ASYMPTOTIC BEHAVIOR OF THE VERTEX FUNCTION IN THE "SINGLE LOGARITHMIC" APPROXIMATION

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The ultrarelativistic asymptotic behavior of the vertex function in quantum electrodynamics is evaluated in the "single-logarithmic" approximation by assuming relation (1) to hold between the momenta. The result is expressed by formula (20).

1. In Sudakov's paper[1] the asymptotic behavior of the vertex function is obtained for the limiting case

$$|pq| \gg |p^2|, \quad |q^2| \gg m^2; \qquad \ln|pq/p^2|, \quad \ln|pq/q^2| \gg 1 \tag{1}$$

in the approximation in which all the "doubly logarithmic" terms of the type $(e^2 \ln|pq/p^2| \ln|pq/q^2|)^n$ are retained. In the same approximation Abrikosov[2] and Baĭer and Kheĭfets[3] have evaluated cross sections for certain scattering processes.

For sufficiently high momenta "single-logarithmic" terms of the type $e^2 \ln|pq/p^2|$ $(e^2 \ln|pq/p^2| \ln|pq/q^2|)^n$ etc. also become important. The present paper is devoted to the calculation of the vertex function in the limiting case (1) taking into account both the doubly logarithmic and the singly logarithmic terms. The method of calculation developed for this purpose, which is a certain generalization of the "doubly logarithmic" method of Sudakov, can also be utilized for evaluating cross sections.

We consider only the real part of the vertex function. Knowledge of the imaginary part of the corresponding matrix elements is not essential in order to obtain the cross sections in the single-logarithmic approximation.

2. We consider first those diagrams in which all the internal photon lines encompass the principal vertex (we shall call these the principal diagrams). Examples of such diagrams are given in the figure.

The diagram $\Gamma_{1\sigma}$ corresponds to the integral

$$\Gamma_{1\sigma} = \frac{e^2}{\pi i} \int \frac{d^4k}{(2\pi)^2} \frac{\gamma_\mu (\hat{q} - \hat{k}) \gamma_\sigma (\hat{p} - \hat{k}) \gamma_\mu}{[(q-k)^2 + i\varepsilon][(p-k)^2 + i\varepsilon](k^2 + i\varepsilon)}. \tag{2}$$

In virtue of the inequalities (1) the mass of the electron is everywhere neglected. Following Sudakov[1] we represent k in the form

$$k = u(p - \beta q) + v(q - \alpha p) + k_\perp, \tag{3}$$

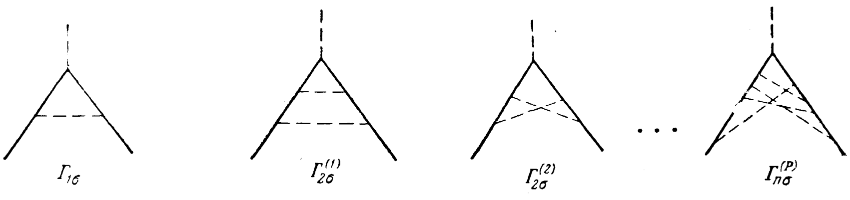
where $k_\perp$ is orthogonal to p and q and is spacelike, and α and β are chosen with the aid of the conditions $(p - \beta q)^2 = 0$, $(q - \alpha p)^2 = 0$ from which we obtain with sufficient accuracy $\alpha = q^2/2pq$, $\beta = p^2/2pq$.

As new variables we adopt $u, v, z = -k_\perp^2/2pq$ and the polar angle φ, which defines the direction of the vector $k_\perp$ in the two dimensional plane orthogonal to p and q. In terms of these variables

$$d^4k = (pq)|pq|\, du\, dv\, dz\, d\varphi, \qquad k^2 = 2pq(uv - z),$$
$$(p-k)^2 = 2pq[(v-\beta)(u-1) - z],$$
$$(q-k)^2 = 2pq[(u-\alpha)(v-1) - z], \tag{4}$$

where u and v vary from $-\infty$ to ∞, φ varies from 0 to 2π, z varies from 0 to ∞ for $pq > 0$ and from 0 to $-\infty$ for $pq < 0$. For the sake of definiteness we assume in our calculations that $pq > 0$; the result can be easily generalized to the case $pq < 0$.

