial \varepsilon\,(\mathcal{L}\mathcal{T}_{s\beta} + \mathcal{M}\tilde{\mathcal{T}}_{s\beta}). \quad (56)$$

In the limiting case of long waves and low frequencies we get from Eq. (56), by using Eq. (33),

$$\tilde{\mathcal{T}}_{s\beta} = 0, \qquad \mathcal{T}_{s\beta} = \sigma_\beta \partial G^{-1}/\partial \varepsilon, \qquad \mathcal{P}^s_{\alpha\beta} = 0.$$

VI. SYSTEMS OF FINITE DIMENSIONS (THE NUCLEUS)

Since the dimensions of the nucleus are large in comparison with the distance between particles $(R/r_0 \sim A^{1/3})$, many results obtained for the infinite system are also valid for nuclei. The only exceptions are phenomena which are due to the pole parts of the one-particle and two-particle Green's functions. The nonpole parts of the Green's functions are constant for all nuclei to the same accuracy that the density of nuclei is constant.

For the same reason the entire program of reduction of distant integrations to integrations near

the Fermi surface, i.e., the program of renormalizations, can be carried out also for finite systems. Therefore in going over from the formulas for the infinite system to the analogous expressions for the finite system we can use the equations in already renormalized form. For example, to obtain the equation for the vertex in the nucleus, which determines the probabilities of transitions, we are to use not Eq. (22′), but the renormalized equation (32). In this equation the integration is near the Fermi surface.

For finite systems the widths of the regions near the Fermi surface that are important in Eq. (32) are of the order $\delta\epsilon \sim v/R$, and therefore, as in the infinite system, the quantities Γ^{ω} and Γ^{ξ} can be regarded as depending only on the angle between the momentum operators.

The quantity $\mathcal{J}^{\omega}$ is determined from the same considerations as in the infinite system. For example, for scalar and vector vertices $\mathcal{J}^{\omega}$ is determined from gauge invariance. In the case of finite systems it is convenient to study the equation for $\mathcal{J}$ not in the momentum representation, but in the representation in which the Green's function is diagonal. As was shown in [2], this representation is the representation of the eigenfunctions of the one-particle Hamiltonian operator

$$H\varphi_\lambda = (p^2/2M_{eff} + U_{eff}(r))\varphi_\lambda = \varepsilon_\lambda\varphi_\lambda,$$

where the effective mass and the potential U_{eff} are expressed in terms of the proper-energy part $\Sigma(r, p)$ in the mixed representation. Thus the functions φ_λ are the eigenfunctions of a one-particle problem with a potential which is chosen by comparison with experiment.

The forms in this representation for the pole parts of the functions G_s and F, which were introduced earlier, are [2]

$$G_{s\lambda\lambda'}(\varepsilon) = \delta_{\lambda\lambda'}a\frac{\varepsilon+\varepsilon_\lambda}{\varepsilon^2-\varepsilon_\lambda^2-\Delta_\lambda^2},$$

$$F_{\lambda\lambda'}(\varepsilon) = -\delta_{\lambda\lambda'}a\frac{\Delta_\lambda}{\varepsilon^2-\varepsilon_\lambda^2-\Delta_\lambda^2}. \qquad (57)$$

One can also obtain an expression for Δ_λ which expresses this quantity in terms of the one constant Γ^{ξ}.

Transition Probabilities

The vertex $\mathcal{J}$ in the representation of the φ_λ is closely connected with the amplitude for the transition of the nucleus under the action of the field, i.e., with the intensity of electromagnetic transitions. For simplicity let us consider a system without pairing. In the coordinate representation the equation (32) (with $F = 0$) takes the form

$$\mathcal{J}(r, r', \xi) = \mathcal{J}^{\omega}(r, r', \xi) + \int\Gamma^{\omega}(r,r', r_1, r_2)\,G\left(r_1,r_3,\varepsilon+\frac{\omega}{2}\right) \times G\left(r_2,r_4,\varepsilon-\frac{\omega}{2}\right)\mathcal{J}(r_3,r_4,\xi)\,dr_1\,dr_2\,dr_3\,dr_4\,\frac{d\varepsilon}{2\pi i}. \qquad (58)$$

Here there remains a Fourier transformation with respect to the fourth variables. As in Eq. (32), we are to take G to be the pole part of the Green's function. The external field of frequency ω is applied at the point ξ.

In the momentum representation Γ^{ω} depended only on the angles between the momenta

$$\Gamma^{\omega} = \sum_l \mathcal{P}_l(\mathbf{pp}'/p_0^2)\,\Gamma_l^{\omega}.$$

In the coordinate representation the form of Γ^{ω} is

$$\Gamma^{\omega}(r, r', r_1, r_2) = \Gamma_0^{\omega}\delta(r-r')\,\delta(r_1-r_2)\,\delta(r-r_1) - \Gamma_1^{\omega}\nabla\delta(r-r')\,\nabla\delta(r_1-r_2)\,\delta\left(\frac{r+r'-r_1-r_2}{2}\right)+\ldots$$

Similarly, the quantity $\mathcal{J}^{\omega}$ is written in the coordinate representation as

$$\mathcal{J}^{\omega}(r, r', \xi) = \delta(r-\xi)\,\mathcal{J}^{\omega}\,\delta(r'-\xi).$$

For a scalar field $\mathcal{J}^{\omega} = 1/a$, and for a vector field $\mathcal{J}^{\omega} = (-i/a)\nabla$.

