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INTEGRAL REPRESENTATION OF A SQUARE DIAGRAM WITH ANOMALOUS MASS RATIO

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The Bergman-Weil integral representation is employed to write a double dispersion relation (in energy and momentum transfer) for a square diagram with arbitrary stable masses. The relation consists of a sum of three integrals the first of which coincides with the usual Mandelstam representation. The last two integrals vanish upon transition to a normal mass ratio ($\theta \to 2\pi$).

1. In a previous paper, [1] the author has considered the most general form of multiple dispersion relations which follow from the Bergman-Weil representation and reduce to the Mandelstam representation in a particular case. As a first example, the triangular diagram for a vertex part has been investigated, for which a double dispersion relation in the squares of the external momenta was written down. In the present paper we consider the next example: the square diagram for a scattering amplitude (Fig. 1). Using the method developed in the previous paper, we write down a double dispersion relation in the energy and momentum transfer for this diagram, which is valid in the presence of anomalous singularities, when the Mandelstam representation breaks down.[1)]

We consider here the most general case of arbitrary internal masses and use the method of the previous paper. [1]

2. Let us consider the diagram for the scattering amplitude of Fig. 1. We shall use the notation (see [4,5])

$$\mu_{ik} = (m_i^2 + m_k^2 - p_{ik}^2)/2m_i m_k,$$
$$\theta_{ik} = \arccos \mu_{ik}, \quad \theta \equiv \theta_{12} + \theta_{23} + \theta_{34} + \theta_{14}$$

(i, k are adjacent indices). Let us also write, for brevity, $x \equiv \mu_{13}$, $y \equiv \mu_{24}$.

In the simplest case, when all $\theta_{ik} + \theta_{kl} \leq \pi$ (and hence $\theta \leq 2\pi$), the physical sheet is defined by the condition that for $y > 1$ the following dispersion relation holds (see, e.g., the paper of Gribov et al. [5]):

$$A(x, y) = \frac{1}{\pi} \int_{-\infty}^{-1} \frac{A_1(x', y)}{x' - x} dx';$$

$$A_1 = \frac{1}{\sqrt{K}} \ln \frac{V_1 - \sqrt{(x^2-1)K}}{V_1 + \sqrt{(x^2-1)K}}, \qquad K = \det |\mu_{ik}|, V_1 \equiv \frac{1}{2}\frac{\partial K}{\partial y}. \tag{1}$$

The Mandelstam representation [6] is easily derived from this:

$$A(x, y) = \frac{1}{\pi^2} \int_{\infty}^{-1} \frac{dx'}{x' - x} \int_{-\infty}^{-1} \frac{dy'}{y' - y} \frac{2\pi\theta(K)}{\sqrt{K(x', y')}} \quad \theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}. \tag{2}$$

The transition to anomalous mass values, for which $\theta_{ik} + \theta_{kl} > \pi$, and to the case $\theta > 2\pi$ is effected by analytic continuation of the relation (1) in μ_{ik}, as was done, for example, in [5].

However, the analytic continuation of the representation (2) in μ_{ik} is not trivial, since the singularity of the integrand passes through the boundary of the region of integration, $x' = -1$, $y' = -1$. We can increase μ_{12} in such a way that the lines $x = \Delta_1 \equiv \cos(\theta_{12} + \theta_{23})$ and $y = \Delta_1' \equiv \cos(\theta_{12} + \theta_{14})$ become singular (for this it is necessary that $\theta_{12} + \theta_{23} > \pi$ and $\theta_{12} + \theta_{14} > \pi$). If also $\theta \leq 2\pi$, a Mandelstam representation can be written down for the function $A(x, y)$ which coincides formally with the analytic continuation of formula (2), if the upper limits of the integration over x' and y' are replaced by Δ_1 and Δ_1', respectively.

As μ_{12} is increased further, so that θ becomes larger than 2π, the amplitude $A(x, y)$ acquires a singularity on a part of the surface $K(x, y) = 0$ which does not any more lie on the usual cuts in the x and y planes.

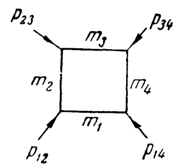

FIG. 1

[1)]Recently, the papers of Fronsdal et al.[2] and Islam [3] appeared, in which the Bergman-Weil representation was obtained for the special cases of equal[2] or pairwise equal[3] cosines μ_{ik}. These representations have a very special character and cannot be generalized to the case of unequal masses or a larger number of variables.