We consider the numerator of the integrand in (2). In averaging over the directions of the vector $k_\perp$ the terms linear in $k_\perp$ will vanish, while the quadratic terms will turn out to be proportional to $(pq)z$. Such terms are not singly logarithmic in the principal domain of small k. In this section we restrict ourselves to the evaluation of the single-logarithmic contribution of this particular region, and, therefore,



we omit $k_\perp$ in the numerator. After this the numerator effectively reduces to $\gamma_\sigma \cdot 4(pq)(1-u)(1-v)$ if we take into account the fact that $\Gamma_\sigma(p, q)$ is flanked both on the left and on the right by the electron propagation functions $\hat{q}/q^2$ and $\hat{p}/p^2$ respectively, and if we neglect in comparison with pq terms p^2 and q^2 arising in the course of this calculation.

Integration over φ yields simply 2π. As a result of this we can write for the real part of $\Gamma_{1\sigma}(p, q)$

$$\mathrm{Re}\,\Gamma_{1\sigma} = -e^2 (2\pi)^{-2} \gamma_\sigma J_1 \{(1-u)(1-v)\}, \quad (5)$$

where $J_1\{\ \}$ is determined by formulas (A.1) and (A.7) of the Appendix.

We now turn to $\Gamma_{2\sigma}^{(1)}$. On writing in accordance with Feynman's rules the expression for $\Gamma_{2\sigma}^{(1)}$ in analogy with (2), and on introducing in place of k_1 and k_2 new variables in accordance with rule (3) we obtain an integral the denominator of whose integrand contains the factors

$$k_{1\perp}^2 = 2pq\,(u_1 v_1 - z_1), \qquad k_{2\perp}^2 = 2pq\,(u_2 v_2 - z_2),$$
$$(p-k_1)^2 = 2pq\,[(v_1-\beta)(u_1-1) - z_1],$$
$$(q-k_1)^2 = 2pq\,[(u_1-\alpha)(v_1-1) - z_1],$$
$$(p-k_1-k_2)^2 = 2pq\,[(v_1+v_2-\beta)(u_1+u_2-1) - z_1 - z_2 - 2\sqrt{z_1 z_2}\cos(\varphi_1-\varphi_2)],$$
$$(q-k_1-k_2)^2 = 2pq\,[(u_1+u_2-\alpha)(v_1+v_2-1) - z_1 - z_2 - 2\sqrt{z_1 z_2}\cos(\varphi_1-\varphi_2)]. \quad (6)$$

A new feature as compared with $\Gamma_{1\sigma}$ is the presence of the terms $2(z_1 z_2)^{1/2}\cos(\varphi_1-\varphi_2) = -2k_1^\perp k_2^\perp$. However, detailed estimates show that in calculations of single-logarithmic accuracy these terms can be neglected. Qualitatively this is explained by the fact that in the principal domain of small variables the term $(z_1 z_2)^{1/2}$ is of the same order of magnitude as z_1+z_2 only in the narrow region in which $z_1 \sim z_2$ and the effect of this is not logarithmic.

By neglecting the terms indicated above and by subjecting the numerator to the same manipulations as was done in the case of $\Gamma_{1\sigma}$ we obtain the following results:

$$\mathrm{Re}\,\Gamma_{2\sigma}^{(1)} = (-1)^2 (e/2\pi)^4 \gamma_\sigma J_2^{(1)} \{(1-v_1)(1-v_1-v_2) \times (1-u_1-u_2)(1-u_1)\}. \quad (7)$$

Similarly we have

$$\mathrm{Re}\,\Gamma_{2\sigma}^{(2)} = (-1)^2 (e/2\pi)^4 \gamma_\sigma J_2^{(2)} \{(1-v_2)(1-v_1-v_2) \times (1-u_1-u_2)(1-u_1)\}. \quad (8)$$

The definition of $J_n^{(P)}\{\ \}$ is given by formula (A.1) of the Appendix.

In the general case we have

$$\mathrm{Re}\Gamma_{n\sigma}^{(P)} = (-1)^n \left(\frac{e}{2\pi}\right)^{2n} \gamma_\sigma J_n^{(P)} \Big\{\prod_{k=1}^{n} 1 - u_1 - \ldots - u_k) \times \prod_{s=1}^{n-m} (1 - v_{p_1} - \ldots - v_{p_s}) \times \prod_{r=0}^{m-1} (1 - v_{p_1} - \ldots - v_{p_{n-m+r}} - v_n)\Big\}, \quad (9)$$

where the numbers p_i and the number m both depend on the number of the permutation P of the right hand ends of the photon lines (we regard their left hand ends to be fixed), and m is the number of brackets in the numerator containing v_n.