In Eq. (58) it is convenient to go over to the representation in which the Green's functions are diagonal

$$\mathcal{J}_{\lambda\lambda'}(\xi) = \mathcal{J}^{\omega}_{\lambda\lambda'}(\xi) + \int\sum_{\lambda_1\lambda_2}\Gamma^{\omega}_{\lambda\lambda'\lambda_1\lambda_2}G_{\lambda_1}G_{\lambda_2}\mathcal{J}_{\lambda_1\lambda_2}(\xi)\,\frac{d\varepsilon}{2\pi i}, \qquad (59)$$

$$\Gamma^{\omega}_{\lambda\lambda'\lambda_1\lambda_2} = \Gamma_0^{\omega}\int\varphi_\lambda(r)\varphi_{\lambda'}(r)\,\varphi_{\lambda_1}(r)\varphi_{\lambda_2}(r)\,dr + \Gamma_1\int j_{\lambda\lambda'}(r)\,j_{\lambda_1\lambda_2}(r)\,dr. \qquad (60)$$

Let us verify that in Eq. (59) the summation and integration are over regions near the Fermi surface. First we note that in the sum of Eq. (59) the energies ϵ_{λ_1} and ϵ_{λ_2} cannot be very different, since when the difference of the numbers of nodes of the functions φ_{λ_1} and φ_{λ_2} is large the integrals (60) are small. Therefore $G_{\lambda_1}G_{\lambda_2}$ has a sharp maximum for $\epsilon \sim \epsilon_{\lambda_1} \sim \epsilon_{\lambda_2}$. Since the functions Γ^{ω}, $\mathcal{J}^{\omega}$, and consequently also $\mathcal{J}$, depend weakly on ϵ, Γ^{ω} and $\mathcal{J}$ can be taken at the point $\epsilon = \epsilon_\lambda$. There remains the integral

$$\int G_{\lambda_1}G_{\lambda_2}\frac{d\varepsilon}{2\pi i} = \frac{n_{\lambda_1}-n_{\lambda_2}}{\varepsilon_{\lambda_1}-\varepsilon_{\lambda_2}+\omega}, \qquad (61)$$

where n_λ are occupation numbers:

$$n_\lambda = \begin{cases} 1, & \varepsilon_\lambda < \varepsilon_0 \\ 0, & \varepsilon_\lambda > \varepsilon_0 \end{cases}.$$

Thus ϵ_{λ_1} and ϵ_{λ_2} lie on opposite sides of the Fermi surface, and consequently are close to ϵ_0. The amplitude for the transition of the nucleus from state λ to state λ' under the action of the field A is connected with the vertex part by the relation

$$M_{\lambda\lambda'} = a\int \mathcal{T}_{\lambda\lambda'}(\xi) A(\xi)\, d\xi. \qquad (62)$$

The field A can be either the electromagnetic field or the β-decay field.

Let us multiply Eq. (59) by $A(\xi)$ and integrate over ξ. Using Eq. (61), we get the equation for the transition amplitude

$$M_{\lambda\lambda'} = M^0_{\lambda\lambda'} + a^2 \sum_{\lambda_1\lambda_2} \Gamma^{\omega}(\lambda, \lambda', \lambda_1, \lambda_2) \frac{n_{\lambda_1} - n_{\lambda_2}}{\varepsilon_{\lambda_1} - \varepsilon_{\lambda_2} + \omega} M_{\lambda_1\lambda_2}, \qquad (63)$$

where $M^0_{\lambda\lambda'}$ is the matrix element calculated with the one-particle model. The kernel of the integral equation (63) can be simplified if we use the quasiclassical approximation for the functions φ_λ.

Thus we find the transition amplitude if we know the one-particle levels ϵ_λ, the wave functions φ_λ, and the renormalized interaction constants Γ^{ω}_{l}. It can be supposed that the Γ^{ω}_{l} fall off rapidly with increasing l, so that to good accuracy one can use only the first one or two of the harmonics. It is important that these constants are not only the same for the various transitions of one nucleus, but are also the same for all nuclei to the same accuracy as the particle density is the same.

Equation (59) is similar to the Schrödinger equation for two interacting particles which are in an external field. Independently of how many particles there are in the last shell, the residual interactions of three or more particles, as well as the interaction with the core, lead only to a renormalization of the two-particle interaction and are included in Γ^{ω}. A detailed study of the equations of the type of Eq. (63) for various transitions in actual nuclei, i.e., with the pair correlation and the interaction between neutrons and protons included, will be made in a subsequent paper.

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