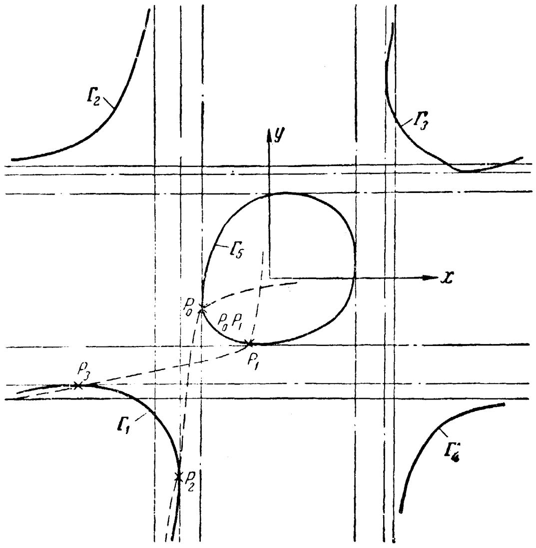

FIG. 2

3. We shall assume, for definiteness, that

$$\theta_{12}+\theta_{23}>\pi, \qquad \theta_{12}+\theta_{14}>\pi,$$

$$\theta_{34}+\theta_{14}<\pi, \qquad \theta_{23}+\theta_{34}<\pi, \qquad \theta>2\pi.$$

In Fig 2 we show, in the plane of real x and y, the lines $x = \pm 1$, $x = \Delta_j = \cos(\theta_{ik} \pm \theta_{kl})$, $y = \pm 1$, $y = \Delta_j'$, and the lines $K(x, y) = 0$, denoted by $\Gamma_1, \ldots, \Gamma_5$.

We may conclude from the analysis of the singularities of a square diagram of Tarski [7] and Vladimirov [10] that the singularities of the function $A(x, y)$ are the following: $x = -1$, $y = -1$, $x = \Delta_2$, $y = \Delta_2'$, the arc P_0P_1 of the curve Γ_5, and the complex surface Σ_1, joining P_0P_1 and the arc P_3P_2 of the curve Γ_1. The curve Γ_1 itself is also singular, but only in the limit $x \to \bar{x} \pm i\epsilon$, $y \to \bar{y} \mp i\epsilon$ (see Fig. 2). The last three singularities are determined by the equation $K(x, y) = 0$.

The function $A(x, y)$ is therefore analytic anywhere outside the hypersurfaces representing the cuts coming from the singularities enumerated above:

$$L_x: \qquad x = \Delta_2 - r, \quad 0 \leqslant r < \infty;$$
$$L_y: \qquad y = \Delta_2' - r', \quad 0 \leqslant r' < \infty;$$
$$M_{xy}: \; K(x, y) = -r, \qquad 0 \leqslant r < \infty.$$

We note that $K(x, y) > 0$ in the regions beyond the curves Γ_j of Fig. 2 and $K(x, y) < 0$ between the curves Γ_j.

If $y = \bar{y} - i\epsilon$, we have two cuts in the x plane: L_x and $M_{x,\bar{y}-i\epsilon}$, which intersect one another at some point x_V (Fig. 3). This implies that in the point x_V the cut M_{xy} migrates from the first Riemann sheet of the branchpoint Δ_2 to the second sheet. We are therefore only interested in the part of the cut M_{xy} from the singular point

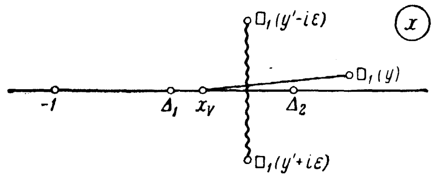

FIG. 3

to the point of intersection. It is this part which is the boundary of the region of analyticity of the function $A(x, y)$. For values of y for which the singular point $\Box_1(y)$ itself disappears from the sheet under consideration, the cut M_{xy} does not belong at all to the manifold of boundaries of the region of analyticity.

Let us determine the curve along which L_x and M_{xy} intersect for $y = \bar{y} + i\epsilon$. Setting $y = \bar{y} - i\epsilon$ in the equation $K(x, y) = -r$ and separating out the imaginary part, we obtain

$$\operatorname{Im} x = \varepsilon \frac{\partial K}{\partial y} \Big/ \frac{\partial K}{\partial x}. \tag{3}$$

From this we find the desired equation for the determination of x_V in the form

$$\partial K / \partial y = 0. \tag{4}$$

Part of the curve satisfying this equation is indicated in Fig. 2 by the dashed line P_0P_2.