In the single-logarithmic approximation we must keep in the numerator only the variables v_n and u_{k_p} (where the subscript k_p depends on P, i.e., on the form of the diagram), which appear in the denominator of the integrand in the combinations $v_n - \beta$ and $u_{k_p} - \alpha$ only once. Moreover, we have

$$\mathrm{Re}\,\Gamma_{n\sigma}^{(P)} = (-1)^n (e/2\pi)^{2n} \gamma_\sigma J_n^{(P)} \{(1-v_n)^m (1-u_{k_p})^{l_p}\} \quad (10)$$

containing a certain l_p which depends on P.

The value of the integral appearing in (10) is given by formula (A.7) of the Appendix. The result is independent of P and, therefore, the summation over P reduces to multiplication by n!. Moreover, in summing over all values of n we obtain

$$\mathrm{Re}\,\Gamma_\sigma = \gamma_\sigma \left\{1 - \frac{e^2}{2\pi}\ln|\alpha\beta|\right\} \times \exp\left(-\frac{e^2}{2\pi}\ln|\alpha|\ln|\beta|\right) + \ldots \quad (11)$$

The repeated dots denote the missing contribution from the region of large momenta of the virtual photons. It is evaluated in the next section.

3. The contribution of the principal diagrams to the single-logarithmic approximation comes not only from the region of low momenta of virtual photons but also from large momenta. We have in mind that region in which the integrals diverge, and the corresponding contribution arises after regularization. We denote this contribution of the diagram $\Gamma_{n\sigma}^{(P)}$ by $L_{n\sigma}^{(P)}$. It is clear that for a diagram of the first order in $\Gamma_{1\sigma}$ we have

$$L_{1\sigma} = \frac{e^2}{\pi i} \int \frac{d^4k}{(2\pi)^2} \frac{\gamma_\mu \hat{k} \gamma_\sigma \hat{k} \gamma_\mu}{(p-k)^2 (q-k)^2 k^2}.$$

By regularizing this expression in accordance with the usual rules [4] it can be easily shown that with logarithmic accuracy we have

$$L_{1\sigma}^R = \gamma_\sigma L_1^R, \quad (12)$$

$$L_1^R = (e^2/4\pi)\{\ln|m^2/pq| - 2\ln(\lambda^2/m^2)\} \quad (13)$$

(λ is the photon "mass").

It can be easily shown that in the n-th order of all the n! diagrams $\Gamma_{n\sigma}^{(P)}$ a contribution resulting from large momenta of the virtual photons will be given only by the $(n-1)!$ diagrams in which the uppermost photon line does not intersect the other lines, and it is this particular line that is characterized by the large momentum, while the momenta of the other photon lines must be small. In such a case we have

$$L_{n\sigma}^{(P)} = \Gamma_{n-1,\sigma}^{(P)} L_1^R. \quad (14)$$

Summing over all such diagrams and utilizing (11) and (13) we obtain

$$L_\sigma \equiv \sum L_{n\sigma}^{(P)} = L_1^R \sum \Gamma_{n-1,\sigma}^{(P)}$$

$$= \frac{e^2}{4\pi}\left\{\ln\left|\frac{m^2}{pq}\right| - 2\ln\frac{\lambda^2}{m^2}\right\}\exp\left(-\frac{e^2}{2\pi}\ln|\alpha|\ln|\beta|\right). \quad (15)$$

4. We now consider diagrams obtained from the principal diagrams (cf., the figure) by the insertion of an electron loop into one of the photon lines. Such a line corresponds to the propagator function

$$D_{\mu\nu}^{(2)}(k^2) = \frac{e^2}{\pi}\frac{I(k^2)}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right),$$

$$I(k^2) = \frac{1}{2}\int_0^1 dz\,(1-z^2)\ln\left[1 - \frac{k^2}{4m^2}(1-z^2)\right]. \quad (16)$$

It can be shown that in using the method of integration adopted by us (cf. Appendix) the single-logarithmic contribution comes from the imaginary part $I(k^2)$, which is given with sufficient accuracy [cf., for example, [5], (32.14)] by $-i\pi\theta(k^2-4m^2)/3$. Below we shall evaluate the contribution of the term $\sim\delta_{\mu\nu}$. An analogous investigation shows that the term $\sim k_\mu k_\nu$ gives no contribution to the single-logarithmic approximation.