After the equations for the cuts have been chosen, it is easy to write down a dispersion relation in a single variable. For example, if $y_{P_0} > y > \Delta_2$, the dispersion relation in x has the form (see Fig. 3)

$$A(x, y) = \frac{1}{\pi} \int_{-\infty}^{-1} \frac{A_1(x', y)}{x' - x} dx'$$
$$- \frac{1}{\pi} \int_{-1}^{\Delta_2} \frac{dx'}{x' - x} \frac{2\pi i}{\sqrt{K}} - \frac{1}{\pi} \int_{x_V}^{\Box_1(y)} \frac{dx'}{x' - x} \frac{4\pi i}{\sqrt{K}}. \tag{5}$$

In the last integral $\arg K = \pi$, $\Box_1(y)$ is the smaller root of the equation $K(x, y) = 0$; in the second integral $\arg K = \pi$ for $x' < x_V$, $\arg K = -\pi$ for $x' > x_V$, where x_V is determined by (4). Equation (5) coincides with (42) of the paper of Gribov et al. [5]

With the help of (3) and Fig. 2 we can now follow the motion of the singular point $\Box(y - i\epsilon)$ as y moves along the real axis. $\Box(y - i\epsilon)$ appears on the physical sheet in the point P_0 [here, according to (3), $\operatorname{Im} \Box(y - i\epsilon)$ changes sign, but P_0 lies on the cut and hence $\Box(y - i\epsilon)$ goes from one sheet of the Riemann surface to another]. As y is decreased, the point $\Box(y - i\epsilon)$ moves along the arc P_0P_1 and subsequently emerges on the upper complex half plane, while $\operatorname{Re} \Box(y - i\epsilon)$ moves along the line P_1P_3, then along the arc P_3P_2, and disappears again from the sheet under consideration in the point P_2. Im $\Box(\bar{y} - i\epsilon) > 0$ everywhere for $y = \bar{y} - i\epsilon$.

We can now write down the Bergman-Weil representation for the function $A(x, y)$ (see [1] and references quoted there):

$$A(x, y) = J_{12} + J_{13} + J_{23}; \tag{6}$$

$$J_{12} = \frac{1}{(2\pi i)^2} \int_{L_x} \frac{dx'}{x' - x} \int_{L_y} \frac{dy'}{y' - y} A(x', y'), \tag{7}$$

$$J_{13} = \frac{1}{(2\pi i)^2} \int_{L_x} \frac{dx'}{x' - x} \int_{M_{xy}} \frac{dy' \, QA(x', y')}{K(x', y') - K(x, y)}, \tag{8}$$

$$J_{23} = \frac{1}{(2\pi i)^2} \int_{L_y} \frac{dy'}{y' - y} \int_{M_{xy}} \frac{dx' PA(x', y')}{K(x', y') - K(x, y)}. \tag{9}$$

The integration along each of the cuts goes along both lips in the positive direction. The functions $P(x, y; x', y')$ and $Q(x, y; x', y')$ are determined by

$$K(x', y') - K(x, y) = (x' - x) P(x', y'; x, y) + (y' - y) Q(x', y'; x, y); \tag{10}$$

their explicit forms are not needed here.

J_{12} and J_{23} can be rewritten in the form

$$J_{12} = \frac{1}{\pi^2} \int_{-\infty}^{\Delta_1} \frac{dx'}{x' - x} \int_{-\infty}^{\Delta_1'} \frac{dy'}{y' - y} \rho(x', y'), \tag{11}$$

$$J_{23} = \frac{1}{\pi^2} \int_{-\infty}^{\Delta_2'} \frac{dy'}{y' - y} \left\{ \int_{x_V'}^{\Box_1(y' + i\varepsilon)} \frac{dx' P\rho'(x', y' + i\varepsilon)}{K' - K} - \int_{x_V'}^{\Box_1(y' - i\varepsilon)} \frac{dx' P\rho'(x', y' - i\varepsilon)}{K' - K} \right\}. \tag{12}$$

Here

$$\rho(x, y) = (2i)^{-2} (A_{++} - A_{+-} - A_{-+} + A_{--}), K' \equiv K(x', y') \tag{13}$$

and, for example, $A_{+-} = A(x + i\epsilon, y - i\epsilon')$; $\rho'(x, y)$ is the analytic continuation of $\rho(x, y)$ in the region of integration in (12). $\rho(x, y)$ can be obtained either from the dispersion relation (5) or by calculating the Feynman integral for the diagram of Fig. 1 with all four denominators replaced by δ functions.[8]