We consider first of all the diagram obtained by the insertion of a loop into $\Gamma_{1\sigma}$. It corresponds to the integral $(\gamma \equiv 2m^2/pq > 0)$

$$\Pi_{1\sigma} = -\gamma_\sigma\frac{e^4}{12\pi^2}$$

$$\times\int_0^\infty dz\iint_{-\infty}^{\infty}\frac{du\,dv\,\theta(uv-z-\gamma)}{[(v-\beta)(u-1)-z][(u-\alpha)(v-1)-z](uv-z)}. \quad (17)$$

In the principal region $(1 \gg |v| \gg \beta,\ 1 \gg |u| \gg |\alpha|)$ we can neglect the quadratic terms and z in the first two factors of the denominator. Integration over z yields $\theta(uv-\gamma)\ln|uv/\gamma|$. As a result of this we see that $\Pi_{1\sigma}$ is obtained from $\Gamma_{1\sigma}$ by the following replacement

$$\theta(uv) \to (e^2/3\pi)\,\theta(uv-\gamma)\ln|uv/\gamma|$$

in the integral $J_1\{1\}$. Similarly, the diagram obtained by the insertion of a loop into the i-th photon line of the diagram $\Gamma_{n\sigma}^{(P)}$ corresponds to the expression obtained from $\Gamma_{n\sigma}^{(P)}$ by the replacement

$$\theta(u_iv_i) \to (e^2/3\pi)\,\theta(u_iv_i-\gamma)\ln|u_iv_i/\gamma|$$

in $J_n^{(P)}\{1\}$.

The subsequent integration can be carried out particularly easily when $|\gamma| \lesssim |\alpha\beta|$. In this case we can omit γ in $\theta(u_iv_i-\gamma)$. Summation over i from unity up to n now leads to the integral obtained from $J_n^{(P)}\{1\}$ by adding the factor $(e^2/3\pi)\times\ln|u_1v_1\ldots u_nv_n/\gamma^n|$. In virtue of the symmetry of the numerator of the integrand the integral is independent of P, since for any P one can perform an interchange of the variables $u_k \to u_i$ which is the inverse of P. Therefore one can take the lines in the order corresponding to a ladder diagram, and then the summation over P reduces to multiplication by n!.

Finally the contribution $\Pi_{n\sigma}$ of all the diagrams obtained from all the $\Gamma_{n\sigma}^{(P)}$ by all possible insertions of an electron loop will turn out to be the product of n! and the expression obtained from $\Gamma_{n\sigma}^{(1)}$ by the addition of the factor $(e^2/3\pi)\ln|u_1v_1\ldots u_nv_n/\gamma^n|$ to $J_n^{(1)}\{1\}$. The evaluation of the integral is not complicated and leads to the following result

$$\Pi_{n\sigma} = \gamma_\sigma\frac{e^2}{6\pi(n-1)!}\ln\left|\frac{\alpha\beta}{\gamma^2}\right|\left(-\frac{e^2}{2\pi}\ln|\alpha|\ln|\beta|\right)^n. \quad (18)$$

Insertions of two or more loops simultaneously need not be taken into account since they lead to a decrease in the logarithmic order by two or more.

The total contribution of the polarization of the vacuum to the single-logarithmic approximation is consequently equal to

$$\Pi_\sigma = \sum_n \Pi_{n\sigma} = -\gamma_\sigma\frac{e^4}{12\pi^2}\ln|\alpha|\ln|\beta|\ln\left|\frac{\alpha\beta}{\gamma^2}\right|$$

$$\times\exp\left(-\frac{e^2}{2\pi}\ln|\alpha|\ln|\beta|\right). \quad (19)$$

5. Insertion into the principal diagrams (cf. the figure) of a photon line which does not encompass the main vertex leads to single-logarithmic diagrams only in those cases when this line either does not encompass any side vertices (electronic self-energy part), or encompasses only one side vertex (vertex part). In this case it is this side photon line that must carry a large momentum, while the remaining proton lines must carry low momenta.

However, it can be easily shown that the single-logarithmic contributions of such diagrams mutually cancel. This circumstance is related to Ward's theorem. The insertion of two or more side photon lines diminishes the logarithmic order of the diagrams by two or more.