We then obtain

$$J_{12} = \frac{1}{\pi^2} \int_{-\infty}^{\Delta_1} \frac{dx'}{x' - x} \int_{-\infty}^{\Delta_1'} \frac{dy'}{y' - y} \frac{2\pi\theta(K)}{\sqrt{K(x', y')}}, \tag{14}$$

$$J_{23} = \frac{1}{\pi^2} \int_{\Delta_1'}^{\Delta_2'} \frac{dy'}{y' - y} \int_{\Box_1(y' - i\varepsilon)}^{\Box_1(y' + i\varepsilon)} \frac{dx' \cdot P \cdot 2\pi}{(K' - K)\sqrt{K(x', y')}},$$

$$\arg K(x', y') = -\pi. \tag{15}$$

The path of integration in the x' plane is indicated in Fig. 3 by a wavy line.

Analogously,

$$J_{13} = \frac{1}{\pi^2} \int_{\Delta_1}^{\Delta_2} \frac{dx'}{x' - x} \int_{\Box_1(x' - i\varepsilon)}^{\Box_1(x' + i\varepsilon)} \frac{dy' \cdot Q \cdot 2\pi}{(K' - K)\sqrt{K(x', y')}}, \qquad \arg K = -\pi. \tag{16}$$

We see that the integration in J_{23} (J_{13}) goes only over that part of the intersection $M_{xy} \cap L_y$ ($M_{xy} \cap L_x$) which extends beyond the limits of L_x (L_y).

4. Let us discuss some of the properties of our representation. Assume that μ_{ik} is varied in such a way that $\theta \to 2\pi$. Then we must necessarily have $\Delta_2 \to \Delta_1$ and $\Delta_2' \to \Delta_1'$. The integrals (15) and (16) vanish since their limits become equal. Thus we obtain the Mandelstam representation (14) for $A(x, y)$.

The discontinuity of the function $A(x, y)$ on the cut L_x for $x < \Delta_1$ is given by the integral (14), and for $\Delta_1 < x < \Delta_2$, by the integral (16). The magnitudes of the discontinuities are calculated directly and are the same as in (5). The discontinuity of the function $A(x, y)$ on the anomalous cut is, by definition, equal to $-4\rho'$, as can be seen from (12). This result again agrees with (5).

Thus the representations (13), (15), and (16) have the necessary analytic properties. The transition from the double dispersion relations (14) to (16) to single variable dispersion relations can be effected in the same way as in [1].

5. We have considered a single but rather general case in which the only singular lines of the triangular type are $y = \Delta_2$ and $x = \Delta_2$. The proposed method of investigation is general and permits us to write down a double dispersion relation for arbitrary values of the internal masses.

In principle, it is possible to write down a representation of the amplitude $A(\mu_{ik})$ in more variables (three, four, five, and six), having in view a future study of the analytic properties of production amplitudes (many-point graphs). Here one must use the analysis of Wu,[9] who investigated the analytic properties of the amplitude $A(\mu_{ik})$ as a function of all six variables (p_{ik}^2 or μ_{ik}).

We note also that our method allows us to write down single-variable dispersion relations in an arbitrary region of variables in an easy manner, without using the rather laborious method of interlacing of contours which was employed, for example, by Gribov et al.[5]

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[1] Yu. A. Simonov, JETP **43**, 2263 (1962), Soviet Phys. JETP **16**, 1599 (1963).

[2] Fronsdal, Mahanthappa, and Norton, Phys. Rev. **127**, 1847 (1962).

[3] I. N. Islam, Univ. of Cambridge, preprint.

[4] Karplus, Sommerfield, and Wichman, Phys. Rev. **114**, 376 (1959).

[5] Gribov, Danilov, and Dyatlov, JETP **41**, 924 (1961), Soviet Phys. JETP **14**, 659 (1962).

[6] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

[7] J. Tarski, J. Math. Phys. **1**, 149 (1960).

[8] R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).

[9] A. Chi-Tai Wu, Mat. Fys. Medd. Dan. Vid. Selsk. **33**, Nr. 3 (1961).

[10] V. S. Vladimirov, preprint Joint Inst. Nuc. Res. R-449.

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