Detailed estimates of diagrams obtained from the principal diagrams by the insertion of "parquet-like" parts show that such diagrams also give no contribution to the single-logarithmic approximation. Here we shall confine ourselves just to this remark since the corresponding calculations are fairly awkward.

6. We have carried out all the calculations using the photon propagator function in the form $k^{-2}\delta_{\mu\nu}$ which corresponds to the choice $d_l = 1$ for the longitudinal part. It can be easily shown that the choice of other numerical values for d_l will change only the contribution of the region of large momenta, viz., the result (15) will simply be multiplied by d_l.

On taking this remark into account and on combining (11), (15), and (19) we obtain the following final result

$$\operatorname{Re}\Gamma_\sigma(p,q) = \gamma_\sigma \Big\{1 - \frac{e^2}{2\pi}\ln|\alpha\beta| + \frac{e^2}{4\pi} d_l \Big(\ln|\gamma| - 2\ln\frac{\lambda^2}{m^2}\Big) - \frac{e^4}{12\pi^2}\ln|\alpha|\ln|\beta|\ln\Big|\frac{\alpha\beta}{\gamma^2}\Big|\Big\}\exp\Big(-\frac{e^2}{2\pi}\ln|\alpha|\ln|\beta|\Big),$$

$$\alpha = \frac{q^2}{2pq}, \qquad \beta = \frac{p^2}{2pq}, \qquad \gamma = \frac{2m^2}{pq}, \qquad |\gamma| \lesssim |\alpha\beta|. \tag{20}$$

In the calculations we have taken $pq > 0$, but the result (20) is valid also for $pq < 0$.

The contribution due to the polarization of the vacuum (19) is calculated for $|\gamma| \lesssim |\alpha\beta|$. The case $|\gamma| \gg |\alpha\beta|$ compatible with (1) does not lead to such simple expressions and requires special consideration.

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APPENDIX

We shall now describe the method of calculating in the single-logarithmic approximation the integrals associated with the principal diagrams. In doing this we take into account only the contribution of the principal region where all the variables of integration are small.

The general structure of the real part of the integrals corresponding to a certain n-th order diagram can be expressed in the following form

$$J_n^{(P)}\{\ \} = \operatorname{Re}\int \frac{\{\ \}\, i\,du_1\,dv_1\,dz_1 \ldots i\,du_n\,dv_n\,dz_n}{\prod_{k=1}^{n}[z_k][v_k]\prod_{s=1}^{n}[u_{p_s}]};$$

$$[z_k] \equiv u_k v_k - z_k + i\varepsilon,$$

$$[v_k] \equiv (v_1 + \ldots + v_k - \beta)(1 - u_1 \ldots - u_k) + z_1 + \ldots + z_k - i\varepsilon,$$

$$[u_{p_s}] \equiv (u_{p_1} + \ldots + u_{p_s} - \alpha)(1 - v_{p_1} - \ldots - v_{p_s}) + z_{p_1} + \ldots + z_{p_s} - i\varepsilon, \tag{A.1}$$

where the symbol P denotes a permutation of the right hand ends of the photon lines which converts the n-th order "ladder" diagram into the given one; the nature of this permutation determines the values of the subscripts p_s which can vary from unity up to n; the curly brackets can contain functions of the type 1, $(1-u_i)(i-v_k)$, z_l etc.; the variables u_k and v_k vary between the limits $(-\infty, \infty)$, while z_k varies between the limits $(0, \infty)$.

We first of all consider the simplest integral

$$J_1\{1\} = \operatorname{Re}\int \frac{i\,du\,dv\,dz}{(uv - z + i\varepsilon)[(v-\beta)(1-u) + z - i\varepsilon][(u-\alpha)(1-v) + z - i\varepsilon]}$$

$$= \pi\int du\,dv \Big\{\frac{\theta[(v-\beta)(u-1)] - \theta(uv)}{[(v-\beta+\beta u) - (u-\alpha+\alpha v)](v-\beta+\beta u)} + \frac{\theta[(u-\alpha)(v-1)] - \theta(uv)}{[(u-\alpha+\alpha v) - (v-\beta+\beta u)](u-\alpha+\alpha v)}\Big\},$$

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}.$$

The integrand has no singularities only if taken as a whole, but if we interpret all the integrals as principal values, we can evaluate each term separately.

We pick out from the whole region of integration the square $(-1 \le u \le 1,\ -1 \le v \le 1)$. It can be shown, and this is a characteristic property of all the integrals of the type $J_n^{(P)}\{1\}$, that: 1) the region outside the square $(-1, 1)$ gives no single-logarithmic contribution; 2) the terms with $\theta[(v-\beta)(u-1)]$ and $\theta[(u-\alpha)(v-1)]$ give no contribution even inside the square; 3) in the term containing $\theta(uv)$ we can neglect both αv and βv. As a result of this we obtain

$$J_1\{1\} = \pi\int_{-1}^{1} du \int_{-1}^{1} dv \frac{\theta(uv)}{(u-\alpha)(v-\beta)}$$

$$= 2\pi \ln|\alpha|\ln|\beta| + O(1). \tag{A.2}$$

We now consider $J_n^{(P)}\{(1-v_n)^m\}$ where m coincides with the number of factors in the denominator of the integrand containing the combination $u_n - \alpha$. If v_n occurs in the numerator then the integration with respect to v_n ceases to be logarithmic and must be carried out more accurately. The principal contribution now comes from the region where v_n is finite, while the remaining variables are small so that they can be neglected in comparison with unity and v_n. For the integration with respect to z_i, u_i, v_i for $i \neq n$ the method which was used for the evaluation of $J_1\{1\}$ remains valid, i.e., we can integrate over u_i and v_i using principal values within the square $(-1, 1)$, and in this case in the integral over z_i an effective contribution is given only by the pole $z_i = u_i v_i$, as a result of which z_i can be rejected in advance everywhere except in the principal bracket $(u_i v_i - z_i + i\epsilon)$, and after this the integration over z_i is elementary and yields $-i\pi\theta(u_i v_i)$. The logarithmic contribution from the integration over u_n also comes from the region of small values of u_n, therefore the effective region for u_n is also $(-1, 1)$.

After carrying out the operations indicated and making the allowable approximations we can write

$$J_n^{(P)}\{(1-v_n)^m\} = \operatorname{Re} \pi^{n-1} \int (1-v_n)^m \, i \, du_n \, dv_n \, dz_n \times \prod_{i=1}^{n-1} \theta(u_i v_i) \, du_i \, dv_i \Big\{(u_n v_n - z_n + i\varepsilon) \times \prod_{k=1}^{n} (v_1 + \ldots + v_k - \beta) \prod_{s=1}^{n-m} (u_{p_1} + \ldots + u_{p_s} - \alpha) \times \prod_{r=0}^{m-1} [(u_{p_1} + \ldots + u_{p_{n-m+r}} + u_n - \alpha) \times (1 - v_n) + z_n - i\varepsilon]\Big\}^{-1},$$

where the integration is taken over the region indicated previously.

Without reducing the degree of accuracy we can simplify the expression obtained by leaving in each of the curved brackets only that variable v_i or u_k (for $i, k = 1, 2, \ldots, n-1$) which appears in the denominator a smaller number of times, provided that in doing so we also change the limits of integration by taking them to be equal to $(v_{n-2}, 1)$, $(v_{n-3}, 1)$, $\ldots, (v_1, 1)$, $(\beta, 1)$ respectively for v_{n-1}, v_{n-2}, $\ldots, v_2, v_1$, and similarly for the variables u_p. On carrying out the integration over z_n we obtain

$$J_n^{(P)}\{(1-v_n)^m\} = \pi^n \int (1-v_n)^m \, du_n \, dv_n \prod_{i=1}^{n-1} \theta(u_i v_i) \, du_i \, dv_i \Big\{(v_{n-1} + v_n) \prod_{k=1}^{n-1} v_k \prod_{s=1}^{n-m} u_{p_s}\Big\}^{-1} \times \sum_{r=0}^{m-1} \frac{\theta(u_n v_n) - \theta[(v_n - 1)(u_n + u_{p_{n-m+r}})]}{(1-v_n)^{m-1} [u_n + u_{p_{n-m+r}} (1-v_n)] (-u_{p_{n-m+r}})^r u_{p_{n-m+r+1}} \cdots u_{p_{n-1}}}.$$

The factor $(1-v_n)^{m-1}$ in the denominator completely cancels a similar factor in the numerator. This cancellation is not accidental, it guarantees the absence of a singularity at the point $v_n = 1$. Now there remains in the numerator only $1 - v_n$. Integration over u_n and v_n gives us with the required degree of accuracy

$$\int \frac{(1-v_n)\, dv_n}{v_n + v_{n-1}} \int_{-1}^{1} du_n \frac{\theta(u_n v_n) - \theta[(v_n - 1)(u_n + u_p)]}{u_n + u_p (1 - v_n)} = 2 (\ln |v_{n-1}| + 1) \ln |u_p|.$$

Therefore we have

$$J_n^{(P)}\{(1-v_n)^m\} = 2\pi^n \int \prod_{i=1}^{n-1} \theta(u_i v_i) \, du_i \, dv_i \, (\ln |v_{n-1}| + 1) \times \Big\{\prod_{k=1}^{n-1} v_k \cdot \prod_{s=1}^{n-m} u_{p_s}\Big\}^{-1} \sum_{r=0}^{m-1} \frac{\ln |u_{p_{n-m+r}}|}{(-u_{p_{n-m+r}})^r u_{p_{n-m+r+1}} \cdots u_{p_{n-1}}}$$

A contribution to the single-logarithmic approximation will be given only by the first two terms of the sum $(r = 0, 1)$. Taking into account the fact that in the given integral we can make the replacement

$$\prod_{i=1}^{n-1} \theta(u_i v_i) \to 2^{n-1} \prod_{i=1}^{n-1} \theta(u_i) \, \theta(v_i)$$

and writing out explicitly the limits of integration we obtain finally

$$J_n^{(P)}\{(1-v_n)^m\} = (2\pi)^n \int_{|\beta|}^{1} \frac{dv_1}{v_1} \int_{v_1}^{1} \frac{dv_2}{v_2} \cdots \int_{v_{n-2}}^{1} \frac{dv_{n-1}}{v_{n-1}} (\ln |v_{n-1}| + 1) \times \int_{|\alpha|}^{1} \frac{du_{p_1}}{u_{p_1}} \int_{u_{p_1}}^{1} \frac{du_{p_2}}{u_{p_2}} \cdots \int_{u_{p_{n-2}}}^{1} \frac{du_{p_{n-1}}}{u_{p_{n-1}}} (\ln |u_{p_{n-m}}| - \ln |u_{p_{n-m+1}}|)$$
$$= (2\pi)^n \left\{\frac{\ln^n |\alpha| \ln^n |\beta|}{n!\, n!} + \frac{\ln^n |\alpha| \ln^{n-1} |\beta|}{n!\,(n-1)!}\right\} + O(\ln^{n-1} |\alpha| \ln^{n-1} |\beta|). \qquad \text{(A.3)}$$

The result is independent of m and p, i.e., it is the same for all n-th order diagrams.

In accordance with the above derivation we can evidently identify the first and second terms with the corresponding parts of the original integral:

$$J_n^{(P)}\{(1-v_n)^{m-1}\} = (2\pi)^n \ln^n|\alpha|\ln^n|\beta|/n!\,n!, \quad \text{(A.4)}$$

$$J_n^{(P)}\{-v_n(1-v_n)^{m-1}\} = (2\pi)^n \ln^n|\alpha|\ln^{n-1}|\beta|/n!\,(n-1)!. \quad \text{(A.5)}$$

From the derivation of (A.4) it can be easily seen that the principal contribution to this integral arises from the region of small values of v_k in which

$$J_n^{(P)}\{(1-v_n)^{m-1}\} = J_n^{(P)}\{1\}.$$

In carrying out similar calculations it is more convenient to make use of this fact from the beginning.

On taking into account the results obtained we can write with single-logarithmic accuracy

$$J_n^{(P)}\{(1-v_n)^m(1-u_{k_p})^{l_p}\} = J_n^{(P)}\{1-v_n(1-v_n)^{m-1} - u_{k_p}(1-u_{k_p})^{l_p-1}\}. \quad \text{(A.6)}$$

On the basis of the results (A.4) and (A.5) and of symmetry considerations with respect to α and β (A.6) enables us to write

$$J_n^{(P)}\{(1-v_n)^m(1-u_{k_p})^{l_p}\} = (2\pi)^n \ln^{n-1}|\alpha|\ln^{n-1}|\beta|(\ln|\alpha|\ln|\beta| + n\ln|\alpha\beta|)/n!\,n!. \quad \text{(A.7)}$$

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